CHRISTIAN BERG

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DIRICHLET FORMS ON SYMMETRIC SPACES

by Christian BERG

CONTENTS

0. Introduction.

Beurling and Deny have reduced the problem of determining all Dirichlet forms on a locally compact space X endowed with a positive Radon measure ξ to the question of determining contraction-semigroups of hermitian operators on $L^2(X, \xi)$, which moreover are submarkovian, cf. [3]. In order to obtain a complete solution of the last problem, it is certainly necessary to impose further structure, and for example in the case of a locally compact abelian group X with Haar measure ξ, the translation invariant semigroups of the above type are characterized by the so-called negative definite functions on the dual group ([3]).

In the present paper we shall extend this result to a more general setting including the symmetric spaces. The results are most satisfactory in the compact case, where we generalize results obtained for the sphere in [1]. Furthermore there is a surprising analogy between the compact case and the symmetric spaces of noncompact type of rank one : The potentials of finite energy in invariant regular Dirichlet spaces are all square integrable with respect to Riemannian measure.

1. Description of the scope.

Let G be a locally compact group with left Haar measure dg, $L^1(G)$ the Banach algebra of complex functions integrable with respect to dg, considered as a subalgebra of $M^1(G)$, which consists f all bounded complex measures on G.

If K denotes a compact subgroup of G, $M^1(G)^\natural$ denotes the subalgebra of measures $\mu \in M^1(G)$, which are bi-invariant under K, i.e.

$$\varepsilon_k * \mu = \mu * \varepsilon_k = \mu \qquad \text{for every } k \in K.$$

In general, for a subset $A \subseteq M^1(G)$ we write $A^\natural$ for the set of elements of A, which are bi-invariant under K. For functions f the bi-invariance amounts to

$$f(gk) = f(kg) = f(g) \qquad \text{for } g \in G,\ k \in K,$$

because the modulus Δ of G is constant 1 on K.

For any continuous function f on G we define a continuous bi-invariant function $f^\natural$ on G as

$$f^\natural(g) = \int_K \int_K f(kgl)\, dk\, dl\,,$$

where dk, dl denote normalized Haar measure on K.

Notice that $(f^\natural)^\vee = (\check{f})^\natural$, where $\check{f}(g) = f(g^{-1})$.

We now fix a compact subgroup K of G and will always assume the fundamental hypothesis :

$M^1(G)^\natural$ *is commutative.* (1)

(1) This implies that G *is unimodular.* In fact if $\mathcal{K}(G)$ denotes the set of continuous functions with compact support, it suffices to verify $\int f(g)\,dg = \int f(g^{-1})\,dg$ for any $f \in \mathcal{K}(G)^\natural$. We choose $\varphi \in \mathcal{K}(G)^\natural$ to be 1 on the compact set $\mathrm{supp}(f) \cup \mathrm{supp}(f)^{-1}$ and get

$$\int f(g)\,dg = f * \varphi(e) = \varphi * f(e) = \int f(g^{-1})\,dg\,.$$

This hypothesis is fullfilled in the following three situations :

a) (G , K) is a Riemannian symmetric pair.

b) G is abelian, K $=\{e\}$.

c) G $= \mathcal{U} \times \mathcal{U}$ where $\mathcal{U}$ is any compact group, and K is the diagonal in G.

Let now X = G/K be the homogeneous space of left cosets, $\pi : G \mapsto X$ the canonical surjection. The action of G on X is denoted $(g, x) \mapsto g \cdot x$, where $g \cdot x = \pi(gg_1)$ if $x = \pi(g_1)$. Functions on G/K are identified with right invariant functions on G. If F is a function on G and $s \in G$, we let $\lambda(s)$ F denote the function $g \mapsto F(s^{-1} g)$, which is right invariant if F is so.

On X there is a unique G-invariant measure ξ fixed by the formula

$$\int_G F(g)\, dg = \int_X \left(\int_K F(gk)\, dk\right) d\xi(\pi(g)) \ . \tag{1}$$

Our main purpose is to characterize Dirichlet forms (Q , V) on $L^2(X, \xi)$ which are *G-invariant,* that is

$$\forall s \in G \quad \forall F \in V\ (\lambda(s)\, F \in V\ ,\ Q(\lambda(s)\, F) = Q(F))\ .$$

Here V is a dense subspace of $L^2(X, \xi)$ and Q is a closed positive hermitian form on V, and we suppose that the normal contractions operate in (Q , V), cf. [3].

In § 2 we give a brief summary of the harmonic analysis of the algebra $L^1(G)^{\natural}$. Everything is more or less in the article of Godement [9]. In the next § we prove that the G-invariant semigroups involved can be viewed as convolution semigroups of bi-invariant measures. The Fourier transformation of such semigroups leads to the notion of negative definite functions defined on the set Ω^+ of positive definite spherical functions.

In the abelian case b), this reduces to Schoenberg's notion of negative definite functions on an abelian group (viz. the dual group of G).

The G-invariant Dirichlet forms are in one to one correspondence with the real negative definite functions on Ω^+. This main result is established in § 4.

Next we give an intrinsic characterization of the negative definite functions. The result (theorem 5.2) is not true in full generality because the function 1 can be an isolated point in Ω^+.

Finally we consider the problem : When does a G-invariant Dirichlet form give rise to a Dirichlet space ? A complete answer is obtained in the compact case and for symmetric spaces of noncompact type of rank one.

2. Harmonic analysis on symmetric spaces.

We shall now summarize, how the Gelfand theory applies to the commutative Banach algebra $L^1(G)^\natural$, which has an involution $*$, $f^*(g) = \overline{f(g^{-1})}$, cf. [9], [11].

A (zonal) spherical function is any non-zero continuous solution $\omega : G \to C$ to the equation

$$\int_K \omega(g_1 k g_2)\, dk = \omega(g_1)\, \omega(g_2) \ , \ g_1, g_2 \in G \ . \tag{2}$$

A spherical function ω is bi-invariant and $\omega(e) = 1$. The maximal ideal space of $L^1(G)^\natural$ can be identified with the set Ω of bounded spherical functions, and the Gelfand transform of $f \in L^1(G)^\natural$ takes the form

$$\mathfrak{F} f(\omega) = \int_G f(g)\, \overline{\omega(g)}\, dg \ , \ \omega \in \Omega \ .$$

We also introduce the co-transform $\overline{\mathfrak{F}} f$ defined as

$$\overline{\mathfrak{F}} f(\omega) = \int_G f(g)\, \omega(g)\, dg \ .$$

Now let Ω^+ be the set of positive definite spherical functions. Then Ω^+ is a closed subset of Ω, and the topology on Ω^+ can be described as the topology of compact convergence over G, [4, th. 13.5.2]. This implies that $(g, \omega) \to \omega(g)$ is a continuous mapping from $G \times \Omega^+$ into C.

The main result in the whole theory is the *Godement-Plancherel theorem* :

For every positive definite measure μ on G there is a uniquely determined positive measure $\hat{\mu}$ on Ω^+ such that $\mathfrak{F}\varphi \in \mathcal{L}^2(\Omega^+, \hat{\mu})$ for all $\varphi \in \mathcal{K}(G)^\natural$, and such that for all $\varphi, \psi \in \mathcal{K}(G)^\natural$.

$$\mu(\varphi * \psi^*) = \int_{\Omega^+} \overline{\mathfrak{F}}\varphi(\omega)\, \overline{\overline{\mathfrak{F}}\psi(\omega)}\, d\hat{\mu}(\omega) \ . \tag{3}$$

(Note that $\overline{\mathfrak{F}}\varphi(\omega) = \mathfrak{F}\check{\varphi}(\omega)$ for $\omega \in \Omega^+$).

In particular if φ is a continuous bi-invariant positive definite function, we obtain the *Bochner theorem* :

$$\varphi(g) = \int_{\Omega^+} \omega(g)\, d\sigma(\omega) \tag{4}$$

where $\sigma = (\varphi dg)$ is a uniquely determined positive bounded measure on Ω^+.

Furthermore, for $\mu = \varepsilon_e$ the measure $\hat{\mu}$ is called *the Plancherel measure* and is denoted $d\omega$. Much trouble is caused by the fact, that the support Ω_0^+ of $d\omega$ can be a proper subset of Ω^+. The formula (3) is specialized to

$$\int_G \varphi(g)\, \overline{\psi(g)}\, dg = \int_{\Omega_0^+} \overline{\mathfrak{F}\varphi(\omega)}\ \overline{\mathfrak{F}\psi(\omega)}\, d\omega \ \ , \ \varphi, \psi \in \mathcal{K}(G)^\natural, \tag{5}$$

and $\mathfrak{F}$ can be extended in a unique way to an isometry of $L^2(G)^\natural$ onto $L^2(\Omega_0^+, d\omega)$.

For $\mu \in M^1(G)^\natural$ we define the transform $\mathfrak{F}\mu : \Omega^+ \to \mathbf{C}$ as the function

$$\mathfrak{F}\mu(\omega) = \int \overline{\omega(g)}\, d\mu(g) \ .$$

This extends $\mathfrak{F}$ from $L^1(G)^\natural$ to a homomorphism of the algebra $M^1(G)^\natural$ into the algebra of bounded continuous functions on Ω^+.

2.1. LEMMA. – *Let* $\mu \in M^1(G)^\natural$. *If* $\mathfrak{F}\mu(\omega) = 0$ *for all* $\omega \in \Omega_0^+$, *then* $\mu = 0$.

Proof. – If $\mu \in L^2(G)^\natural$ this is an immediate consequence of (5), and the general case is reduced to this situation by convolution with an approximative identity $(\varphi_V^\natural)$ in $L^1(G)^\natural$, which is obtained in the following way. For any neighbourhood V of the origin in G we choose a positive function $\varphi_V \in \mathcal{K}(G)$ with support in V and with integral 1. We next form the bi-invariant function $\varphi_V^\natural$. ‖

In particular it follows that $L^1(G)^\natural$ and $M^1(G)^\natural$ are semi-simple algebras.

2.2. *The inversion* THEOREM. – *Let* $\mu \in M^1(G)^\natural$ *and suppose that* $\mathfrak{F}\mu \in \mathcal{L}^1(\Omega_0^+, d\omega)$. *Then* μ *has a continuous density* φ *with respect to* dg *and*

$$\varphi(g) = \int_{\Omega_0^+} \omega(g)\, \mathfrak{F}\mu(\omega)\, d\omega\ .$$

Proof. – Suppose first that $f \in L^1(G)^\natural$ is continuous and bounded, and that $\mathfrak{F}f \in \mathcal{L}^1(\Omega_0^+, d\omega)$. The function

$$\varphi(g) = \int_{\Omega_0^+} \omega(g)\, \mathfrak{F}f(\omega)\, d\omega$$

is continuous, bi-invariant and bounded. Since also $f \in L^2(G)^\natural$ (5) gives

$$\int_G \psi(g)\, \overline{\varphi(g)}\, dg = \int_{\Omega_0^+} \mathfrak{F}\psi(\omega)\, \overline{\mathfrak{F}f(\omega)}\, d\omega = \int_G \psi(g)\, \overline{f(g)}\, dg$$

for all $\psi \in \mathcal{K}(G)^\natural$, and we get $\varphi = f$.

This result applies to $\mu * \varphi_V^\natural$, where as before $(\varphi_V^\natural)$ is an approximative identity, and we get

$$\mu * \varphi_V^\natural(g) = \int_{\Omega_0^+} \omega(g)\, \mathfrak{F}\mu(\omega)\, \mathfrak{F}\varphi_V^\natural(\omega)\, d\omega\ .$$

This implies the desired result, because $\mathfrak{F}\varphi_V^\natural \mapsto 1$ uniformly over compact subsets of Ω^+ as V shrinks to e. ‖

2.3. COROLLARY. – *Suppose that* $\mu \in M^1(G)^\natural$. *Then* μ *is positive definite if and only if* $\mathfrak{F}\mu$ *is positive on* Ω_0^+.

If this is the case, the measure $\hat{\mu}$ *in the Godement-Plancherel theorem has* $\mathfrak{F}\mu$ *as density with respect to the Plancherel measure* $d\omega$, *i.e.* $\hat{\mu} = \mathfrak{F}\mu(\omega)\, d\omega$.

Proof. – Suppose first that μ is positive definite. We will show that the measure $\mathcal{F}\mu(\omega)\,d\omega$ satisfies (3).

For $\varphi, \psi \in \mathcal{K}(G)^{\natural}$, $h = \varphi * \psi^*$, the inversion theorem gives

$$h(g) = \int_{\Omega_0^+} \overline{\omega(g)}\, \mathcal{F}\check{h}(\omega)\, d\omega ,$$

and then we have

$$\int_{\Omega_0^+} \overline{\mathcal{F}}h(\omega)\, \mathcal{F}\mu(\omega)\, d\omega = \int_{\Omega_0^+} \left(\mathcal{F}\check{h}(\omega) \int_G \overline{\omega(g)}\, d\mu(g) \right) d\omega =$$

$$= \int_G h(g)\, d\mu(g) ,$$

which shows that $\hat{\mu} = \mathcal{F}\mu(\omega)\,d\omega$. Since $\hat{\mu}$ is a positive measure, it follows that $\mathcal{F}\mu(\omega) \geqslant 0$ for all $\omega \in \Omega_0^+$.

Suppose next that $\mathcal{F}\mu$ is positive on Ω_0^+. If $\mathcal{F}\mu$ is integrable with respect to the Plancherel measure, the inversion theorem implies that

$$\mu(f * f^*) = \int_G \left(f * f^*(g) \int_{\Omega_0^+} \omega(g)\, \mathcal{F}\mu(\omega)\, d\omega \right) dg \geqslant 0 ,$$

for all $f \in \mathcal{K}(G)$, i.e. μ is positive definite.

If as before $(\varphi_V^{\natural})$ denotes an approximative identity, we find that $\mu * \varphi_V^{\natural} * (\varphi_V^{\natural})^*$ is positive definite, because $\mathcal{F}\mu\, |\mathcal{F}\varphi_V^{\natural}|^2$ is integrable and positive on Ω_0^+. If we let V shrink to e, we get the desired result. ‖

For each $\omega \in \Omega^+$ we have a canonically defined Hilbert space H_ω with a unit vector e_ω, and a continuous irreducible representation π_ω of G in H_ω such that e_ω is a cyclic vector. Moreover

$$\omega(g) = (e_\omega , \pi_\omega(g) e_\omega) ,$$

and $\pi_\omega(k)\, e_\omega = e_\omega$ for all $k \in K$, i.e. π_ω is of class one.

The representation π_ω can in the usual way be extended to a representation of $M^1(G)$.

2.4. Lemma. – *For $\mu \in M^1(G)^{\natural}$ the operator $\pi_\omega(\mu)$ is given by*

$$\pi_\omega(\mu)\, a = \mathcal{F}\mu(\omega)\, (a , e_\omega)\, e_\omega \ , \ a \in H_\omega , \omega \in \Omega^+ .$$

In particular $\pi_\omega(\mu)$ is in the trace class and $tr\, \pi_\omega(\mu) = \mathcal{F}\mu(\omega)$.

Proof. – Let $g_1, g_2 \in G$, $a = \pi_\omega(g_1)e_\omega$; $b = \pi_\omega(g_2^{-1})e_\omega$. It suffices to show

$$(\pi_\omega(\mu)\, a\, , b) = \mathcal{F}\mu(\omega)\, (a, e_\omega)\, (e_\omega\, , b) = \mathcal{F}\mu(\omega)\, \overline{\omega(g_1)}\; \overline{\omega(g_2)}\ .$$

We have

$$(\pi_\omega(\mu)\, a\, , b) = \int_G (\pi_\omega(g)\, a\, , b)\, d\mu(g) = \int_G \overline{\omega(g_2 g g_1)}\, d\mu(g)\ ,$$

but since μ is bi-invariant, we also have

$$\int_G \overline{\omega(g_2 g g_1)}\, d\mu(g) = \int_G \overline{\omega}(g_2\, kg\, lg_1)\, d\mu(g)$$

for all $k, l \in K$. Thus, integrating over K with respect to k and l, we obtain

$$(\pi_\omega(\mu)\, a\, , b) = \overline{\omega(g_1)}\; \overline{\omega(g_2)} \int_G \overline{\omega(g)}\, d\mu(g)\ . \ \|$$

2.5. THEOREM. – *For any function* $F \in L^1 \cap L^2(X\, , \xi)$ *and any* $\omega \in \Omega^+$, $\pi_\omega(F)$ *is a Hilbert-Schmidt operator in* H_ω. *The function* $\omega \to \|\pi_\omega(F)\|_{H.S.}$ *is continuous and square integrable with respect to the Plancherel measure, and satisfies*

$$\int_G |F(g)|^2\, dg = \int_{\Omega_0^+} \|\pi_\omega(F)\|_{H.S.}^2\, d\omega\ . \tag{6}$$

The mapping $F \to (\pi_\omega(F))_\omega$ *can be extended uniquely to an isometry of* $L^2(X\, , \xi)$ *into the Hilbert space of square integrable vector fields of Hilbert-Schmidt operators.*

Proof. – The function $h = F^* * F$ is bi-invariant and positive definite. The measure $\mathcal{F}h(\omega)\, d\omega$ in the Godement-Plancherel theorem is bounded, so $\mathcal{F}h$ is integrable. By the inversion theorem we then have

$$\int_G |F(g)|^2\, dg = h(e) = \int_{\Omega_0^+} \mathcal{F}h(\omega)\, d\omega\ ,$$

but since $\pi_\omega(h) = \pi_\omega(F)^*\, \pi_\omega(F)$ is in the trace class, $\pi_\omega(F)$ is a Hilbert-Schmidt operator and

$$\mathcal{F}h(\omega) = tr\,\pi_\omega(h) = \|\pi_\omega(\mathrm{F})\|^2_{\mathrm{H.S.}}\ .$$

The extension to $\mathrm{L}^2(\mathrm{X}\,,\xi)$ is classical, cf. [8]. ∎

3. Characterization of G-invariant semigroups.

We shall now deal with the question of determining the strongly continuous contraction-semigroups of submarkovian operators in $\mathrm{L}^2(\mathrm{X}\,,\xi)$, which commute with the action of G in X.

3.1. LEMMA. – *In order that a bounded operator* T *in* $\mathrm{L}^2(\mathrm{X}\,,\xi)$ *satisfies*

i) $\mathrm{T}(\lambda(s)\,\mathrm{F}) = \lambda(s)\,\mathrm{T}(\mathrm{F})$ *for all* $s \in \mathrm{G}$, $\mathrm{F} \in \mathrm{L}^2(\mathrm{X}\,,\xi)$.

ii) *If* $0 \leqslant \mathrm{F} \leqslant 1$ ξ-a.e., *then* $0 \leqslant \mathrm{TF} \leqslant 1$ ξ-a.e. (T *is submarkovian*)

it is necessary and sufficient that there is a positive bi-invariant measure μ *on* G *with* $\|\mu\| \leqslant 1$, *such that* $\mathrm{TF} = \mathrm{F} * \mu$ *for all* $\mathrm{F} \in \mathrm{L}^2(\mathrm{X}\,,\xi)$.

The measure μ *is uniquely determined.*

Proof. – It is immediate to verify that $\mathrm{TF} = \mathrm{F} * \mu$ defines a bounded operator with the properties i) and ii).

For the converse we proceed as follows :

a) We first assume that TF is continuous for every $\mathrm{F} \in \mathrm{L}^2(\mathrm{X}\,,\xi)$. It is easy to see that i) implies that $\mathrm{T}(\mathrm{F})^\natural = \mathrm{T}(\mathrm{F}^\natural)$ for all $\mathrm{F} \in \mathcal{K}(\mathrm{X})$. The mapping $\mathrm{F} \mapsto \mathrm{T}((\mathrm{F}^\natural)^\vee)(e)$ is a positive linear form on $\mathcal{K}(\mathrm{G})$, so it defines a positive Radon measure μ on G, which is clearly bi-invariant and of total mass $\leqslant 1$.

For $\mathrm{F} \in \mathcal{K}(\mathrm{X})$ we have

$$\mathrm{F} * \mu(e) = \int \check{\mathrm{F}}\, d\mu = \mathrm{T}(\mathrm{F}^\natural)(e) = \mathrm{T}(\mathrm{F})^\natural(e) = \mathrm{T}(\mathrm{F})(e)\ ,$$

and finally for any $g \in \mathrm{G}$

$$\mathrm{F} * \mu(g^{-1}) = [\lambda(g)\,\mathrm{F}] * \mu(e) = \mathrm{T}(\lambda(g)\,\mathrm{F})(e) = \lambda(g)\,\mathrm{T}(\mathrm{F})(e) = \\ = \mathrm{T}(\mathrm{F})(g^{-1})\ .$$

By the density of $\mathcal{K}(X)$ in $L^2(X, \xi)$ it follows that $TF = F * \mu$ for all $F \in L^2(X, \xi)$.

b) In the general case we consider the operators

$$T_V F = TF * \varphi_V^{\natural} \quad , \quad F \in L^2(X, \xi) \, ,$$

where $(\varphi_V^{\natural})$ is an approximative identity. Now T_V satisfy i) and ii) and the condition of case a), so there is a positive bi-invariant measure μ_V with $\|\mu_V\| \leqslant 1$ such that $T_V F = F * \mu_V$. If μ is a vague accumulation point for the net (μ_V), it is easy to see that $TF = F * \mu$ (in $L^2(X, \xi)$) for all $F \in \mathcal{K}(X)$, and then for all $F \in L^2(X, \xi)$. ∎

If in the above correspondence μ is associated with T, the adjoint operator T^* is associated with $\check{\mu}$, and if S is another operator of the same type associated with ν, then ST is associated with $\mu * \nu$. This proves in particular that two bounded operators in $L^2(X, \xi)$ commute, if they satisfy the conditions of the lemma. The identity operator is associated with the normalized Haar measure ω_K of K (considered as a measure on G).

DEFINITION. – *A vaguely continuous convolution semigroup on* G *is a family* $(\mu_t)_{t \geqslant 0}$ *of positive bi-invariant measures on* G *satisfying*

i) $\mu_s * \mu_t = \mu_{s+t}$, $s, t \geqslant 0$, $\mu_0 = \omega_K$.

ii) $\|\mu_t\| \leqslant 1$.

iii) $\mu_t \to \omega_K$ vaguely as $t \to 0$.

Under these conditions we even have $\mu_t(f) \to \omega_K(f)$ for $t \to 0$ for all continuous bounded functions f. It is then easy to obtain the following result :

3.2. THEOREM. – *There is a one to one correspondence between strongly continuous contraction-semigroups* $(T_t)_{t \geqslant 0}$ *of operators in* $L^2(X, \xi)$ *satisfying* i) *and* ii) *of lemma 3.1, and vaguely continuous convolution semigroups* $(\mu_t)_{t \geqslant 0}$ *on* G. *The correspondence is given by*

$$T_t F = F * \mu_t \quad , \quad t \geqslant 0 \quad , \quad F \in L^2(X, \xi) \, .$$

DEFINITION. – *A continuous function* $p : \Omega^+ \to \mathbf{C}$ *is called positive definite,* if $p = \mathcal{F}\mu$ *for some (necessarily unique) positive measure* $\mu \in M^1(G)^{\natural}$.

Notice that $|p(\omega)| \leqslant p(1) = \|\mu\|$ for all $\omega \in \Omega^+$.

A continuous function $q : \Omega^+ \to \mathrm{C}$ is called *negative definite,* if $q(1) \geqslant 0$ and if $\exp(-tq)$ is positive definite for all $t > 0$.

Notice that

$$|e^{-tq(\omega)}| \leqslant e^{-tq(1)} \leqslant 1 \quad , \quad \omega \in \Omega^+ \ ,$$

for all $t \geqslant 0$. This implies that $\mathrm{Re}\, q \geqslant q(1) \geqslant 0$.

The sets $\mathcal{P}$ and $\mathcal{Q}$ of positive and negative definite functions are convex cones containing the positive constant functions. The cone $\mathcal{P}$ is even stable under multiplication.

In § 5 we are concerned with an intrinsic characterization of these cones.

3.3. THEOREM. – *There is a one to one correspondence between negative definite functions q on Ω^+ and vaguely continuous convolution semigroups $(\mu_t)_{t \geqslant 0}$ on* G. *The correspondence is given by*

$$\mathcal{F}\mu_t(\omega) = e^{-tq(\omega)} \quad , \quad t \geqslant 0 \quad , \quad \omega \in \Omega^+ \ . \tag{7}$$

Proof. – If $(\mu_t)_{t \geqslant 0}$ is given, and $\omega \in \Omega^+$ is fixed, there is a uniquely determined complex number $q(\omega)$ such that (7) is fullfilled for all $t \geqslant 0$. Since all the functions $\exp(-tq)$, $t \geqslant 0$ are continuous on the locally compact space Ω^+, the next lemma shows that q is continuous. Since $\exp(-tq(1)) = \|\mu_t\| \leqslant 1$ for all $t \geqslant 0$, we have $q(1) \geqslant 0$.

Conversely, if q is negative definite, we have by definition a family $(\mu_t)_{t \geqslant 0}$ of positive bi-invariant measures satisfying (7). Consequently we have

$$\mathcal{F}(\mu_s * \mu_t) = \mathcal{F}(\mu_s)\,\mathcal{F}(\mu_t) = e^{-(s+t)q} = \mathcal{F}(\mu_{s+t}) \ ,$$

which shows the semigroup property. The formula

$$\mu_t(\varphi * \psi^*) = \int_{\Omega_0^+} \mathcal{F}\varphi(\omega)\,\overline{\mathcal{F}\psi(\omega)}\, e^{-tq(\omega)}\, d\omega \quad , \quad \varphi\,, \psi \in \mathcal{K}(\mathrm{G})^{\natural},$$

implies that $\mu_t(\varphi * \psi^*) \to \omega_{\mathrm{K}}(\varphi * \psi^*)$ for $t \to 0$, because

$$|e^{-tq(\omega)}| = |\mathcal{F}\mu_t(\omega)| \leqslant \|\mu_t\| = e^{-tq(1)} \leqslant 1 \ .$$

This is sufficient to ensure the continuity property of the semigroup. ‖

3.4. LEMMA. – *Let* $f : \mathrm{Y} \to \mathbf{R}$ *be a real function on a compact Hausdorff space* Y. *If* $f_t : \mathrm{Y} \to \mathbf{C}$ *given as*

$$f_t(y) = \exp(itf(y))$$

is continuous for every $t \in \mathbf{R}$, *then* f *is continuous.*

Proof. – Let $\mathbf{T}$ be the group of complex numbers with absolute value 1 and define

$$\beta : \mathbf{R} \to \mathbf{T}^{\mathbf{R}} \quad \text{as} \quad \beta(x)(t) = \exp(itx) \ .$$

By definition $\beta \circ f$ is continuous, so $\beta(f(\mathrm{Y}))$ is compact. A theorem of Glicksberg [7] then shows, that $f(\mathrm{Y})$ is compact in $\mathbf{R}$, and now it is easy to prove that f is continuous. ‖

Remarks. – The lemma extends to k-spaces, in particular to locally compact spaces. On the other hand, if we put $\mathrm{Y} = \beta(\mathbf{R})$, $f = \beta^{-1}$ we get an example which shows, that the lemma is false for topological spaces in general.

Note that a positive definite function $p = \mathcal{F}\mu$ on Ω^+ is real if and only if μ is symmetric ($\mu = \check{\mu}$), and consequently a negative definite function q on Ω^+ is real (and then positive) if and only if the corresponding convolution semigroup $(\mu_t)_{t>0}$ consists of symmetric measures. It follows from corollary 2.3, that the measures μ_t are positive definite in this case.

4. Characterization of G-invariant Dirichlet forms.

4.1. THEOREM. – *There is a one to one correspondence between G-invariant Dirichlet forms* (Q , V) *on* $\mathrm{L}^2(\mathrm{X}, \xi)$ *and real negative definite functions* q *on* Ω^+. *The correspondence is given by*

$$\mathrm{Q(F)} = \int_{\Omega_0^+} \| \pi_\omega(\mathrm{F}) \|^2_{\mathrm{H.S.}}\, q(\omega)\, d\omega \quad \textit{for} \quad \mathrm{F} \in \mathrm{V} \ , \tag{8}$$

and V *is the set of functions* $F \in L^2(X, \xi)$ *for which the integral in* (8) *is finite.*

Proof. – By the general theorem of Beurling and Deny (cf. [3]), there is a one to one correspondence between the Dirichlet forms (Q, V) on $L^2(X, \xi)$ and the strongly continuous contraction-semigroups $(T_t)_{t>0}$ of hermitian and submarkovian operators in $L^2(X, \xi)$. The correspondence is given by

$$\lim_{t\to 0}\left(\frac{1}{t}(F - T_t F), F\right) = \begin{cases} Q(F) & \text{for } F \in V \\ \infty & \text{for } F \in L^2(X, \xi)\backslash V . \end{cases}$$

The Dirichlet form (Q, V) is G-invariant if and only if each of the operators T_t satisfy i) of lemma 3.1. (For the "only if" part one considers the semigroup $T_t^{(s)}F = \lambda(s^{-1})\, T_t(\lambda(s) F)$, $s \in G$).

The correspondence is now proved by theorem 3.2 and 3.3. In order to prove (8), we use the expressions

$$T_t F = F * \mu_t \quad , \quad \mathcal{F}\mu_t(\omega) = e^{-tq(\omega)} \tag{9}$$

For $F \in L^1 \cap L^2(X, \xi)$ we get by theorem 2.5 and lemma 2.4

$$\left(\frac{1}{t}(F - T_t F), F\right) = \left(F * \frac{1}{t}(\omega_K - \mu_t), F\right) =$$

$$= \int_{\Omega_0^+} tr\left(\pi_\omega(F)^* \pi_\omega\left(F * \frac{1}{t}(\omega_K - \mu_t)\right)\right) d\omega$$

$$= \int_{\Omega_0^+} tr\, \pi_\omega\left((F^* * F) * \frac{1}{t}(\omega_K - \mu_t)\right) d\omega =$$

$$= \int_{\Omega_0^+} \frac{1}{t}(1 - e^{-tq(\omega)})\, \|\pi_\omega(F)\|_{H.S.}^2\, d\omega .$$

By continuity, this formula holds for all $F \in L^2(X, \xi)$. When t decreases to zero, $t^{-1}(1 - \exp(-tq(\omega)))$ increases to $q(\omega)$, and the proof is finished. ‖

If G is compact, H_ω is of finite dimension N_ω and can be taken to be the subspace of $L^2(X, \xi)$ spanned by the functions $\lambda(s)\,\omega$, $s \in G$. Furthermore $L^2(X, \xi)$ is the Hilbert sum of the spaces H_ω, $\omega \in \Omega^+$, so $F \in L^2(X, \xi)$ has a L^2-expansion

$$F = \sum_{\omega \in \Omega^+} P_\omega F \ ,$$

where P_ω is the orthogonal projection on H_ω. It turns out that

$$P_\omega F = N_\omega F * \omega \quad , \quad \| P_\omega F \|^2 = N_\omega \| \pi_\omega(F) \|^2_{\mathrm{H.S.}} \ .$$

Formula (8) is now reduced to

$$Q(F) = \sum_{\omega \in \Omega^+} \| P_\omega F \|^2 \, q(\omega) \ , \tag{10}$$

because Ω^+ is discrete, and the Plancherel measure has the mass N_ω in ω. (Note that $\Omega = \Omega^+ = \Omega_0^+$). This generalizes results for the sphere [1].

In the case $G = \mathcal{U} \times \mathcal{U}$ where $\mathcal{U}$ is a compact group and K is the diagonal in G, we obtain a characterization of the Dirichlet forms on the compact group $\mathcal{U}$, which are invariant under the inner automorphisms of $\mathcal{U}$.

Finally, in the case where G is abelian, $K = \{e\}$, theorem 4.1 reduces to the theorem of Beurling and Deny [3 p. 190].

5. Positive and negative definite functions on Ω^+.

We now introduce intrinsic definitions of positive and negative definite functions.

DEFINITION. – *A continuous function $p : \Omega^+ \to \mathbf{C}$ is called a* PD-*function, if the following property holds :*

$$\forall a_1, \ldots, a_n \in \mathbf{C} , \forall \omega_1, \ldots, \omega_n \in \Omega^+$$

$$\left(\mathrm{Re}\left(\sum_{i=1}^n a_i \omega_i\right) \geq 0 \ \text{ on } \ G \Rightarrow \mathrm{Re}\left(\sum_{i=1}^n a_i p(\omega_i)\right) \geq 0\right) .$$

A continuous function $q : \Omega^+ \to \mathbf{C}$ is called a ND-*function* if $q(1) \geq 0$, and if the following property holds :

$\forall a_1, \ldots, a_n \in \mathbf{C}, \forall \omega_1, \ldots, \omega_n \in \Omega^+$

$$\left(\sum_{i=1}^{n} a_i = 0 , \operatorname{Re}\left(\sum_{i=1}^{n} a_i \omega_i\right) \geqslant 0 \text{ on } \mathrm{G} \Rightarrow \operatorname{Re}\left(\sum_{i=1}^{n} a_i \, q(\omega_i)\right) \leqslant 0\right).$$

5.1. THEOREM. – *Every positive (resp. negative) definite function on Ω^+ is a* PD- *(resp.* ND-*) function.*

Proof. – If $p = \mathcal{F}\mu$ for $\mu \in \mathrm{M}^1_+(\mathrm{G})^\natural$, and if

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i \, \omega_i\right) \geqslant 0 \quad \text{on} \quad \mathrm{G} ,$$

we have

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i p(\omega_i)\right) = \int \operatorname{Re}\left(\sum_{i=1}^{n} a_i \, \omega_i(g)\right) d\check{\mu}(g) \geqslant 0 .$$

Suppose next that q is negative definite and that

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i \, \omega_i\right) \geqslant 0 \quad \text{on G} \quad \text{with } \sum_{i=1}^{n} a_i = 0 .$$

This implies that for all $t > 0$

$$\operatorname{Re}\left(\sum_{i=1}^{n} a_i \frac{1}{t}\left(1 - e^{-tq(\omega_i)}\right)\right) \leqslant 0 ,$$

and for $t \to 0$ we get $\operatorname{Re}\left(\sum_{i=1}^{n} a_i \, q(\omega_i)\right) \leqslant 0$. ▯

The converse to theorem 5.1 is not true in general, because there are symmetric spaces of rank one, for which 1 is an isolated point in Ω^+, [12](1). On the other hand it is simple to check that PD- and ND-functions are usual continuous positive and negative definite functions on Ω^+ in the case of G abelian, $\mathrm{K} = \{e\}$, in the case of which Ω^+ is the character group of G.

We shall now prove that the converse of theorem 5.1 is true, whenever G is compact.

(1) In these cases the function $p(\omega) = 0$ for $\omega \neq 1$, $p(1) = 1$ is a PD-function, but not a positive definite function.

5.2. THEOREM. – [1] *If* G *is compact, then every* PD- (*resp.* ND-) *function is a positive (resp. negative) definite function.*

Proof. – a) The subspace E of $\mathcal{K}(G)^\natural$ spanned by the spherical functions $\omega \in \Omega^+$ is dense in $\mathcal{K}(G)^\natural$ under the uniform norm (lemma 2.1). If p is given to be a PD-function, we can in a unique way extend p to a linear form $L_p : E \to C$, namely

$$L_p\Big(\sum_{i=1}^n a_i \omega_i\Big) = \sum_{i=1}^n a_i p(\omega_i) .$$

The definition of a PD-function implies that L_p is real and positive, and consequently continuous. This shows that there is a uniquely determined positive bi-invariant measure μ on G such that

$$L_p(f) = \int f d\mu \quad \text{for all } f \in E ,$$

in particular $\mathcal{F}\check{\mu}(\omega) = p(\omega)$ for all $\omega \in \Omega^+$.

b) Let $q : \Omega^+ \to C$ be a ND-function. We define the operator A in E by

$$A\Big(\sum_{i=1}^n a_i \omega_i\Big) = -\sum_{i=1}^n a_i q(\omega_i)\, \omega_i$$

and shall prove that A satisfies *the maximum principle* :

If $f = \sum_{i=1}^n a_i \omega_i$ satisfies $\operatorname{Re} f(g_0) = \sup_G \operatorname{Re} f \geq 0$ for some $g_0 \in G$, then $\operatorname{Re} Af(g_0) \leq 0$.

To see this put $h = \sum_{i=1}^n a_i \omega_i(g_0)\, \omega_i$, so that

$$\sup \operatorname{Re} h \geq \operatorname{Re} h(e) = \operatorname{Re} f(g_0) \geq 0 .$$

Next, note that $p_g(\omega) = \omega(g_0)\, \omega(g)$ is a positive definite function on Ω^+ for fixed g, so since

[1] A mild modification of the proof gives that the theorem is valid in the non-compact case if all the functions $\omega \in \Omega^+ \backslash \{1\}$ tend to 0 at infinity, and if 1 is not an isolated point in Ω^+. These conditions are satisfied for instance for the symmetric spaces of euclidean type and the real and complex hyperbolic spaces.

$$\operatorname{Re}\Big(f(g_0)\, 1 - \sum_{i=1}^{n} a_i\, \omega_i \Big) \geqslant 0 \quad \text{on } G \,,$$

we get

$$\operatorname{Re}\ \Big(f(g_0) - \sum_{i=1}^{n} a_i\, p_g(\omega_i)\Big) \geqslant 0 \,,$$

that is $\operatorname{Re} h \leqslant \operatorname{Re} f(g_0)$.

This shows that $\sup \operatorname{Re} h = \operatorname{Re} h(e) \geqslant 0$, so we have

$$\operatorname{Re}\ \Big(h(e)\cdot 1 - \sum_{i=1}^{n} a_i\, \omega_i(g_0)\, \omega_i\Big) \geqslant 0 \quad \text{on } G \,,$$

which implies that

$$\operatorname{Re}\ \Big(h(e)\, q(1) - \sum_{i=1}^{n} a_i\, \omega_i(g_0)\, q(\omega_i)\Big) \leqslant 0 \,,$$

i.e. $\operatorname{Re} Af(g_0) \leqslant - q(1) \operatorname{Re} h(e) \leqslant 0$.

c) We next show that $(\lambda I - A)\, E = E$ for $\lambda > 0$, where I is the identity operator on E.

For any $\omega \in \Omega^{+}$ we have $\operatorname{Re}(1 - \omega) \geqslant 0$ on G, which implies that $\operatorname{Re} q(\omega) \geqslant \operatorname{Re} q(1) = q(1) \geqslant 0$. Since

$$(\lambda I - A)\Big(\sum_{i=1}^{n} a_i\, \omega_i\Big) = \sum_{i=1}^{n} (\lambda + q(\omega_i))\, a_i\, \omega_i \,,$$

it is clear that $(\lambda I - A)\, E = E$ for $\lambda > 0$ and that

$$(\lambda I - A)^{-1}\Big(\sum_{i=1}^{n} a_i\, \omega_i\Big) = \sum_{i=1}^{n} \frac{a_i}{\lambda + q(\omega_i)}\, \omega_i \,.$$

In particular $(\lambda I - A)\, E$ is dense in $\mathcal{K}(G)^{\natural}$ for $\lambda > 0$, and then one knows (cf. [2] or [13]), that the closure of A is the infinitesimal generator of a Feller semigroup $(P_t)_{t\geqslant 0}$ on $\mathcal{K}(G)^{\natural}$. (One should think of $\mathcal{K}(G)^{\natural}$ as the space of continuous functions on the double coset space $K\backslash G/K$).

The resolvent $(V_\lambda)_{\lambda>0}$ of the semigroup is given as $V_\lambda = (\lambda I - A)^{-1}$ on E. We therefore obtain

$$\exp(t\lambda(\lambda V_\lambda - I))\Big(\sum_{i=1}^{n} a_i\, \omega_i\Big) = \sum_{i=1}^{n} \exp\Big(- \frac{t\lambda\, q(\omega_i)}{\lambda + q(\omega_i)}\Big)\, a_i\, \omega_i \,.$$

For λ tending to ∞ we get

$$\mathrm{P}_t\left(\sum_{i=1}^{n} a_i\,\omega_i\right) = \sum_{i=1}^{n} \exp(-\,tq(\omega_i))\,a_i\,\omega_i\ ,$$

which shows that $\exp(-\,tq)$ is a PD-function for all $t > 0$. ‖

In the important case where G is a noncompact connected semisimple Lie group with finite center, and K a maximal compact subgroup of G, the negative definite functions on Ω^+ can be expressed by a Levi-Khinchine formula due to Gangolli [6].

In this case we have

5.3. Lemma. – *Let* $q : \Omega^+ \to \mathrm{R}$ *be a real negative definite function. If* q *is not identical zero, then* $q(\omega) > 0$ *for all* $\omega \in \Omega_0^+$.

Proof. – For $\omega \in \Omega_0^+$ it is well known that $\omega(x) \to 0$ for $x \to \infty$ in G (Theorem 2 p. 585 in [10]). Consequently $\{x \in \mathrm{G} \mid \omega(x) = 1\}$ is a compact set, but just by the fact that ω is positive definite, it follows that $\{x \in \mathrm{G} \mid \omega(x) = 1\}$ is a subgroup. Thus, by the maximality of K, we conclude that $\{x \in \mathrm{G} \mid \omega(x) = 1\} = \mathrm{K}$.

If we suppose that $q(\omega) = 0$ for some $\omega \in \Omega_0^+$, we have by (7) that

$$1 = \mathscr{F}\mu_t(\omega) = \int_{\mathrm{G}} \operatorname{Re}\omega(x)\,d\mu_t(x) \quad \text{for all } t > 0\ ,$$

which implies that $\operatorname{supp}\mu_t \subseteq \mathrm{K}$, $\|\mu_t\| = 1$ for all $t > 0$. Since μ_t is bi-invariant, we must have $\mu_t = \omega_{\mathrm{K}}$ for all $t > 0$ i.e. $q \equiv 0$. ‖

If we furthermore suppose that X is of rank 1 the following holds :

For $x \in \mathrm{G}\backslash\mathrm{K}$, *the function* $\omega \mapsto \omega(x)$ *tends to zero at infinity on the locally compact space* Ω_0^+.

For a proof, see f. ex. [5, th. 2]. It seems to be unknown whether the same property holds for higher rank.

This has the following consequence :

5.4. Corollary. – *Suppose that* X *is a symmetric space of noncompact type and of rank* 1, *and that* $q : \Omega^+ \to \mathrm{R}$ *is a real negative* *definite function not identical zero. Then there is a constant $a > 0$ such that $q(\omega) \geqslant a$ for all $\omega \in \Omega_0^+$.*

Proof. – Put $a = \inf\{q(\omega) \mid \omega \in \Omega_0^+\}$ and suppose that $a = 0$. We then have a sequence $\omega_n \in \Omega_0^+$ such that $\lim q(\omega_n) = 0$. Suppose that some subsequence ω_{n_p} is contained in a compact subset of Ω_0^+. For a cluster point ω_0 of ω_{n_p} we would then have $q(\omega_0) = 0$, and then q is identical zero. Thus $\omega_n \to \infty$ in Ω_0^+. By (7) we have

$$\int_G \omega_n(x)\, d\mu_t(x) = e^{-tq(\omega_n)} \quad \text{for all } t > 0 .$$

The above property of the spherical functions implies that the integral tends to $\mu_t(\mathrm{K})$, so we get $\mu_t(\mathrm{K}) = 1$ for all $t > 0$. Consequently we have $\mu_t = \omega_\mathrm{K}$ for all $t > 0$ and thus $q \equiv 0$, which is a contradiction. ‖

6. Characterization of G-invariant Dirichlet spaces.

Let (Q, V) be a G-invariant Dirichlet form on $\mathrm{L}^2(\mathrm{X}, \xi)$. It is known (cf. [3]) that Q is positive definite and that the completion $\hat{\mathrm{V}}$ of V under the norm $\mathrm{Q}^{1/2}$ is a Dirichlet space if and only if

$$\int_0^\infty (\mathrm{T}_t \mathrm{F}, \mathrm{F})\, dt < \infty \quad \text{for all } \mathrm{F} \in \mathcal{K}_+(\mathrm{X}) , \tag{11}$$

$(\mathrm{T}_t)_{t>0}$ being the corresponding semigroup.

Using the expressions (9) this condition amounts to the requirement that

$$\int_{\Omega_0^+} \|\pi_\omega(\mathrm{F})\|_{\mathrm{H.S.}}^2 \; \frac{1}{q(\omega)}\, d\omega < \infty \tag{12}$$

for all $\mathrm{F} \in \mathcal{K}_+(\mathrm{X})$. ($\|\pi_\omega(\mathrm{F})\|_{\mathrm{H.S.}}^2$ is a function in $\mathrm{C}_0(\Omega^+) \cap \mathcal{L}^1(\Omega_0^+, d\omega)$).

For any $\omega \in \Omega_0^+$ there is $\varphi \in \mathcal{K}_+(\mathrm{G})^\natural$ such that $|\mathcal{F}\varphi(\omega)|^2 > 0$, and (12) then implies that $1/q$ is integrable over some neighbourhood of ω with respect to the Plancherel measure. We have thus proved :

6.1. Lemma. – *Let (Q, V) be a G-invariant Dirichlet form and suppose that Q is positive definite and that the completion $\hat{\mathrm{V}}$ of V under $\mathrm{Q}^{1/2}$ is a Dirichlet space. Then $1/q \in \mathcal{L}^1_{\mathrm{loc}}(\Omega_0^+, d\omega)$.*

We do not know whether the converse of lemma 6.1 is valid in general, i.e. if $1/q \in \mathcal{L}^1_{\mathrm{loc}}(\Omega_0^+, d\omega)$, is it then true that Q is positive definite and that $\hat{\mathrm{V}}$ is a (necessarily G-invariant) Dirichlet space ?

This is known however in the abelian case (see [3]), and we shall now prove it in the compact case and when X is a symmetric space of noncompact type of rank 1.

6.2. THEOREM. – *Suppose that* G *is compact and let* (Q , V) *be a* G-*invariant Dirichlet form on* $\mathrm{L}^2(\mathrm{X}, \xi)$ *with associated negative definite function* q. *Then* $1/q \in \mathcal{L}^1_{\mathrm{loc}}(\Omega_0^+, d\omega)$ *if and only if* $q(1) > 0$, *and in this case* Q *is positive definite. Moreover,* V *is a regular,* G-*invariant Dirichlet space under the norm* $\mathrm{Q}^{1/2}$.

Proof. – Recall that Ω^+ is discrete and that $q(\omega) \geqslant q(1)$ for all $\omega \in \Omega^+$, which by (10) implies that

$$\mathrm{Q}(\mathrm{F}) \geqslant q(1) \, \|\mathrm{F}\|^2 \quad \text{for all } \mathrm{F} \in \mathrm{V} .$$

This shows that Q is positive definite, and since V is complete under the norm $(\|\mathrm{F}\|^2 + \mathrm{Q}(\mathrm{F}))^{1/2}$, V is also complete under $\mathrm{Q}^{1/2}$, and this proves that V is a Dirichlet space. It is obvious that V is regular, because all the functions $\lambda(s)\,\omega$, $s \in \mathrm{G}$, $\omega \in \Omega^+$ are contained in V. ‖

6.3. THEOREM. – *Suppose that* $\mathrm{X} = \mathrm{G}/\mathrm{K}$ *is a symmetric space of noncompact type of rank* 1, *and let* (Q , V) *be a* G-*invariant Dirichlet form on* $\mathrm{L}^2(\mathrm{X}, \xi)$ *with associated negative definite function* q *not identical zero.*

Then Q *is positive definite and* V *is a regular* G-*invariant Dirichlet space under the norm* $\mathrm{Q}^{1/2}$.

Proof. – By corollary 5.4 there is some constant $a > 0$ such that $q(\omega) \geqslant a$ for all $\omega \in \Omega_0^+$, and from (8) we then get the inequality

$$\mathrm{Q}(\mathrm{F}) \geqslant a \, \|\mathrm{F}\|^2 \quad \text{for all } \mathrm{F} \in \mathrm{V}.$$

This proves as above that Q is positive definite and that V is a Dirichlet space under $\mathrm{Q}^{1/2}$. The regularity of V is proved like in [3]. ‖

Let $(\mu_t)_{t \geqslant 0}$ be the vaguely continuous convolution semigroup corresponding to q satisfying the conditions in one of the two theorems. We have

$$\mu_t(\varphi * \psi^*) = \int_{\Omega_0^+} \overline{\mathfrak{F}}\varphi(\omega)\, \overline{\overline{\mathfrak{F}}\psi(\omega)}\, e^{-tq(\omega)}\, d\omega$$

for all $\varphi, \psi \in \mathcal{K}(G)^\natural$, and since $1/q$ is bounded over Ω_0^+, we get

$$\int_0^\infty \mu_t(\varphi * \psi^*)\, dt = \int_{\Omega_0^+} \overline{\mathfrak{F}}\varphi(\omega)\, \overline{\overline{\mathfrak{F}}\psi(\omega)}\, \frac{1}{q(\omega)}\, d\omega$$

for all $\varphi, \psi \in \mathcal{K}(G)^\natural$.

Now, since any function $f \in \mathcal{K}_+(G)$ can be dominated by a function of the form $\varphi * \psi^*$, where $\varphi, \psi \in \mathcal{K}_+(G)^\natural$, the formula

$$\nu = \int_0^\infty \mu_t\, dt$$

defines a positive definite, positive and bi-invariant measure ν on G satisfying

$$\nu(\varphi * \psi^*) = \int_{\Omega_0^+} \overline{\mathfrak{F}}\varphi(\omega)\, \overline{\overline{\mathfrak{F}}\psi(\omega)}\, \frac{1}{q(\omega)}\, d\omega \quad , \quad \varphi, \psi \in \mathcal{K}(G)^\natural.$$

The measure ν is the potential kernel of the Dirichlet space V : The potential generated by $F \in \mathcal{K}(X)$ is (represented by) the function $F * \nu$.

Since $V \subseteq L^2(X, \xi)$, we have proved that the potentials of finite energy are square integrable with respect to ξ under the hypothesis of theorem 6.2 or 6.3. This in turn implies, that any function in $L^2(X, \xi)$ is a measure of finite energy.

BIBLIOGRAPHY

[1] Chr. BERG, Suites définies négatives et espaces de Dirichlet sur la sphère, Séminaire Brelot-Choquet-Deny, Paris, 13e année 1969/70.

[2] Chr. BERG, Semi-groupes de Feller et le théorème de Hunt, Preprint Series no, 12, Matematisk Institut, University of Copenhagen, 1971.

[3] J. DENY, Méthodes hilbertiennes en théorie du potentiel, Potential theory (C.I.M.E., I ciclo, Stresa), Ed. Cremonese, Rome, 1970.

[4] J. DIXMIER, Les C*-algèbres et leurs représentations, Gauthier-Villars, Paris, 1964.

[5] M. FLENSTED-JENSEN, Paley-Wiener type theorems for a differential operator connected with symmetric spaces, Arkiv för Matematik. Vol. 10 (1972) n° 1.

[6] R. GANGOLLI, Isotropic infinitely divisible measures on symmetric spaces, Acta Math. 111 (1964), 213-246.

[7] I. GLICKSBERG, Uniform boundedness for groups, Canad. J. Math. 14 (1962), 269-276.

[8] R. GODEMENT, Sur la théorie des représentations unitaires, Ann. Math. 53 (1951), 68-124.

[9] R. GODEMENT, Introduction aux travaux de A. Selberg, Séminaire Bourbaki, Paris 1957, exposé 144.

[10] HARISH-CHANDRA, Spherical functions on a semi-simple Lie group II, *Amer. J. Math.* 80 (1958), 553-613.

[11] S. HELGASON, Differential Geometry and Symmetric spaces (Pure and Applied Mathematics 12), Academic Press, New York, London, 1962.

[12] B. KOSTANT, On the existence and irreducibility of certain series of representations, *Bull. Amer. Math. Soc.* 75 (1969), 627-642.

[13] K. SATO, T. UENO, Multidimensional diffusion and the Markov process on the boundary, *J. Math. Kyoto Univ.* 4 (1965), 529-605.

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Christian BERG
Department of Mathematics
University of Copenhagen
Denmark