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THE HAMILTON-CARTAN FORMALISM IN THE CALCULUS OF VARIATIONS

by Hubert GOLDSCHMIDT and Shlomo STERNBERG (*)

In this paper, we give an exposition of the geometry of the calculus of variations in several variables. The main emphasis is on the Hamiltonian formalism via the use of a linear differential form studied in detail by Cartan. We present an overall survey of the subject. Many of the ideas are to be found in the books of Caratheodory [1], Cartan [2] and De Donder [3], and the papers by Hill [6], Lepage [7], Van Hove [12] and Weyl [13]. Expositions of certain aspects of our subject may be found in the books [5], [8], [9] and [11].

The main innovations in our treatment consist of the consistent use of fibered manifolds and the affine structure of jet bundles (cf. [4]), and the introduction of the Hamilton-Cartan form which makes possible an invariant treatment of the Hamiltonian formalism. In particular, the Hamiltonian as a *function* is not an invariant concept and depends on a trivialization of a fibered manifold. We include discussions of Noether's theorem, the Hamilton-Jacobi equation and the second variation.

1. Differential operators and jet bundles.

Let X be a differentiable manifold of dimension n whose tangent bundle and cotangent bundle we denote by $T = T(X)$ and $T^* = T^*(X)$ respectively. Let $C^\infty(X)$ be the space of real-valued differentiable functions on X. If E is a vector bundle over X, we denote by $C^\infty(E)$ the space of differentiable sections of E over X. Let E, F be vector bundles over X and let $D : C^\infty(E) \to C^\infty(F)$ be a first-order differentiel op-

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erator. The symbol $\sigma(D) : T^* \otimes E \to F$ of D is the unique morphism of vector bundles such that

$$D(fs) = f Ds + \sigma(D)(df \otimes s)$$

for all $f \in C^\infty(X)$, $s \in C^\infty(E)$. The adjoint operator

$$D^* : C^\infty(F^* \otimes \Lambda^n T^*) \to C^\infty(E^* \otimes \Lambda^n T^*)$$

is the unique first-order linear differential operator such that

$$\int_X \langle De, \beta \rangle = \int_X \langle e, D^*\beta \rangle$$

for all $\beta \in C^\infty(F^* \otimes \Lambda^n T^*)$ and all sections e of E of compact support. The symbol $\sigma(D)$ of D gives rise to a map $E \to T \otimes F$ which we also denote by $\sigma(D)$. If $e \in E$, $\beta \in F^* \otimes \Lambda^n T^*$, the image of $(\sigma(D)\, e) \otimes \beta$ under the map

$$T \otimes F \otimes F^* \otimes \Lambda^n T^* \to \Lambda^{n-1} T^*$$

$$\xi \otimes f \otimes f^* \otimes \alpha \mapsto \langle f, f^* \rangle \; \xi \lrcorner \alpha$$

where $\xi \in T$, $f \in F$, $f^* \in F^*$, $\alpha \in \Lambda^n T^*$, will be denoted by $(\sigma(D)\, e) \lrcorner \beta$. The following is the integration by parts formula for D :

$$\langle De, \beta \rangle = \langle e, D^*\beta \rangle + d((\sigma(D)\, e) \lrcorner \beta) \tag{1.1}$$

where $e \in C^\infty(E)$, $\beta \in C^\infty(F^* \otimes \Lambda^n T^*)$; the proof of this formula is standard and will therefore be omitted.

We now recall certain facts, in particular about jet bundles, which we shall need. If Z, Z_1, Z_2 are differentiable manifolds, $f_i : Z \to Z_i$ is a differentiable map and W_i is a vector bundle over Z_i, with $i = 1, 2$, we shall generally use the notation $W_1 \otimes_Z W_2$ for the tensor product of induced bundles $f_1^{-1} W_1 \otimes f_2^{-1} W_2$.

Let $\varphi_t : X \to Z$ be a one-parameter family of differentiable mappings and set $\varphi = \varphi_0$. Let $\xi : X \to T(Z)$ be the vector field tangent to the family φ_t at $t = 0$ defined by

$$\xi(x) = \left. \frac{d\varphi_t(x)}{dt} \right|_{t=0} .$$

If α is a p-form on Z, then $\xi(x) \lrcorner \alpha(\varphi(x))$ belongs to $\Lambda^{p-1} T^*_{\varphi(x)}(Z)$ and we have thus a $(p-1)$-form $\varphi^*(\xi \lrcorner \alpha)$ on X defined by

$$\varphi^*(\xi \lrcorner \alpha)(x) = \varphi^*(\xi(x) \lrcorner \alpha(\varphi(x))) \,.$$

The following formula is easily seen to hold :

$$(1.2) \qquad \left.\frac{d\varphi_t^* \alpha}{dt}\right|_{t=0} = d\varphi^*(\xi \lrcorner \alpha) + \varphi^*(\xi \lrcorner d\alpha) \,.$$

Let $\pi : Y \to X$ be a fibered manifold ; that is, we require π to be a surjective submersion. Let $T(Y)$ be the tangent bundle of Y and $T(Y/X)$ the sub-bundle of $T(Y)$ of vertical vectors. We shall sometimes use the notation $V(Y)$ for the bundle of vertical vectors over X. We denote by $J_k(Y)$ the fibered manifold over X of k-jets of (local) sections of Y, by $j_k(s)$ the k-jet of a section s of Y, and by $\pi_{k-1} : J_k(Y) \to J_{k-1}(Y)$, $\pi : J_k(Y) \to X$ the natural projections ; we shall identify $J_0(Y)$ with Y. We recall the following properties of $J_k(Y)$ (see [4]) :

(1.3) $J_k(Y)$ is an affine bundle over $J_{k-1}(Y)$ whose associated vector bundle is $S^k T^* \otimes_{J_{k-1}(Y)} V(Y)$.

(1.4) There is a canonical isomorphism

$$V(J_k(Y)) \cong J_k(V(Y))$$

sending

$$\left.\frac{d}{dt} j_k(s_t)(x)\right|_{t=0} \quad \text{into} \quad j_k\left(\left.\frac{d}{dt} s_t\right|_{t=0}\right)(x)$$

if s_t is a one-parameter family of sections of Y defined on some neighborhood of $x \in X$.

(1.5) We have an exact sequence

$$0 \to S^k T^* \otimes_{J_k(Y)} V(Y) \to V(J_k(Y)) \xrightarrow{\pi_{k-1*}} \pi_{k-1}^{-1} V(J_{k-1}(Y)) \to 0 \,.$$

(1.6) If Y is a vector bundle E, then $J_k(E)$ is also a vector bundle.

Let $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ be a coordinate system for Y on an open set U such that $(x^1, \ldots, x^n)$ is a coordinate system for X on πU. Then there is a natural coordinate system for $J_k(Y)$ on $\pi_0^{-1} U$

(1.7) $(x^i, y^j, y^j_\alpha) \quad , \quad 1 \leqslant i \leqslant n \quad , \quad 1 \leqslant j \leqslant m \quad , \quad 1 \leqslant |\alpha| \leqslant k \,,$

where $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a multi-index, such that

$$y^j_\alpha(j_k(s)(x)) = \frac{\partial^{|\alpha|} s^j}{\partial x^\alpha}(x) \quad , \quad s^j = y^j \circ s \,,$$

for any section s of Y defined on a neighborhood of $x \in \pi U$ satisfying $s(x) \in U$.

PROPOSITION 1.1. – *There exists a unique* $T(J_{k-1}(Y))$ *-valued 1-form* ω_k *on* $J_k(Y)$ *such that* :

i) $\langle j_k(s)_* v, \omega_k \rangle = 0$, *for all sections* s *of* Y *over an open set* $U \subset X$ *and all* $v \in T_x$, $x \in U$;

ii) $\langle \xi, \omega_k \rangle = \pi_{k-1*} \xi$, *for all* $\xi \in V(J_k(Y))$.

Proof. – i) Uniqueness. Suppose that ω_k, $\bar{\omega}_k$ are two 1-forms satisfying the above conditions. Then the restriction of $\omega_k - \bar{\omega}_k$ to $V(J_k(Y))$ vanishes and this 1-form satisfies condition i). Let p be an arbitrary point of $J_k(Y)$; then we can write $p = j_k(s)(x)$, for some section s of Y over a neighborhood of $x = \pi(p)$. If $\xi \in T_p(J_k(Y))$, then $\xi - j_k(s)_* \pi_* \xi$ is a vertical vector ; hence both terms of the right-hand side of the equation

$$\langle \xi, \omega_k - \bar{\omega}_k \rangle = \langle \xi - j_k(s)_* \pi_* \xi, \omega_k - \bar{\omega}_k \rangle + \langle j_k(s)_* \pi_* \xi, \omega_k - \bar{\omega}_k \rangle$$

vanish, and so $\omega_k = \bar{\omega}_k$.

ii) Existence. Let p be a point of $J_k(Y)$; if s is a section of Y over a neighborhood of $x = \pi(p)$, with $j_k(s)(x) = p$, we denote by $p_* : T_x \to T_{\pi_{k-1}(p)}(J_{k-1}(Y))$ the map $j_{k-1}(s)_*$ which depends only on p. Define ω_k by

$$\langle \xi, \omega_k \rangle = \pi_{k-1*} \xi - p_* \pi_* \xi \tag{1.8}$$

if $p \in J_k(Y)$, $\xi \in T_p(J_k(Y))$. Conditions i) and ii) are trivially verified. In fact $\langle \xi, \omega_k \rangle$ is just the projection of $\pi_{k-1*} \xi$ onto $V_{\pi_{k-1}(p)}(J_{k-1}(Y))$ along the horizontal subspace of $T_{\pi_{k-1}(p)}(J_{k-1}(Y))$ determined by p.

The following is the converse of condition i) of the above proposition.

PROPOSITION 1.2. – *If* u *is a section of* $J_k(Y)$ *over* $U \subset X$, *satisfying* $\langle u_* v, \omega_k \rangle = 0$, *for all* $v \in T_x$, $x \in U$, *then* $u = j_k(\pi_0 u)$.

Proof. – We proceed by induction on k. Let $k = 1$ and u be a section of $J_1(Y)$ over U satisfying the above condition. If $x \in U$, then for all $v \in T_x$,

$$0 = \langle u_* v, \omega_1 \rangle = ((\pi_0 u)_* - u(x)_*) v$$

which implies that $u(x) = j_1(\pi_0 u)(x)$. Assume that the proposition holds for $k-1$ and let u be a section of $J_k(Y)$ satisfying the hypothesis of the proposition. By the uniqueness of ω_{k-1} we have $\pi_{k-2*}\, \omega_k = \omega_{k-1}$. Thus by induction, $\pi_{k-1} u = j_{k-1}(\pi_0 u)$. If $x \in U$, then for all $v \in T_x$,

$$0 = \langle u_* v, \omega_k \rangle = ((\pi_{k-1} u)_* - u(x)_*) v$$

which implies that $u(x)_* = (\pi_{k-1} u)_*$ and hence that $u(x) = j_k(\pi_0 u)(x)$.

Let $\varphi : Y \to Y$, $\overline{\varphi} : X \to X$ be mappings such that $\pi \circ \varphi = \overline{\varphi} \circ \pi$; if s is a section of Y over a neighborhood of $x \in X$, then $\varphi \circ s \circ \overline{\varphi}^{-1}$ is a section of Y over a neighborhood of $\overline{\varphi}(x)$, and we define a map $\varphi_k : J_k(Y) \to J_k(Y)$ by

$$\varphi_k(j_k(s)(x)) = j_k(\varphi \circ s \circ \overline{\varphi}^{-1})(\overline{\varphi}(x)) .$$

We have

$$\pi \circ \varphi_k = \overline{\varphi} \circ \pi$$

and

$$\pi_l \circ \varphi_k = \varphi_l \circ \pi \quad , \quad \text{for } 0 \leqslant l \leqslant k ,$$

where $\varphi_0 = \varphi$. Therefore, to each projectable vector field ξ on Y, we can associate a projectable vector field ξ_k on $J_k(Y)$. In fact, ξ_k is π_l-projectable, for $0 \leqslant l \leqslant k$, and

$$\pi_{l*}\, \xi_k = \xi_l, \quad \text{for } 0 \leqslant l \leqslant k,$$

where $\xi_0 = \xi$. If η is another projectable vector field on Y, then

$$[\xi , \eta]_k = [\xi_k , \eta_k] .$$

If E is any vector bundle over X, we denote by

(1.9) $d_{Y/X} : C^\infty(E \otimes_Y \Lambda^p T^*(Y/X)) \to C^\infty(E \otimes_Y \Lambda^{p+1} T^*(Y/X))$

the exterior derivative on $\pi : Y \to X$, or fiber derivative on Y (relative to X). The restriction of $s \in C^\infty(E \otimes_Y \Lambda^p T^*(Y/X))$ to $\pi^{-1}(x)$, if $x \in X$, is an E_x-valued p-form on $\pi^{-1}(x)$; the restriction of $d_{Y/X}\, s$ to $\pi^{-1}(x)$

is simply the usual exterior derivative of this vector-valued p-form. If f is a real-valued function on Y and s_t is a one-parameter family of sections of Y, then

$$\langle \left.\frac{ds_t}{dt}\right|_{t=0}, (d_{Y/X} f) \circ s_0 \rangle = \left.\frac{d}{dt} f(s_t)\right|_{t=0} .$$

The differential operator $d_{Y/X}$ has the following properties :

(1.10) $$d_{Y/X} \circ d_{Y/X} = 0 \ ;$$

(1.11) $$\xi \cdot f = \langle \xi , d_{Y/X} f \rangle$$

where ξ is a section of $T(Y/X)$ over Y and f is an E-valued function on Y ;

(1.12) $$d_{Y/X}(\omega_1 \wedge \omega_2) = (d_{Y/X}\, \omega_1) \wedge \omega_2 + (-1)^p \omega_1 \wedge d_{Y/X}\, \omega_2$$

for all sections ω_1 of $E \otimes \Lambda^p T^*(Y/X)$ and ω_2 of $\Lambda^q T^*(Y/X)$ over Y.

Assume that Y is an affine bundle over X whose associated vector bundle we denote by E. An element $e \in E_x$, induces a vector field $\widetilde{e}$ on the fiber Y_x of Y over x, and an isomorphism $\alpha_y : E_x \to T_y(Y/X)$, for $y \in Y_x$. Let f be a real-valued function on Y. For $y \in Y_x$, we define the bilinear form $(d^2_{Y/X} f)(y)$ on $T_y(Y/X)$ by

$$(d^2_{Y/X} f)(y)(\xi_1 , \xi_2) = \widetilde{e}_2 \cdot \langle \widetilde{e}_1 , d_{Y/X} f \rangle$$

where $e_i = \alpha_y^{-1}(\xi_i)$, $i = 1, 2$. These bilinear forms are symmetric and so define a section $d^2_{Y/X} f$ of $S^2\, T^*(Y/X)$.

2. The calculus of variations.

Assume now that X is orientable ; let ω be a nowhere zero n-form on X. Let $\pi : Y \to X$ be a fibered manifold and let L be a real-valued function, called the *Lagrangian,* defined on $J_1(Y)$. For any compact subset $A \subset X$ and any section s of Y over a neighborhood of A, write

$$I_A[s] = \int_A L(j_1(s))\, \omega \ .$$

The main problem of the calculus of variations is to find sections s which minimize, or extremize, this integral relative to a suitable family of variations. For instance, if A is a submanifold with smooth boundary of X we may seek a section s satisfying the condition :

$$\mathrm{I}_{\mathrm{A}}[s] \leqslant \mathrm{I}_{\mathrm{A}}[s'] \tag{2.1}$$

for all sections s' over a neighborhood of A such that $s' = s$ on ∂A. Such a section s is called a minimum of I_{A} for fixed boundary values. For the remainder of this section, we shall only study the fixed boundary value problem and so shall omit the phrase "for fixed boundary values".

A weaker condition on s is to require that (2.1) holds for all sections s' in some C^0-neighborhood of s such that $s' = s$ on ∂A. A section s verifying this condition is called a *strong local minimum.*

A still weaker condition to impose on s is to require that (2.1) holds for all sections s' in some C^1-neighborhood of s such that $s' = s$ on ∂A. A section s verifying this condition is called a *weak local minimum.*

A consequence of any one of these three conditions on s is

$$\left.\frac{d}{dt}\mathrm{I}_{\mathrm{A}}[s_t]\right|_{t=0} = 0 \tag{2.2}$$

for any smooth one-parameter family of sections of Y such that $s_0 = s$ and $s_t = s$ on ∂A for all t. A section s satisfying (2.2) is called an *extremal* of I_{A} ; if this condition holds for all subsets $\mathrm{A} \subset\subset \mathrm{X}$, then we say that s is an extremal.

We suppose that A is a compact submanifold of X with smooth boundary ∂A of dimension $n-1$. If we compute the derivative in (2.2), we obtain

$$\frac{d}{dt}\mathrm{I}_{\mathrm{A}}[s_t] = \frac{d}{dt}\int_{\mathrm{A}} \mathrm{L}(j_1(s_t))\,\omega$$

$$= \int_{\mathrm{A}} (d_{\mathrm{J}_1(\mathrm{Y})/\mathrm{X}}\mathrm{L} \circ j_1(s_t))\left(\frac{d}{dt}\,j_1(s_t)\right)\omega\ .$$

Let $V_s(Y)$ denote the vector bundle over X induced from $V(Y)$ by the map s. We may identify

$$\dot{s}_0 = \left.\frac{ds_t}{dt}\right|_{t=0}$$

with a section of $V_s(Y)$ and thus $j_1(\dot{s}_0)$ becomes a section of $J_1(V_s(Y))$, which is identified with

$$\left.\frac{d}{dt}\, j_1(s_t)\right|_{t=0} \cdot$$

With this interpretation

$$(d_{J_1(Y)/X}\, L) \circ j_1(s)$$

gives rise to a first-order linear differential operator $\mathcal{L}[s]$ on $V_s(Y)$. We evaluate the preceding equation at $t = 0$ and obtain

$$\left.\frac{d}{dt}\, I_A[s_t]\right|_{t=0} = \int_A (\mathcal{L}[s]\, \dot{s}_0)\, \omega \cdot$$

Let ν be any section of $V_s(Y)$ such that

$$\operatorname{supp} \nu \subset \mathring{A} \cdot$$

Extend ν to a vertical vector field ξ on Y of compact support, and let φ_t be the corresponding flow. Then $s_t = \varphi_t \circ s$ is a one-parameter family of sections of Y such that $s_t = s$ outside $\operatorname{supp} \nu$ and $\dot{s}_0 = \nu$. According to (2.2), we must have

$$\int_A (\mathcal{L}[s]\, \nu)\, \omega = 0 \cdot$$

Let

$$\mathcal{E}[s] : C^\infty(\Lambda^n T^*) \to C^\infty(V_s^*(Y) \otimes \Lambda^n T^*)$$

denote the adjoint differential operator, i.e., $\mathcal{E}[s] = \mathcal{L}[s]^*$. Then the integration by parts formula (1.1) says that

$$(2.3) \qquad (\mathcal{L}[s]\, w)\, \omega = \langle\, w\, ,\, \mathcal{E}[s]\, \omega\, \rangle + d((\sigma(\mathcal{L}[s]\, w)) \lrcorner\, \omega)$$

for any section w of $V_s(Y)$. Since ν vanishes on ∂A, (2.3) and Stokes' theorem imply that

$$\int_A \langle\, \nu\, ,\, \mathcal{E}[s]\, \omega\, \rangle = 0 \cdot$$

Since this last equation must hold for *all* sections ν of $V_s(Y)$ satisfying $\operatorname{supp} \nu \subset \mathring{A}$, we conclude that on $\mathring{A}$

$$\mathcal{E}[s]\,\omega = 0 \tag{2.4}$$

Equation (2.4) is known as the *Euler-Lagrange equation* and is equivalent to the fact that s is an extremal.

Let us examine the form of (2.3) in terms of a local coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ on an open subset U of Y of the type considered in § 1, for which

$$\omega = dx^1 \wedge \ldots \wedge dx^n$$

on πU. In terms of the coordinates (1.7) (x^i, y^j, y^j_i), $i = 1, \ldots, n$, $j = 1, \ldots, m$, on $J_1(Y)$, we have

$$d_{J_1(Y)/X} L = \sum_j \frac{\partial L}{\partial y^j}(j_1(s))\, d_{J_1(Y)/X}\, y^j + \sum_{i,j} \frac{\partial L}{\partial y^j_i}(j_1(s))\, d_{J_1(Y)/X}\, y^j_i\,,$$

and therefore

$$\mathcal{L}[s]\,\nu = \sum_j \frac{\partial L}{\partial y^j}(j_1(s))\,\nu^j + \sum_{i,j} \frac{\partial L}{\partial y^j_i}(j_1(s))\,\frac{\partial \nu^j}{\partial x^i}\,,$$

where

$$\nu = \sum_j \nu^j \left(\frac{\partial}{\partial y^j}\right)(s)$$

is a section of $V_s(Y)$.

Then $\mathcal{E}[s]\,\omega = \alpha \otimes \omega$, where α is the section of $V^*_s(Y)$ determined by

$$\left\langle \left(\frac{\partial}{\partial y^j}\right)(s)\,, \alpha \right\rangle = \frac{\partial L}{\partial y^j}(j_1(s)) - \sum_{i=1}^{n} \frac{\partial}{\partial x^i}\left(\frac{\partial L}{\partial y^j_i}(j_1(s))\right)$$

with $j = 1, \ldots, m$. Thus (2.4) can be written as the system of second-order partial differential equations for the section $s(x) = (x^i, s^j(x))$

$$\frac{\partial L}{\partial y^l}\left(x^i, s^j(x), \frac{\partial s^j}{\partial x^i}(x)\right) = \sum_{k=1}^{n} \frac{\partial}{\partial x^k}\left(\frac{\partial L}{\partial y^l_k}(x^i, s^j(x), \frac{\partial s^j}{\partial x^i}(x))\right)$$

(2.4′)

$(l = 1, \ldots, m)$. This is the classical expression of the Euler-Lagrange equation. We shall not have much occasion to use this local expression.

An extremely important special case is that of "quadratic Lagrangians". In this case, Y is a vector bundle E over X and L has the following form. Let F be a second vector bundle ; assume that E, F are both equipped with scalar products which we shall denote by $\langle\langle\ ,\ \rangle\rangle$. Suppose that $D : C^\infty(E) \to C^\infty(F)$ is a first-order differential operator and that the function L is determined by

$$L(j_1(s)) = \frac{1}{2} \langle\langle Ds , Ds \rangle\rangle \quad , \quad \text{for } s \in C^\infty(E) .$$

In terms of the vector bundle map $p(D) : J_1(E) \to F$ induced by D satisfying $D = p(D) \circ j_1$, the function L is given by

$$L(z) = \frac{1}{2} \langle\langle p(D) z , p(D) z \rangle\rangle \quad , \quad \text{for } z \in J_1(E) .$$

Thus L is indeed a quadratic function on $J_1(E)$. Therefore, its fiber derivative

$$(d_{J_1(E)/X} L)(z) \in T_z^*(J_1(E)/X) \quad , \quad \text{at } z \in J_1(E)_x ,$$

sends $w \in J_1(E)_x$ into $\langle\langle p(D) z , p(D) w \rangle\rangle$. Thus, identifying $V_s(E)$ with E, the differential operator $\mathcal{L}[s]$ is given by

$$\mathcal{L}[s] v = \langle\langle Ds , Dv \rangle\rangle \quad , \quad \text{for } v \in C^\infty(E) .$$

It is clear that

$$\langle v , \mathcal{E}[s] \omega \rangle = \langle\langle v , D^* Ds \rangle\rangle \omega ,$$

where D^* denotes the metric adjoint of D relative to the scalar products $\langle\langle\ ,\ \rangle\rangle$ and the volume element ω. Therefore, the Euler-Lagrange equation is equivalent to

$$D^* Ds = 0 . \tag{2.5}$$

Let us now return to the general situation. Let s_t be a one-parameter family of sections of Y with $s_0 = s$ and let $\overline{\varphi}_t$ be a flow on X whose infinitesimal generator is $\overline{\xi}$. Suppose that the integrand considered is equivariant with respect to these two one-parameter families in the sense that

$$\overline{\varphi}_t^*(L(j_1(s_t))\, \omega) = L(j_1(s))\, \omega . \tag{2.6}$$

Geometrically, this means that for any compact set $B \subset X$

$$I_{\overline{\varphi}_t(B)}[s_t] = \int_{\overline{\varphi}_t(B)} L(j_1(s_t))\,\omega = I_B[s]\,.$$

If we set $\nu = \dot{s}_0$ and differentiate (2.6) with respect to t and set $t = 0$, we obtain, using (2.3)

$$d\{\overline{\xi} \lrcorner L(j_1(s))\,\omega \;+ \sigma(\mathcal{L}[s])\,\nu \lrcorner \omega\} + \langle \nu, \mathcal{E}[s]\,\omega\rangle = 0\,.$$

If s is an extremal, then the last term vanishes and we obtain

$$d\{(L(j_1(s))\,\overline{\xi} + \sigma(\mathcal{L}[s])\,\nu) \lrcorner \omega\} = 0\,. \tag{2.7}$$

Notice that we can consider the more general situation where (2.6) is replaced by

$$\overline{\varphi}_t^*(L(j_1(s_t))\,\omega) = L(j_1(s))\,\omega + d\alpha_t \tag{2.6'}$$

where α_t is a one-parameter family of $(n-1)$ forms on X. If we set

$$\dot{\alpha} = \left.\frac{d\alpha_t}{dt}\right|_{t=0},$$

then (2.7) is replaced by

$$d\{(L(j_1(s))\,\overline{\xi} + \sigma(\mathcal{L}[s]\,\nu)) \lrcorner \omega - \dot{\alpha}\} = 0\,. \tag{2.7'}$$

In particular, suppose that φ_t is a one-parameter family of automorphisms of the fibered manifold Y satisfying $\pi \circ \varphi_t = \overline{\varphi}_t \circ \pi$ and that $s_t = \varphi_t \circ s \circ \overline{\varphi}_t^{-1}$. If ξ is the infinitesimal generator of φ_t and $\overline{\xi}$ the infinitesimal generator of $\overline{\varphi}_t$, then

$$\nu = \xi \circ s - s_* \overline{\xi}\,.$$

We have thus proved *Noether's Theorem.* Let s be a section of Y. Suppose that φ_t is a one-parameter family of automorphisms of the fibered manifold Y which preserve the action in the sense that

$$\overline{\varphi}_t^*(L(j_1(s_t))\,\omega) = L(j_1(s))\,\omega\,,$$

where $\overline{\varphi}_t$ are diffeomorphisms of X such that $\pi \circ \varphi_t = \overline{\varphi}_t \circ \pi$ and $s_t = \varphi_t \circ s \circ \overline{\varphi}_t^{-1}$. Then if s is an extremal, the $(n-1)$-form on X

$$(L(j_1(s))\,\overline{\xi} + \sigma(\mathcal{L}[s])\,(\xi \circ s - s_*\overline{\xi})) \lrcorner \omega \tag{2.8}$$

is closed.

If Y is a vector bundle E, if φ_t is a one-parameter family of vector bundle automorphisms, and if L is a quadratic Lagrangian in the sense described above, then

$$\nu = [\xi , s]$$

considering the section s of E as a vertical vector field on E, and identifying ν with a section of E. Also

$$\sigma(\mathcal{R}[s])\,\nu = \langle\langle Ds , \sigma(D)\,\nu \rangle\rangle ,$$

where we have set

$$\langle\langle f_1 , t \otimes f_2 \rangle\rangle = \langle\langle f_1 , f_2 \rangle\rangle\, t \quad , \quad \text{for } f_1 , f_2 \in F , t \in T .$$

Then the form (2.8) is

$$(2.9) \qquad (L(j_1(s))\,\bar{\xi} + \langle\langle Ds , \sigma(D)\,[\xi , s]\rangle\rangle) \lrcorner\, \omega .$$

3. The Hamiltonian formalism for the first variation.

Let W be an affine space whose associated vector space is V. The affine structure on W identifies the tangent space $T_p(W)$ of W at any point p with V. It therefore also identifies $T_p^*(W)$ with V^*. If f is any function on W, then $(df)(p) \in V^*$ and so we have a map $W \to V^*$ sending p into $(df)(p)$. Similarly, if W is an affine bundle over X, whose associated vector bundle is V, we have a bundle map $W \to V^*$ sending p into $(d_{W/X} f)(p)$.

We apply these considerations to the case of the affine bundle $W = J_1(Y)$ over Y whose associated vector bundle is $T^* \otimes_Y V(Y)$. We let L be a function on $J_1(Y)$ as in the previous section. Then setting $\sigma(L)\,p = (d_{J_1(Y)/X} L)(p)$, for $p \in J_1(Y)$, we obtain a map of fibered manifolds

$$\sigma(L) : J_1(Y) \to T \otimes_Y V^*(Y)$$

over Y, which is called the *Legendre transformation*. In fact, according to (1.5), $d_{J_1(Y)/Y} L$ can be identified with a section of $T \otimes_{J_1(Y)} V^*(Y)$, which is precisely the section corresponding to $\sigma(L)$.

If Y is the trivial bundle X × M, where M is an m-dimensional manifold, then we may identify $J_1(Y)$ with the manifold of 1-jets of mappings of X into M, and hence with the vector bundle

$$\mathrm{Hom}(\mathrm{T}, \mathrm{T}(\mathrm{M})) = \mathrm{T}^* \otimes_{\mathrm{Y}} \mathrm{T}(\mathrm{M})$$

over Y ; the 1-jet at $x \in \mathrm{X}$ of a function $f : \mathrm{X} \to \mathrm{M}$ corresponds to $f_* : \mathrm{T}_x \to \mathrm{T}_{f(x)}(\mathrm{M})$. The vector bundle $\mathrm{T} \otimes_{\mathrm{Y}} \mathrm{V}^*(\mathrm{Y})$ is canonically isomorphic to $\mathrm{T} \otimes_{\mathrm{Y}} \mathrm{T}^*(\mathrm{M})$, and the Legendre transformation is therefore a map

$$\sigma(\mathrm{L}) : \mathrm{T}^* \otimes_{\mathrm{Y}} \mathrm{T}(\mathrm{M}) \to \mathrm{T} \otimes_{\mathrm{Y}} \mathrm{T}^*(\mathrm{M}) .$$

We now define a linear differential form θ on $\mathrm{J}_1(\mathrm{Y})$ with values in T as follows. For any $\xi \in \mathrm{T}_p(\mathrm{J}_1(\mathrm{Y}))$, let $\langle \xi , \theta \rangle \in \mathrm{T}_x$ with $\pi(p) = x$, be given by

$$\langle \xi , \theta \rangle = \frac{1}{n} \mathrm{L}(p) \pi_* \xi + \sigma(\mathrm{L}) p \cdot (\langle \xi , \omega_1 \rangle) \tag{3.1}$$

where ω_1 is the 1-form on $\mathrm{J}_1(\mathrm{Y})$ with values in V (Y) defined in § 1. Since $\sigma(\mathrm{L})_p \in \mathrm{T} \otimes_{\mathrm{Y}} \mathrm{V}^*(\mathrm{Y})$, and $\langle \xi , \omega_1 \rangle \in \mathrm{V}(\mathrm{Y})$, the element $\sigma(\mathrm{L})_p . (\langle \xi , \omega_1 \rangle)$ lies in T_x .

As this form plays a crucial role in the remainder of this paper, let us give its expression in local coordinates in the classical case of one independent variable. At the end of this section, we will give the general expression. The manifold X = R, whose coordinate we denote by t, has a distinguished vector field $\partial/\partial t$, so that $\omega = dt$ is a non-vanishing one-form on X. If $(t , q^1 , \ldots , q^m)$ is a coordinate system for Y, and if we set $x^1 = t$, $y^j = q^j$, then

$$(t , q^1 , \ldots , q^m , \dot{q}^1 , \ldots , \dot{q}^m) \cdot$$

is a coordinate system for $\mathrm{J}_1(\mathrm{Y})$, where $\dot{q}^j = y^j_1$. In terms of these coordinates,

$$\omega_1 = \sum_j (dq^j - \dot{q}^j\, dt) \otimes \frac{\partial}{\partial q^j} \cdot$$

$$\theta = \mathrm{L}\, dt \otimes \frac{\partial}{\partial t} + \sum_j \frac{\partial \mathrm{L}}{\partial \dot{q}^j} (dq^j - \dot{q}^j\, dt) \otimes \frac{\partial}{\partial t} \cdot$$

$$= \Big(\sum_j \frac{\partial \mathrm{L}}{\partial \dot{q}^j} dq^j - \mathrm{H}\, dt\Big) \otimes \frac{\partial}{\partial t} ,$$

where we have set

$$H = \sum_j \frac{\partial L}{\partial \dot{q}^j} \dot{q}^j - L .$$

We now return to general considerations.

If s is a section of Y, then by Proposition 1.1 we have

$$j_1(s)^* \theta = \frac{1}{n} L(j_1(s))\, id . \tag{3.2}$$

Let W be an n-dimensional vector space. Let $u \in W^* \otimes W$; then u acts as a derivation on the exterior algebra ΛW^* of W^* by considering u as an element of Hom(W , W) and extending the adjoint $u^* : W^* \to W^*$ of u as a derivation of ΛW^*. Following Frölicher-Nijenhuis, we denote by $u \,\bar{\wedge}\, \beta$ the image of $\beta \in \Lambda W^*$ under this derivation. If $u = \alpha \otimes \xi \in W^* \otimes W$, $\beta \in \Lambda W^*$, then

$$u \wedge \beta = \alpha \wedge (\xi \lrcorner \beta) \ ;$$

indeed, this formula defines a derivation of ΛW^* which is equal to the one considered above when restricted to W^*. Therefore, if $\beta \in \Lambda^n W^*$

$$u \,\bar{\wedge}\, \beta = (\mathrm{Tr}\, u^*) \beta = (\mathrm{Tr}\, u) \beta . \tag{3.3}$$

If ψ is a one-form on $J_1(Y)$ with values in T, that is a section over $J_1(Y)$ of $T^*(J_1(Y)) \otimes_{J_1(Y)} T$, and β is a section of $\Lambda^p T^*$ over X, we denote by $\psi \,\bar{\wedge}\, \pi^* \beta$ the p-form on $J_1(Y)$ defined by the formula

$$\psi \,\bar{\wedge}\, \pi^* \beta = \alpha \wedge \pi^* (\xi \lrcorner \beta)$$

if $\psi = \alpha \otimes \xi$.

We now apply this to θ and ω and obtain an n-form

$$\Theta = \theta \,\bar{\wedge}\, \pi^* \omega$$

on $J_1(Y)$. In the one-dimensional case described above, setting $p^j = \partial L / \partial \dot{q}^j$, we see that

$$\Theta = \sum_j p^j \, dq^j - H\, dt .$$

PROPOSITION 3.1. — *The form Θ is the unique n-form on $J_1(Y)$ satisfying*

(3.4) $$j_1(s)^* \Theta = \mathrm{L}(j_1(s))\, \omega$$

for all sections s *of* Y *and*

(3.5) $$\eta \lrcorner \Theta = \pi^*((\sigma(\mathrm{L})\, p \cdot (\pi_{0*}\eta)) \lrcorner \omega)$$

if $p \in \mathrm{J}_1(\mathrm{Y})$ *and* $\eta \in \mathrm{V}_p(\mathrm{J}_1(\mathrm{Y}))$.

Proof. – The uniqueness part of the proposition is trivial, since (3.4) determines Θ on the subspace $j_1(s)_*(\Lambda^n \mathrm{T}_x)$ of $\Lambda^n \mathrm{T}_p(\mathrm{J}_1(\mathrm{Y}))$, if $p = j_1(s)(x)$ for some section s of Y over a neighborhood of $x \in \mathrm{X}$, while (3.5) determines Θ on the ideal of $\Lambda\, \mathrm{T}_p(\mathrm{J}_1(\mathrm{Y}))$ generated by $\mathrm{V}_p(\mathrm{Y})$. Now

$$j_1(s)^* \Theta = (j_1(s)^* \theta) \barwedge \omega \ ,$$

where the operation $\barwedge$ on the right-hand side is the one described above letting W be the vector space T_x, with $x \in \mathrm{X}$. Hence by (3.2) and (3.3)

$$j_1(s)^* \Theta = \frac{1}{n}\, \mathrm{L}(j_1(s))\, id \barwedge \omega = \mathrm{L}(j_1(s))\, \omega \ .$$

If η is a vertical tangent vector to $\mathrm{J}_1(\mathrm{Y})$, then $\eta \lrcorner \pi^* \omega = 0$, so that

$$\eta \lrcorner \Theta = \pi^*(\langle \eta , \theta \rangle \lrcorner \omega) \ .$$

By (3.1),

$$\langle \eta , \theta \rangle = \sigma(\mathrm{L})\, p \cdot \pi_{0*}\, \eta$$

proving (3.5).

If u is a section of $\mathrm{J}_1(\mathrm{Y})$ of the form $u = j_1(s)$ for some section s of Y, equation (3.4) implies that

(3.6) $$\int_{\mathrm{A}} u^* \Theta = \mathrm{I}_{\mathrm{A}}[s] \ .$$

We can now pose the problem of finding extremals of the integral of the left-hand side of (3.6) among *all* sections of $\mathrm{J}_1(\mathrm{Y})$, not just those which are of the form $j_1(s)$. Let u_t be a one-parameter family of sections of $\mathrm{J}_1(\mathrm{Y})$ with $u_0 = u$ and let ξ be the tangent vector field along u to u_t at $t = 0$, so that $\xi(x) \in \mathrm{V}_{u(x)}(\mathrm{J}_1(\mathrm{Y}))$ is the tangent vector to the curve $u_t(x)$. Then, by formula (1.2)

$$\frac{d}{dt}\, u_t^* \Theta \Big|_{t=0} = u^*(\xi \lrcorner d\Theta) + du^*(\xi \lrcorner \Theta) \ .$$

Now suppose that

$$\frac{d}{dt} \int_{A} u_t^* \Theta \Big|_{t=0}$$

vanishes for all variations u_t of u which agree with u outside a compact subset of $\mathring{A}$, that is, that u is an extremal of the integral on the left of (3.6). Then for such a variation, ξ has compact support and so by Stokes' theorem

$$\int_{A} u^*(\xi \lrcorner\, d\Theta) = 0 \cdot$$

Since this must hold for all vertical vector fields ξ along u of compact support, we conclude that on $\mathring{A}$

(3.7) $\qquad u^*(\xi \lrcorner\, d\Theta) = 0 \quad , \quad \text{for all } \xi \in C^\infty(V(J_1(Y)))$.

This condition is equivalent to the fact that u is an extremal of the integral on the left of (3.6). The main observation in the Hamilton-Cartan formulation of the calculus of variations is

THEOREM 3.1. – *If the map* $\sigma(L) : J_1(Y) \to T \otimes_Y V^*(Y)$ *is an immersion, then equation* (3.7) *is equivalent to the pairs of equations*

$$u = j_1(s) \quad \text{for some section } s \text{ of } Y$$

and

$$\mathcal{E}[s]\,\omega = 0 \cdot$$

In other words, a section u of $J_1(Y)$ is an extremal of the integral on the left of (3.6) in the sense that (3.7) holds if and only if u is of the form $u = j_1(s)$ and s is an extremal for the integral I_A of § 2.

As a first step in the proof of Theorem 3.1 we prove :

LEMMA 3.1. – *Let* s *be a section of* Y *over* X, *then if* $u = j_1(s)$

(3.8) $\qquad u^*(\eta \lrcorner\, d\Theta) = 0 \quad$ *for all* $\eta \in C^\infty(T(J_1(Y)/Y))$.

If $\sigma(L) : J_1(Y) \to T \otimes_Y V^*(Y)$ *is an immersion, then a section* u *of* $J_1(Y)$ *over* X *satisfying* (3.2) *for all* $\eta \in C^\infty(T(J_1(Y)/Y))$ *is of the form* $u = j_1(s)$ *for some section* s *of* Y.

Proof. – Let η be a section of $T(J_1(Y)/Y)$; then $\langle \eta , \theta \rangle = 0$ and $\eta \lrcorner \pi^* \omega = 0$. Hence $\eta \lrcorner \Theta = 0$ and $\mathcal{L}_\eta \Theta = \eta \lrcorner d\Theta$. Also, since $\barwedge$ is an algebraic operation and $\mathcal{L}_\eta \pi^* \omega = 0$, we see that

$$\mathcal{L}_\eta \Theta = (\mathcal{L}_\eta \theta) \barwedge \pi^* \omega \ .$$

Now, since ω_1 and θ both vanish on all tangent vectors $\eta \in T(J_1(Y)/Y)$, we can think of ω_1 and θ as functions from $J_1(Y)$ to $T^*(Y) \otimes_Y T$ given by

$$\langle \xi , \omega_1(p) \rangle = \xi - p_* \pi_* \xi$$

and

$$\langle \xi , \theta(p) \rangle = \frac{1}{n} \mathrm{L}(p) \pi_* \xi + \sigma(\mathrm{L}) p \cdot (\langle \xi , \omega_1(p) \rangle$$

for all $p \in J_1(Y)$ and $\xi \in T_y(Y)$, where $y = \pi_0(p)$. In order to compute $\mathcal{L}_\eta \theta$ and $\mathcal{L}_\eta \omega_1$, we shall determine the Lie derivatives of these functions. The affine structure of $J_1(Y)$ is determined by the map

$$\alpha : \begin{array}{l} J_1(Y) \to T^* \otimes_Y V(Y) \\ \\ \qquad p \mapsto p_* \end{array}$$

then $\mathcal{L}_\eta \alpha = \eta$, where we have identified

$$T(J_1(Y)/Y) \text{ with } T^* \otimes_{J_1(Y)} V(Y)$$

according to (1.5). Therefore for $p \in J_1(Y)$ and $\xi \in T_y(Y)$, with $y = \pi_0(p)$, we have

$$\langle \xi , (\mathcal{L}_\eta \omega_1)(p) \rangle = - \langle \pi_* \xi , \eta(p) \rangle \ . \tag{3.9}$$

We thus obtain for $p \in J_1(Y)$, $\xi \in T_y(Y)$ with $y = \pi_0(p)$

$$\begin{aligned} \langle \xi , (\mathcal{L}_\eta \theta)(p) \rangle = \frac{1}{n} \langle \eta(p) , \sigma(\mathrm{L}) p \rangle \pi_* \xi \\ - \sigma(\mathrm{L}) p \cdot (\langle \pi_* \xi , \eta(p) \rangle) \\ + (\mathcal{L}_\eta \sigma(\mathrm{L}))(p) \cdot \langle \xi , \omega_1 \rangle \ . \end{aligned} \tag{3.10}$$

In the first term of the right-hand side of (3.10), we have identified $\eta(p)$ with an element of $T^* \otimes_{J_1(Y)} V(Y)$. Therefore.

$$u^*(\eta \lrcorner d\Theta) = u^*(\mathcal{L}_\eta \theta) \barwedge \omega = \psi \barwedge \omega$$

where ψ is the section of $T^* \otimes T = \mathrm{Hom}(T, T)$ given by

$$\psi(v) = \frac{1}{n} \langle \eta(u(x)), \sigma(L)\, u(x) \rangle\, id - \sigma(L)\, u(x) \cdot \langle v, \eta(u(x)) \rangle + (\mathcal{L}_\eta \sigma(L))\,(u(x)) \cdot \langle v, u^* \omega_1 \rangle$$

for $v \in T_x$, or, in other words by

$$\psi = \frac{1}{n} \langle \eta \cdot u, \sigma(L)\, u \rangle\, id - (\sigma(L)\, u) \circ (\eta \cdot u) + ((\mathcal{L}_\eta\, \sigma(L)) \cdot u) \circ u^* \omega_1 .$$

Now by (3.3), $\psi \wedge \omega = (\mathrm{Tr}\, \psi)\, \omega$, and the pairing between

$$\eta(p) \in T^* \otimes_{J_1(Y)} V(Y) \quad \text{and} \quad \sigma(L)\, p \in T \otimes_{J_1(Y)} V^*(Y)$$

is exactly $\langle \eta(p), \sigma(L)\, p \rangle = \mathrm{Tr}[(\sigma(L)\, p) \circ \eta(p)]$, if $p \in J_1(Y)$. Thus in computing $\mathrm{Tr}\, \psi$, the contribution of the first two terms of the expression for ψ cancel and we obtain

$$\mathrm{Tr}\, \psi = \mathrm{Tr}[((\mathcal{L}_\eta\, \sigma(L)) \cdot u) \circ u^* \omega_1]$$

and

$$u^*(\eta \lrcorner\, d\Theta) = \mathrm{Tr}[((\mathcal{L}_\eta\, \sigma(L)) \cdot u) \circ u^* \omega_1] ,$$

which implies, by Proposition 1.1, the first part of the lemma. Now, if $p \in J_1(Y)$

$$(\mathcal{L}_\eta\, \sigma(L))\,(p) = \sigma(L)_*\,(\eta(p)) .$$

The assertion that $\sigma(L)$ is an immersion means that

$$\sigma(L)_* : T_p(J_1(Y)/Y) \to T_x \otimes V_y^*(Y)$$

is an isomorphism for all $p \in J_1(Y)$, where $x = \pi(p)$, $y = \pi_0(p)$. If $\sigma(L)$ is an immersion, then (3.8) is equivalent to

$$\mathrm{Tr}(\chi \circ u^* \omega_1) = 0$$

for all sections χ of $T \otimes_Y V^*(Y)$ and hence to $u^* \omega_1 = 0$. By Proposition 1.2, the remainder of the lemma follows.

Because of the discussion preceding Theorem 3.1, to complete the proof of Theorem 3.1 it is sufficient to show that for a section s of Y the condition $\mathcal{E}[s]\, \omega = 0$ implies that $u = j_1(s)$ satisfies (3.7). Let ξ be any vertical vector field along u, i.e., $\xi(x) \in V_{u(x)}(J_1(Y))$

for $x \in X$, of compact support. Then $\pi_{0*} \xi$ is a vertical vector field along s. Extend $\pi_{0*} \xi$ to a vertical vector field on Y of compact support, and let φ_t be the corresponding flow on Y which satisfies $\pi\varphi_t = id$. Let ξ' be the vector field along u tangent to the family of sections $j_1(\varphi_t \circ s)$ of $J_1(Y)$ at $t = 0$ whose support is contained in an open subset B of X with compact closure. By construction $\pi_{0*} \xi' = \pi_{0*} \xi$ and thus $(\xi' - \xi)(x)$ belongs to $T_{u(x)}(J_1(Y)/Y)$. Hence by Lemma 3.1, we have

$$u^*((\xi' - \xi) \lrcorner\, d\Theta) = 0 \ .$$

Since ξ' vanishes on ∂B by (1.2) and (3.6), we have

$$\int_B u^*(\xi' \lrcorner\, d\Theta) = \frac{d}{dt} I_B[\varphi_t \circ s]\Big|_{t=0} \quad ;$$

this integral vanishes since s is an extremal. Hence

$$\int_B u^*(\xi \lrcorner\, d\Theta) = 0$$

and (3.7) holds, completing the proof of Theorem 3.1.

Notice that in our discussion of (3.7), we restricted our attention to vector fields ξ along u which are vertical. However, this restriction is superfluous. Indeed, if ζ is a vector field on X, then $u_* \zeta$ is a vector field along u and

$$u^*(u_* \zeta \lrcorner\, d\Theta) = \zeta \lrcorner\, u^* d\Theta = 0$$

since $d\Theta$ is an $(n + 1)$-form and X is n-dimensional. If ξ is an arbitrary vector field along u, then $\pi_* \xi$ is a well-defined vector field on X and $\xi - u_* \pi_* \xi$ is a vertical vector field along u. By the above equation,

$$u^*(\xi \lrcorner\, d\Theta) = u^*((\xi - u_* \pi_* \xi) \lrcorner\, d\Theta) = 0$$

and thus

$$u^*(\xi \lrcorner\, d\Theta) = 0$$

holds for *all* vector fields ξ along u if and only if it holds for vertical ξ.

Let γ be the unique section of Λ^n T over X such that $\langle \gamma, \omega \rangle = 1$. That this last equation holds for all vector fields ξ along u is equivalent to

$$u_* \gamma \lrcorner\, d\Theta = 0 \ . \tag{3.12}$$

Equation (3.12) is Hamilton's form of the Euler-Lagrange equation.

We are now in a position to state a more general version of Noether's theorem. Let u_t be a one-parameter family of sections of $J_1(Y)$, let $\overline{\varphi}_t$ be a one-parameter family of diffeomorphisms of X and let α_t be a one-parameter family of $(n-1)$-forms on X such that

$$\overline{\varphi}_t^* u_t^* \Theta = u^* \Theta + d\alpha_t \quad \text{where } u = u_0 \, . \tag{3.13}$$

Let $\overline{\xi}$ be the infinitesimal generator of $\overline{\varphi}_t$ and let ξ denote the vector field along $u = u_0$ tangent to u_t at $t = 0$ and $\dot{\alpha} = \left.\dfrac{d\alpha_t}{dt}\right|_{t=0}$.

We have according to (1.2)

$$d\dot{\alpha} = \left.\frac{d}{dt}\,\overline{\varphi}_t^* u_t \Theta\right|_{t=0} = d(\overline{\xi} \lrcorner \Theta) + du^*(\xi \lrcorner \Theta) + u^*(\xi \lrcorner d\Theta) \, .$$

If u is of the form $u = j_1(s)$ for some section s of Y which is an extremal, then the last term on the right-hand side of the above equation vanishes and

$$d\{(L(j_1(s))\,\overline{\xi} \lrcorner \omega + j_1(s)^* \; (\xi \lrcorner \Theta) - \dot{\alpha}\} = 0 \; .$$

We have thus proved :

THEOREM 3.2. – *Let s be a section of* Y. *Suppose that u_t is a one-parameter family of sections of* $J_1(Y)$ *with* $u_0 = j_1(s)$, *and* $\overline{\varphi}_t$ *is a one-parameter family of diffeomorphisms of* X *and* α_t *is a one-parameter family of* $(n-1)$*-forms on* X *such that* (3.13) *holds. Then if s is an extremal, the* $(n-1)$*-form on* X

$$L(j_1(s))\,\overline{\xi} \lrcorner \omega + j_1(s)^* \; (\xi \lrcorner \Theta) - \dot{\alpha} \tag{3.14}$$

is closed.

By Proposition 3.1, if $u_t = j_1(s_t)$, where s_t is a one-parameter family of sections of Y, then equation (3.13) reduces to (2.6) and this closed $(n-1)$-form is the one obtained by Noether's theorem.

Let us compute the forms θ, Θ, and $d\Theta$ in terms of a local coordinate system $(x^1, \ldots, x^n, y^1, \ldots, y^m)$ on an open subset U of Y of the type considered in § 1, for which

$$\omega = dx^1 \wedge \ldots \wedge dx^n$$

on πU. The coordinates (1.7) (x^i, y^j, y^j_i), $i = 1, \ldots, n$, $j = 1, \ldots, m$, on $J_1(Y)$, induce coordinates on all bundles considered. In terms of these coordinates

$$\omega_1 = \sum_j \left(dy^j - \sum_i y^j_i dx^i\right) \otimes \frac{\partial}{\partial y^j}$$

$$\theta = \frac{1}{n} \sum_i L\, dx^i \otimes \frac{\partial}{\partial x^i} + \sum_{i,j} \frac{\partial L}{\partial y^j_i} \left(dy^j - \sum_k y^j_k dx^k\right) \otimes \frac{\partial}{\partial x^i}$$

$$\Theta = L\, dx^1 \wedge \ldots \wedge dx^n$$

$$+ \sum_{i,j} (-1)^{i+1} \frac{\partial L}{\partial y^j_i} dy^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n \tag{3.15}$$

$$- \sum_{i,j} \frac{\partial L}{\partial y^j_i} y^j_i dx^1 \wedge \ldots \wedge dx^n$$

$$d\Theta = - dH \wedge dx^1 \wedge \ldots \wedge dx^n$$

$$+ \sum_{i,j} (-1)^{i+1} dp^i_j \wedge dy^j \wedge dx^1 \wedge \ldots \wedge \widehat{dx^i} \wedge \ldots \wedge dx^n$$

where we have set

$$H = \sum_{i,j} \frac{\partial L}{\partial y^j_i} y^j_i - L \tag{3.16}$$

and

$$p^i_j = \frac{\partial L}{\partial y^j_i} \quad . \tag{3.17}$$

If $\sigma(L)$ is a diffeomorphism of an open subset W of π_0^{-1}U onto $\sigma(L)$W, then (x^i, y^j, p^i_j) are also coordinates on W. Hamilton's form (3.12) of the Euler-Lagrange equation is equivalent to the equations

$$\frac{\partial H}{\partial y^j}(u(x)) = - \sum_i \frac{\partial u^i_j}{\partial x^i}(x) \tag{3.18}$$

$(j = 1, \ldots, m)$ and

$$\frac{\partial H}{\partial p^i_j}(u(x)) = \frac{\partial u^j}{\partial x^i}(x) \tag{3.19}$$

$(i = 1, \ldots, n\ ;\ j = 1, \ldots, m)$, if $u^j = y^j \circ u$ and $u^i_j = p^i_j \circ u$.

Let us end this section with a rapid discussion of the "variable end point" problem. Let N be a submanifold of Y with $\pi N = \partial A$ for which $\pi : N \to \partial A$ is a submersion. Suppose that we allow variations s_t with $s_t(\partial A) \subset N$ and require that $s = s_0$ be an extremal with respect to all such variations. Then s is certainly an extremal with respect to fixed boundary values. In addition, if $u_t = j_1(s_t)$, $u = j_1(s)$ and ξ is the tangent vector field along u to u_t at $t = 0$, then

$$\int_A \frac{du_t^* \Theta}{dt}\bigg|_{t=0} = \int_A u^*(\xi \lrcorner d\Theta) + \int_{\partial A} u^*(\xi \lrcorner \Theta) \cdot$$

The first term on the right-hand side vanishes because of (3.7), while the second integral vanishes for all such vector fields ξ satisfying $\pi_{0*}\, \xi(x) \in V_{s(x)}(N)$. Thus by (3.5), we obtain the condition

$$((\sigma(L)\, j_1(s)\,(x)) \cdot w) \lrcorner\, \omega\,|_{\partial A} = 0 \tag{3.20}$$

for all $w \in V_y(N)$, $y \in N$, with $\pi(y) = x \in \partial A$. To illustrate the meaning of this condition, consider the following example. Suppose that X and M are Riemannian manifolds, that $Y = M \times X$ and that ω is the volume form on X. Then $J_1(Y)$ is identified with $T^* \otimes_Y T(M)$, which is a vector bundle over Y equipped with a scalar product. Let V be a real-valued function defined on Y and let L be the Lagrangian defined by

$$L(p) = \frac{1}{2}\, \| p \|^2 + V(\pi_0(p)) \,, \quad p \in J_1(Y) \,.$$

Then $\sigma(L) : T^* \otimes_Y T(M) \to T \otimes_Y T^*(M)$ is the natural identification of a vector bundle with its dual given by a scalar product, i.e.

$$\langle p(v) \,, w \rangle = \langle v \,, (\sigma(L)\, p)\, w \rangle$$

for $p \in J_1(Y)$, $\pi_0(p) = (x, m)$, $v \in T_x$ and $w \in T_m(M)$. Let $N = \partial A \times M'$, where M' is a submanifold of M ; then $V_y(N)$ can be identified with $T_m(M')$, if $y = (x, m) \in N$. If $f : X \to Y$ is the function whose graph is s, then $j_1(s)\,(x)$ is identified with $f_*(x) \in \mathrm{Hom}(T_x, T_{f(x)}(M))$. Hence (3.20) is equivalent to

$$(\sigma(L)\, j_1(s)\,(x)) \cdot w \in T_x(\partial A)$$

for all $x \in \partial A$, $w \in T_{f(x)}(M')$, and so condition (3.20) becomes

$$\langle f_*(x)\,\nu\,,w\rangle = 0$$

for all $x \in \partial A$, $\nu \in T_x^{\perp}(\partial A)$, $w \in T_{f(x)}(M')$, where $T_x^{\perp}(\partial A)$ is the space of vectors of T_x orthogonal to ∂A.

4. The Poisson bracket.

The $(n+1)$-form $\Omega = d\Theta$ plays a crucial role in theoretical mechanics. A vector field ξ on $J_1(Y)$ is called locally Hamiltonian if

$$\mathcal{L}_\xi \Omega = 0 \ .$$

Since $d\Omega = 0$, this condition is equivalent to

$$d(\xi \,\lrcorner\, \Omega) = 0 \ .$$

Thus, locally we can write $\xi \,\lrcorner\, \Omega = d\tau$, for some $(n-1)$-form τ on $J_1(Y)$. A vector field ξ on $J_1(Y)$ is called (globally) Hamiltonian if there exists a $(n-1)$-form τ on $J_1(Y)$ such that

$$\xi \,\lrcorner\, \Omega = d\tau \ . \tag{4.1}$$

Suppose that ξ_1, ξ_2 are locally Hamiltonian vector fields on $J_1(Y)$. Then

$$[\xi_1\,,\xi_2] \,\lrcorner\, \Omega = \mathcal{L}_{\xi_1}(\xi_2 \,\lrcorner\, \Omega)$$

and

$$d([\xi_1\,,\xi_2] \,\lrcorner\, \Omega) = \mathcal{L}_{\xi_1}(d(\xi_2 \,\lrcorner\, \Omega)) = 0 \ .$$

Hence $[\xi_1\,,\xi_2]$ is locally Hamiltonian and the set of locally Hamiltonian vector fields on $J_1(Y)$ is a Lie algebra. If ξ_2 is Hamiltonian, then

$$\xi_2 \,\lrcorner\, \Omega = d\tau_2$$

for some $(n-1)$-form τ_2 on $J_1(Y)$. We have

$$[\xi_1\,,\xi_2] \,\lrcorner\, \Omega = \mathcal{L}_{\xi_1} d\tau_2 = d(\mathcal{L}_{\xi_1}\tau_2) \tag{4.2}$$

and so $[\xi_1\,,\xi_2]$ is globally Hamiltonian. Thus the set $\mathfrak{A}$ of globally Hamiltonian vector fields on $J_1(Y)$ is an ideal of the Lie algebra of locally Hamiltonian vector fields on $J_1(Y)$. If $\xi_2 \,\lrcorner\, \Omega = 0$, then, by (4.2), $[\xi_1\,,\xi_2] \,\lrcorner\, \Omega = 0$. Thus the vector fields ξ on $J_1(Y)$ satisfying

$$\xi \lrcorner \Omega = 0 \tag{4.3}$$

also form an ideal in the algebra of locally Hamiltonian vector fields on $J_1(Y)$ (and a fortiori in $\mathfrak{A}$). The quotient algebra $\mathfrak{H}$ of $\mathfrak{A}$ by this ideal is called the Hamiltonian algebra. We denote by $[\xi]$ the image in $\mathfrak{H}$ of an element ξ of $\mathfrak{A}$.

Let $\mathfrak{P}$ be the space of $(n-1)$-forms τ satisfying (4.1), for some element ξ of $\mathfrak{A}$. The element τ of $\mathfrak{P}$ determines $[\xi] \in \mathfrak{H}$ and so we have a surjective map $\mathfrak{P} \to \mathfrak{H}$. The set of closed $(n-1)$-forms $Z^{n-1}(J_1(Y))$ on $J_1(Y)$ is a subspace of $\mathfrak{P}$ and it is clear that the sequence

$$0 \to Z^{n-1}(J_1(Y)) \to \mathfrak{P} \to \mathfrak{H} \to 0 \tag{4.4}$$

is exact. We denote by P the quotient of $\mathfrak{P}$ by the subspace $B^{n-1}(J_1(Y))$ of exact $(n-1)$-forms. The map $\mathfrak{P} \to \mathfrak{H}$ factors through P and hence (4.4) gives us the exact sequence

$$0 \to H^{n-1}(J_1(Y), \mathbf{R}) \to P \to \mathfrak{H} \to 0 \,. \tag{4.5}$$

We now define a skew-symmetric bilinear operation

$$\mathfrak{P} \times \mathfrak{P} \to \mathfrak{P}$$

called the Poisson bracket. If $\tau_1, \tau_2 \in \mathfrak{P}$ and

$$d\tau_1 = \xi_1 \lrcorner \Omega$$
$$d\tau_2 = \xi_2 \lrcorner \Omega \,,$$

where $\xi_1, \xi_2 \in \mathfrak{A}$, set

$$\{\tau_1, \tau_2\} = \xi_1 \lrcorner d\tau_2 = \xi_1 \lrcorner \xi_2 \lrcorner \Omega \,. \tag{4.6}$$

The last of these expressions shows that the Poisson bracket is skew-symmetric and so $\{\tau_1, \tau_2\}$ is a well-defined $(n-1)$-form. Notice that (4.2) implies that

$$d\{\tau_1, \tau_2\} = [\xi_1, \xi_2] \lrcorner \Omega \,. \tag{4.7}$$

Now let us examine Jacobi's identity. We have, by (4.7)

$$\begin{aligned}\{\{\tau_1, \tau_2\}, \tau_3\} &= [\xi_1, \xi_2] \lrcorner d\tau_3 \\ &= \mathcal{L}_{\xi_1}(\xi_2 \lrcorner d\tau_3) - \xi_2 \lrcorner \mathcal{L}_{\xi_1} d\tau_3 \\ &= \mathcal{L}_{\xi_1}\{\tau_2, \tau_3\} - \xi_2 \lrcorner d\{\tau_1, \tau_3\} \\ &= \xi_1 \lrcorner d\{\tau_2, \tau_3\} + d(\xi_1 \lrcorner \{\tau_2, \tau_3\}) - \{\tau_2, \{\tau_1, \tau_3\}\}\end{aligned}$$

or

$$\{\{\tau_1, \tau_2\}, \tau_3\} = \{\tau_1, \{\tau_2, \tau_3\}\} + \{\{\tau_1, \tau_3\}, \tau_2\}$$
$$+ d(\xi_1 \lrcorner \xi_2 \lrcorner \xi_3 \lrcorner \Omega) . \tag{4.8}$$

Thus Jacobi's identity holds for this Poisson bracket on $\mathfrak{P}$ up to the term $d(\xi_1 \lrcorner \xi_2 \lrcorner \xi_3 \lrcorner \Omega)$. Notice that if $n = 1$, then this term vanishes automatically since Ω is a two-form, and so in this case $\mathfrak{P} = \mathrm{P}$ is a Lie algebra.

The Poisson bracket on $\mathfrak{P}$ induces according to (4.6) a Poisson bracket on P. Now by (4.8), this bracket on P satisfies Jacobi's identity, and so P together with the Poisson bracket is a Lie algebra, the Poisson algebra. By (4.7), the sequence (4.5) is an exact sequence of Lie algebras, if $\mathrm{H}^{n-1}(\mathrm{J}_1(\mathrm{Y}), \mathrm{R})$ is considered as an abelian Lie algebra.

If $n = 1$, the orientation of X trivializes the line bundle T and so the Legendre transformation can be viewed as a map

$$\sigma(\mathrm{L}) : \mathrm{J}_1(\mathrm{Y}) \to \mathrm{V}^*(\mathrm{Y}) .$$

Let $t \in \mathrm{X}$; then $\sigma(\mathrm{L})$ maps the $2m$-dimensional fiber $\mathrm{J}_1(\mathrm{Y})_t$ of $\mathrm{J}_1(\mathrm{Y})$ over t into $\mathrm{T}^*(\mathrm{Y}_t)$, if Y_t is the fiber of Y over t. According to (3.5), for all $p \in \mathrm{J}_1(\mathrm{Y})$, $\eta \in \mathrm{V}_p(\mathrm{J}_1(\mathrm{Y}))$

$$\langle \eta , \Theta \rangle = \langle \pi_{0*} \eta , \sigma(\mathrm{L}) p \rangle ,$$

so that, if α_t denotes the canonical 1-form on $\mathrm{T}^*(\mathrm{Y}_t)$ and

$$i_t : \mathrm{J}_1(\mathrm{Y})_t \to \mathrm{J}_1(\mathrm{Y})$$

is the inclusion map, we have

$$i_t^* \Theta = \sigma(\mathrm{L})^* \alpha_t .$$

We now assume that $\sigma(\mathrm{L})$ is an immersion. Since $d\alpha_t$ is the standard symplectic form on $\mathrm{T}^*(\mathrm{Y}_t)$, the 2-form $i_t^* d\Theta$ defines a symplectic structure on $\mathrm{J}_1(\mathrm{Y})_t$. The space $\mathrm{C}^\infty(\mathrm{J}_1(\mathrm{Y})_t)$ of real-valued differentiable functions on $\mathrm{J}_1(\mathrm{Y})_t$ together with the usual Poisson bracket defined in terms of $i_t^* d\Theta$ is a Lie algebra. The map

$$\mathfrak{P} \to \mathrm{C}^\infty(\mathrm{J}_1(\mathrm{Y})_t) \tag{4.9}$$

sending a 0-form $f \in \mathfrak{P}$ into its restriction to $\mathrm{J}_1(\mathrm{Y})_t$ is a homomorphism of Lie algebras. Indeed, there is a unique vector field ζ, the Euler field, on $J_1(Y)$ determined by

$$\zeta \lrcorner \Omega = 0$$

and

$$\langle \zeta , \pi^* \omega \rangle = 1 \ .$$

Thus a real-valued function f on $J_1(Y)$ belongs to $\mathfrak{P}$ if and only if there exists a *vertical* vector field ξ on $J_1(Y)$ such that

$$df = \xi \lrcorner \Omega \ .$$

In fact, for a given $f \in \mathfrak{P}$ there exists a unique such ξ. It follows now from the definitions of Poisson brackets that our restriction map is a homomorphism of Lie algebras.

According to the Euler-Lagrange equation (3.12), a section u of $J_1(Y)$ over X is of the form $u = j_1(s)$ for some extremal s if and only if ζ is tangent to u. A real-valued function belongs to $\mathfrak{P}$ if and only if $\zeta \cdot f = 0$, that is if f is locally constant along all such u. In terms of the coordinates $(t , q^1 , \dots , q^m , p^1 , \dots , p^m)$ on $J_1(Y)$,

$$\zeta = \frac{\partial}{\partial t} + \sum_{j=1}^{m} \frac{\partial H}{\partial p^j} \frac{\partial}{\partial q^j} - \sum_{j=1}^{m} \frac{\partial H}{\partial q^j} \frac{\partial}{\partial p^j}$$

and f belongs to $\mathfrak{P}$ if and only if $df = \xi \lrcorner \Omega$, where

$$\xi = - \sum_{j=1}^{m} \frac{\partial f}{\partial p^j} \frac{\partial}{\partial q^j} + \sum_{j=1}^{m} \frac{\partial f}{\partial q^j} \frac{\partial}{\partial p^j} \cdot$$

If $f, g \in \mathfrak{P}$

$$\{ f , g\} = \sum_{j=1}^{m} \left(\frac{\partial f}{\partial q^j} \frac{\partial g}{\partial p^j} - \frac{\partial f}{\partial p^j} \frac{\partial g}{\partial q^j} \right) \cdot$$

Thus, the algebra P may be thought of as a space of "functions on extremals".

Assume that $X = \mathbf{R}$ and $\omega = dt$, and that ζ generates a one-parameter group of diffeomorphisms φ_t of $J_1(Y)$. If $s, t \in \mathbf{R}$, the map φ_s induces a diffeomorphism $J_1(Y)_t \to J_1(Y)_{t+s}$. In this case, the map (4.9) is an isomorphism of Lie algebras, and so P is infinite dimensional. The underlying operators of quantum mechanics (the "canonical quantization conditions") are the operators arising from a (faithful) representation of the Lie algebra $C^\infty(J_1(Y)_t)$ and hence of the infinite dimensional Lie algebra P.

We now return to the general case. In higher dimensions ($n > 1$), the restriction (4.1) tends to be more severe. For the free fields (i.e. quadratic Lagrangians) that arise in quantum field theory, the algebra P is infinite dimensional and provides enough elements to yield the operators of the associated free quantum fields. However, computations done jointly with Professor S. Coleman, to whom we are very grateful, seem to indicate that if $n \geqslant 3$, then for "interacting Lagrangians", i.e. those containing higher order terms, the algebra P is finite dimensional, and hence does not provide enough operators for quantization.

However, in the general case, in a purely formal sense, we may still think of P as a space of "functions on extremals". For any extremal s and $\tau \in \mathcal{R}$, we have by (3.12), if $u = j_1(s)$

$$du^* \tau = u^* d\tau = u^*(\xi \lrcorner \Omega) = 0$$

if τ satisfies (4.1) for some vector field ξ on $J_1(Y)$, and so $u^*\tau$ is a closed $(n-1)$-form on X, whose cohomology class in $H^{n-1}(X, R)$, denoted by $[\tau](s)$, depends only on the image of τ in P. Thus an element of P defines a function on the set of extremals with values in $H^{n-1}(X, R)$. Now suppose that $X = R \times X_0$, where X_0 is an oriented $(n-1)$-dimensional manifold, and that τ satisfies the following assumption on its support. If $pr_1 : R \times X_0 \to R$ is the projection onto the first factor, we assume that $pr_1 : \pi(\operatorname{supp} \tau) \to R$ is proper. Consider the "space-like surfaces" $Z_t = \{t\} \times X_0$, if $t \in R$. If ι_t is the injection of Z_t into X, we obtain a closed $(n-1)$-form $\iota_t^* u^* \tau$ on Z_t with compact support, and the integral

$$\hat{\tau}(s) = \int_{Z_t} \iota_t^* u^* \tau$$

is independent of t ; hence, for each such element of P, we obtain a well-defined real-valued function $\hat{\tau}$ on the set of extremals.

Suppose that a one-parameter family of diffeomorphisms $\psi_t : J_1(Y) \to J_1(Y)$ is a global symmetry of the system in the sense that

$$\psi_t^* \Theta = \Theta .$$

If ξ is the infinitesimal generator of ψ_t, then

$$0 = \mathcal{L}_\xi \Theta = \xi \lrcorner \Omega + d(\xi \lrcorner \Theta) ,$$

so that (4.1) holds with $\tau = -\xi \lrcorner \Theta$. Thus any global symmetry gives rise to an element $\tau \in \mathcal{P}$. Assume that ψ_t satisfies $\pi \circ \psi_t = \overline{\varphi}_t \circ \pi$, where $\overline{\varphi}_t$ is a one-parameter family of diffeomorphisms of X. If u is a section of $J_1(Y)$ over X, setting $u_t = \psi_t \circ u \circ \overline{\varphi}_t^{-1}$, equation (3.13) holds with $\alpha_t = 0$. If $\overline{\xi}$ is the infinitesimal generator of $\overline{\varphi}_t$, then the vector field along u tangent to u_t at $t = 0$ is

$$\xi \circ u - u_* \overline{\xi} \ ;$$

hence, if s is an extremal and $u = j_1(s)$, then the closed $(n-1)$-form (3.14) given by Theorem 3.2 is precisely $-u^* \tau$.

Let φ be a diffeomorphism of $J_1(Y)$ satisfying $\varphi^* \Omega = \Omega$. Then if ξ is a vector field on $J_1(Y)$, then $\varphi_* \xi$ is a well-defined vector field on $J_1(Y)$ and

$$\varphi^{-1*}(\xi \lrcorner \Omega) = \varphi_* \xi \lrcorner \Omega \ .$$

It follows that φ induces Lie algebra automorphisms of $\mathcal{A}$ and $\mathcal{H}$; moreover, if $\tau_1, \tau_2 \in \mathcal{P}$, then $\varphi^{-1*}\tau_1, \varphi^{-1*}\tau_2 \in \mathcal{P}$ and

$$\varphi^{-1*}\{\tau_1, \tau_2\} = \{\varphi^{-1*}\tau_1, \varphi^{-1*}\tau_2\} \ ,$$

so that φ induces a Lie algebra automorphism of P. If $n = 1$ and $\sigma(L)$ is an immersion, and if the Euler field ζ generates a one-parameter group of diffeomorphisms φ_t, then

$$\mathcal{L}_\zeta \Omega = d(\zeta \lrcorner \Omega) = 0 \ ;$$

hence

$$\varphi_t^* \Omega = 0 \ .$$

Let G be a Lie group acting on $J_1(Y)$,

$$G \times J_1(Y) \to J_1(Y)$$

$$(a, p) \mapsto \varphi_a(p) \ ,$$

with $\varphi_{ab} = \varphi_a \varphi_b$, for all $a, b \in G$. We say that G is a group of dynamical symmetries if $\varphi_a^* \Theta = \Theta$, for all $a \in G$. If $\mathfrak{g}$ is the Lie algebra of G, the action of G on $J_1(Y)$ associates a vector field $\hat{\xi}$ on $J_1(Y)$ to each $\xi \in \mathfrak{g}$; since $\mathcal{L}_{\hat{\xi}} \Theta = 0$, we obtain a Lie algebra homomorphism. Since $\varphi_a^* \Omega = \Omega$, for all $a \in G$, by the above remarks, $\mathcal{A}, \mathcal{H}, \mathcal{P}$, P are naturally G-modules, (4.5) is an exact sequence of G-modules and the map $\mathcal{A} \to \mathcal{H}$ is a G-homomorphism. According to the identity

$$\widehat{\mathrm{Ad}\,a \cdot \xi} = \varphi_{a*}\,\hat{\xi}\ ,$$

for $a \in \mathrm{G}$, $\xi \in \mathfrak{g}$, if we consider $\mathfrak{g}$ as a G-module via the adjoint representation, the map $\mathfrak{g} \to \mathcal{Q}$ is a G-homomorphism. Thus we obtain a G-homomorphism $\mathfrak{g} \to \mathcal{H}$.

Let $\lambda : \mathfrak{g} \to \mathrm{P}$ be an arbitrary lifting of the linear map $\mathfrak{g} \to \mathcal{H}$; then $a \cdot \lambda \cdot \mathrm{Ad}\,a^{-1} - \lambda$ is a map of $\mathfrak{g}$ into $\mathrm{H}^{n-1}(\mathrm{J}_1(\mathrm{Y}), \mathrm{R})$ and $f : \mathrm{G} \to \mathfrak{g}^* \otimes \mathrm{H}^{n-1}(\mathrm{J}_1(\mathrm{Y}), \mathrm{R})$ defined by

$$f(a) = a \cdot \lambda \cdot \mathrm{Ad}\,a^{-1} - \lambda\ , \quad a \in \mathrm{G}\ ,$$

is a 1-cocycle for the G-module $\mathfrak{g}^* \otimes \mathrm{H}^{n-1}(\mathrm{J}_1(\mathrm{Y}), \mathrm{R})$. According to extension theory, the map $\mathfrak{g} \to \mathcal{H}$ can be lifted to a G-homomorphism $\mathfrak{g} \to \mathrm{P}$ if and only if the cohomology class of f in

$$\mathrm{H}^1(\mathrm{G}, \mathfrak{g}^* \otimes \mathrm{H}^{n-1}(\mathrm{J}_1(\mathrm{Y}), \mathrm{R}))$$

vanishes.

Furthermore, if this cohomology class vanishes, there exists a map $\mathfrak{g} \to \mathcal{G}$ sending ξ into a $(n-1)$-form τ_ξ satisfying

$$d\tau_\xi = \hat{\xi} \,\lrcorner\, \Omega$$

and such that

$$\varphi_a^{-1*}\,\tau_\xi - \tau_{\mathrm{Ad}\,a \cdot \xi}$$

is an exact form, for all $a \in \mathrm{G}$. Then for $\xi, \eta \in \mathfrak{g}$

$$\mathcal{L}_{\hat{\xi}}\,\tau_\eta + \tau_{[\xi,\eta]}\ ,$$

or equivalently

$$\{\tau_\xi, \tau_\eta\} + \tau_{[\xi,\eta]}$$

is exact. Therefore the map $\mathfrak{g} \to \mathrm{P}$, sending ξ into the image in P of τ_ξ, lifts the map $\mathfrak{g} \to \mathcal{H}$, is a G-homomorphism and is an anti-homomorphism of Lie algebras. If we set $\tau'_\xi = -\tau_\xi$, for $\xi \in \mathfrak{g}$, the $(n-1)$-form

$$\{\tau'_\xi, \tau'_\eta\} - \tau'_{[\xi,\eta]}\ , \quad \xi, \eta \in \mathfrak{g}\ ,$$

is exact. If s is any extremal and $u = j_1(s)$, we obtain a map $\mathfrak{g} \to \mathrm{H}^{n-1}(\mathrm{X}, \mathrm{R})$ sending ξ into the cohomology class of the closed $(n-1)$-form $u^*\tau'_\xi$ on X, such that, for $\xi, \eta \in \mathfrak{g}$, the forms $u^*\tau'_{[\xi,\eta]}$ and $u^*\{\tau'_\xi, \tau'_\eta\}$ define the same cohomology class in $\mathrm{H}^{n-1}(\mathrm{X}, \mathrm{R})$.

5. The Hamilton-Jacobi equation.

A section w of $J_1(Y)$ over Y will be called a slope field, and a section s of Y over X will be said to be embedded in the slope field w if $w \circ s = j_1(s)$.

THEOREM 5.1. – *Let s be a section of* Y *over* X *and w a slope field. Then the following conditions are equivalent :*

i) *s is embedded in the slope field w and s is an extremal ;*

ii) *s is embedded in the slope field w and*

$$s^*(\eta \lrcorner w^* d\Theta) = 0 \quad \text{for all } \eta \in C^\infty(T(Y)) . \tag{5.1}$$

If $\sigma(L)$ is an immersion, either of the above conditions is equivalent to

iii) (3.7) *holds for $u = w \circ s$.*

Proof. – If $u = w \circ s$, then

$$s^*(\eta \lrcorner w^* d\Theta) = u^*(w_* \eta \lrcorner d\Theta)$$

for all vector fields η on Y. If ξ is any vector field along u, then $\pi_{0*} \xi$ is a vector field along s and $w_* \pi_{0*} \xi$ is a vector field along u and

$$u^*(\xi \lrcorner d\Theta) = u^*(w_* \pi_{0*} \xi \lrcorner d\Theta) + u^*((\xi - w_* \pi_{0*} \xi) \lrcorner d\Theta) .$$

According to Lemma 3.1, if $u = w \circ s = j_1(s)$, the second term on the right hand side of this equation vanishes since $\pi_{0*}(\xi - w_* \pi_{0*} \xi) = 0$ and so ii) $\Rightarrow$ iii). The remaining implications all follow now from Theorem 3.1.

DEFINITION. – *A slope field w is to be called a geodesic field if there exists an $(n-1)$-form α on* Y *such that the Hamilton-Jacobi equation*

$$w^* \Theta = d\alpha \tag{5.2}$$

holds.

Notice that (5.2) implies (5.1) for any section s of Y over X ; hence if s is embedded in the geodesic field w, the section is an extremal.

Let w be a geodesic field. For a section s of Y over X, define

$$W_A[s] = \int_A s^* w^* \Theta \ ;$$

this integral is called Hilbert's independent integral since it depends only on the values of s on ∂A. Indeed, by Stokes' theorem

$$W_A[s] = \int_A s^* d\alpha = \int_{\partial A} s^* \alpha \ .$$

PROPOSITION 5.1. – *If a section* s *of* Y *over* X *is embedded in a geodesic field* w, *then*

$$I_A[s] = W_A[s]$$

depends only on the values of s *on* ∂A *and* s *is an extremal.*

The proposition follows from the above remarks and the equality

$$L(j_1(s))\,\omega = (w \circ s)^* \Theta$$

if $w \circ s = j_1(s)$.

We now compute the integrand of the integral $W_A[\sigma]$ for any section σ of Y over X not necessarily embedded in the field w. Now

$$\sigma^* w^* \Theta = ((w \circ \sigma)^* \theta) \barwedge \omega$$

where the operation $\barwedge$ on the right-hand side is the one described in § 3 letting $W = T_x$, with $x \in X$. Hence

$$\sigma^* w^* \Theta = \left(\frac{1}{n} L(w \circ \sigma)\, id + \psi \right) \barwedge \omega$$

where ψ is the section of $T \otimes T^*$ given by

$$\psi(x)\,\zeta = \sigma(L)\,((w \circ \sigma)(x)) \circ (j_1(\sigma)(x)_* - (w \circ \sigma)(x)_*)\,\zeta$$

$$= \sigma(L)\,((w \circ \sigma)(x)) \circ ((j_1(\sigma)(x) - (w \circ \sigma)(x))\,\zeta$$

for $\zeta \in T_x$. In this last equation we are considering $j_1(\sigma)(x) - (w \circ \sigma)(x)$ as the element of $T^* \otimes_Y V(Y)$ given by the affine structure of $J_1(Y)$ over Y. Since

$$Tr\,\psi = \langle j_1(\sigma) - w \circ \sigma \ , \ \sigma(L)\,(w \circ \sigma) \rangle \ ,$$

by (3.3) we have

(5.3) $\sigma^* w^* \Theta = (L(w \circ \sigma) + \langle j_1(\sigma) - w \circ \sigma , \sigma(L)(w \circ \sigma)\rangle)\,\omega$.

Hence

(5.4) $W_A[\sigma] = \int_A L(w \circ \sigma) + \langle j_1(\sigma) - w \circ \sigma , \sigma(L)(w \circ \sigma)\rangle\,\omega$.

Let s be a section of Y embedded in a geodesic field w and let s' be an arbitrary section of Y with $s' = s$ on ∂A. Then $W_A[s] = W_A[s']$ and

$$I_A[s'] - I_A[s] = (I_A[s'] - W_A[s']) - (I_A[s] - W_A[s])$$
$$= I_A[s'] - W_A[s']$$

by Proposition 5.1. Hence by (5.4)

$$I_A[s'] - I_A[s] = \int_A E(j_1(s'))\,\omega$$

where E is the real-valued function on $J_1(Y)$, the Weierstrass E-function, defined by

(5.5) $E(p) = L(p) - L((w \circ \pi_0)p) - \langle p - (w \circ \pi_0)p , \sigma(L)(w \circ \pi_0)p\rangle$

for $p \in J_1(Y)$.

We therefore obtain the following sufficient condition for the extremal s to be a strong local minimum :

THEOREM 5.2. — *(Weierstrass' criterion). Let s be a section of* Y *embedded in a geodesic field w. If, for some open neighborhood* W *of* $s(A)$ *in* Y, *the function* E *defined by* (5.5) *is* ≥ 0 *on the open subset* $\pi_0^{-1}(W)$ *of* $J_1(Y)$, *then s is a strong local minimum of* I_A.

Since $\pi_0 : J_1(Y) \to Y$ is an affine bundle whose associated vector bundle is $T^* \otimes_Y V(Y)$, we may define, according to § 1, the section

$$d^2_{J_1(Y)/Y} L$$

of $S^2 T^*(J_1(Y)/Y)$. We say that L is *positive definite* if

$$(d^2_{J_1(Y)/Y} L)(p)$$

is a positive definite quadratic form on $T_p(J_1(Y)/Y)$ for all $p \in J_1(Y)$. If $p \in J_1(Y)$, then as $T_p(J_1(Y)/Y)$ is isomorphic to $T^*_x \otimes V_y(Y)$, where

$y = \pi_0(p)$, $x = \pi(p)$, we say that L satisfies the *Legendre-Hadamard condition* at p if

$$(d^2_{J_1(Y)/Y}\,L)\,(p)\,(\lambda \otimes \nu\,,\,\lambda \otimes \nu) \geqslant 0 \tag{5.6}$$

for all $\lambda \in T^*_x$, $\nu \in V_y(Y)$, where $x = \pi(p)$, $y = \pi_0(p)$, and that L is *regular* if, for all $p \in J_1(Y)$, strict inequality holds in (5.6) for all $\lambda \neq 0$ and $\nu \neq 0$. If L is positive definite, then clearly L is regular.

The coordinates (1.7) $(x^i\,,y^j\,,y^j_i)$ on $J_1(Y)$ give us a basis

$$\frac{\partial}{\partial y^j_i}$$

of $T_p(J_1(Y)/Y)$, if $p \in \pi_0^{-1}\,U$; in terms of this basis

$$(d^2_{J_1(Y)/Y}\,L)\,(p)$$

is given by the matrix

$$\left(\frac{\partial^2 L}{\partial y^j_i\;\partial y^l_k}\,(p)\right) \tag{5.7}$$

$(i,\ k = 1, \ldots, n\ ;\ j,\ l = 1, \ldots, m)$, and L is positive definite at p if and only if this matrix is positive definite. Observe that $\sigma(L)$ is an immersion on a neighborhood of p if and only if the matrix (5.7) is non-singular. Hence if L is positive definite, $\sigma(L)$ is an immersion. Furthermore, the Legendre-Hadamard condition holds at p if

$$\sum_{\substack{i,k\\ j,l}} \frac{\partial^2 L}{\partial y^j_i\;\partial y^l_k}\,(p)\,\lambda^i\,\lambda^k\,\nu^j\,\nu^l \geqslant 0$$

for all $\lambda^1, \ldots, \lambda^n, \nu^1, \ldots, \nu^m$.

If L is positive definite, we have the following sufficient condition for s to be a weak local minimum.

THEOREM 5.3. – *Let s be a section of* Y *embedded in a geodesic field w. If* L *is positive definite, then s is a weak local minimum for* I_A.

Proof. – Consider the Weierstrass E-function (5.5). We have $E \circ w = 0$, $(d_{J_1(Y)/Y}\,E) \circ w = 0$ and

$$(d^2_{J_1(Y)/Y}\,E) \circ w = (d^2_{J_1(Y)/Y}\,L) \circ w \ .$$

If L is positive definite, then E is positive definite on a neighborhood of $w(Y)$; hence by Taylor's formula, there exists a neighborhood W of $w(Y)$ on which E is $\geqslant 0$. If s' is a section of Y over A with $s' = s$ on ∂A and $j_1(s') \subset W$, then $I_A[s'] \geqslant I_A[s]$.

We also remark that if L is positive definite then the Euler-Lagrange equations are "strongly elliptic" in the sense to be described below.

If Z is a fibered manifold over X and if $\Phi : J_k(Y) \to Z$ is a differentiable map over X, then, according to (1.3), the map

$$\Phi_* : T(J_k(Y)/J_{k-1}(Y)) \to T(Z/X)$$

induces a map over Φ

$$\sigma(\Phi) : S^k T^* \otimes_{J_k(Y)} V(Y) \to T(Z/X) \ ,$$

the symbol of Φ. If $Z = V^*(Y)$ and Φ is a map over $\pi_0 : J_k(Y) \to Y$, then $\sigma(\Phi)$ induces a map

$$\sigma(\Phi) : S^k T^* \otimes_{J_k(Y)} V(Y) \to V^*(Y)$$

over π_0. If $x \in X$, $\lambda \in T^*_x$, then, for $p \in J_k(Y)$ with $\pi(p) = x$, we denote by $\sigma_\lambda(\Phi) : V_y(Y) \to V^*_y(Y)$ the linear map sending v into

$$\sigma(\Phi)\left(p\,,\ \frac{\lambda^k}{k!} \otimes v\right)$$

where $y = \pi_0(p)$. We say that Φ is strongly elliptic if k is even and if for all non-zero $\lambda \in T^*_x$ and all $p \in J_k(Y)$ with $\pi(p) = x$, the map $(-1)^{k/2}\,\sigma(\Phi)$ is positive self-adjoint, i.e. if $y = \pi_0(p)$

a) $\langle w\,, (-1)^{k/2}\,\sigma_\lambda(\Phi)\,v\rangle = \langle v\,, (-1)^{k/2}\,\sigma_\lambda(\Phi)\,w\rangle$ for all $v, w \in V_y(Y)$;

b) $\langle v\,, (-1)^{k/2}\,\sigma_\lambda(\Phi)\,v\rangle > 0$ for all $v \in V_y(Y)$, $v \neq 0$.

If Y is a vector bundle E over X and Z is equal to E^*, and $\Phi : J_k(E) \to E^*$ is a morphism of vector bundles, then $\sigma(\Phi)$ induces a map $\sigma(\Phi) : S^k T^* \otimes E \to E^*$; we say that Φ or the differential operator $\Phi \circ j_k : C^\infty(E) \to C^\infty(E^*)$ is strongly elliptic if k is even and if, for all $x \in X$, $\lambda \in T_x^*$, with $\lambda \neq 0$, the map $(-1)^{k/2}\, \sigma_\lambda(\Phi) : E_x \to E_x^*$ is positive self-adjoint.

If $\Phi : J_k(Y) \to V^*(Y) \otimes_Y \Lambda^n T^*$ is a map over Y (resp. $\Phi : J_k(E) \to E^* \otimes \Lambda^n T^*$ is a morphism of vector bundles), we say that Φ or $\Phi \circ j_k$ is strongly elliptic if the map $\Phi' : J_k(Y) \to V^*(Y)$ (resp. $\Phi : J_k(E) \to E^*$) defined by $\Phi'(p) \otimes \omega(x) = \Phi(p)$, where $p \in J_k(Y)$ and $x = \pi(p)$, is strongly elliptic.

The Euler-Lagrange equation gives us a map

$$\Phi : J_2(Y) \to V^*(Y) \otimes_Y \Lambda^n T^*$$

over $\pi_0 : J_2(Y) \to Y$ sending $j_2(s)(x)$ into $(\mathcal{E}[s]\,\omega)(x)$, if s is a section of Y over a neighborhood of $x \in X$. In fact, $\Phi(j_2(s)) = 0$ is the Euler-Lagrange equation (2.4).

We identify an element $u \in S^2 T^*$ with a symmetric bilinear form $\widetilde{u}$ on T according to

$$\widetilde{u}(\xi_1, \xi_2) = 2\langle \xi_1, \lambda\rangle\langle \xi_2, \lambda\rangle$$

if $u = \lambda^2$, where $\lambda \in T^*$, $\xi_1, \xi_2 \in T$. Thus the symbol of Φ

$$\sigma(\Phi) : S^2 T^* \otimes_{J_2(Y)} V(Y) \to V^*(Y) \otimes_Y \Lambda^n T^*$$

determines a map

$$\sigma(\Phi)(p) : T_x \otimes T_x \to V_y^*(Y) \otimes V_y^*(Y) \otimes \Lambda^n T_x^*$$

for $p \in J_2(Y)$, with $y = \pi_0(p)$, $x = \pi(p)$ and is thus determined by the maps $\sigma'(\Phi)(p) : T_x \otimes T_x \to V_y^*(Y) \otimes V_y^*(Y)$ for $p \in J_2(Y)$, defined by

$$\sigma'(\Phi)(p) \otimes \omega(x) = \sigma(\Phi)(p)\,.$$

We may also consider

$$(d^2_{J_1(Y)/Y} L)(p)$$

if $p \in J_1(Y)$, with $y = \pi_0(p)$, $x = \pi(p)$, as a map

$$T_x \otimes T_x \to V_y^*(Y) \otimes V_y^*(Y)\,,$$

as $T_p(J_1(Y)/Y)$ is isomorphic to $T_x^* \otimes V_y(Y)$.

PROPOSITION 5.2. — *The symbol $\sigma(\Phi)$ of the Euler-Lagrange operator Φ is determined by*

$$\sigma'(\Phi)(p) = -(d^2_{J_1(Y)/Y} L)(\pi_1 p)$$

for $p \in J_2(Y)$. If L is regular, then the Euler-Lagrange equation is strongly elliptic.

The proportion follows directly from the computation of the Euler-Lagrange equation in terms of local coordinates.

6. Solving the Hamilton-Jacobi equations.

Given an extremal s, let us seek a geodesic field w in which s is embedded, that is, a section w of $J_1(Y)$ over Y satisfying the Hamilton-Jacobi equation

$$w \circ s = j_1(s)$$

$$w^* \Theta = d\alpha$$

for some $(n-1)$-form α on Y. In this section, we give a method for constructing local solutions following [13] and the simplifications introduced by [12]. It is of course sufficient for the local theory to solve the equation

$$w^* d\Theta = 0 \tag{6.1}$$

with the initial condition

$$w \circ s = j_1(s) \cdot \tag{6.2}$$

Let w_0 be an arbitrary auxiliary slope field ; then we can define the Hamiltonian $H : J_1(Y) \to \mathbf{R}$ in terms of w_0 by

$$H(p) = \langle p - w_0(\pi_0(p)), \sigma(L)p\rangle - L(p), \quad p \in J_1(Y).$$

If Y is the trivial bundle X x M, where M is an m-dimensional manifold, then the 0-section of the vector bundle $J_1(Y) \simeq T^* \otimes_Y T(M)$ is such an auxiliary slope field. The Hamiltonian defined in terms of the 0-section of $T^* \otimes_Y T(M)$ is given by

$$H(p) = < p, \sigma(L)p > - L(p)$$

where $p \in T^* \otimes_Y T(M)$ and $< \ , \ >$ is the bilinear form expressing $T \otimes_Y T^*(M)$ as the dual bundle of $T^* \otimes_Y T(M)$.

We now return to the general case. If we consider the computation of § 3 in terms of the coordinates (x^i, y^j, y^j_i) on the open set $\pi_0^{-1} U \subset J_1(Y)$, the function H defined by (3.16) is precisely the Hamiltonian defined in terms of the slope field

$$w_0(x, y) = (x, y, 0).$$

We obtain the following local expressions for equation (6.1), if we use this choice of auxiliary field w_0 :

$$(6.3) \qquad \frac{\partial}{\partial y^j}(\mathrm{H} \circ w) = - \sum_i \frac{\partial}{\partial x^i}(p^i_j \circ w) \quad , \quad j = 1, \ldots, m,$$

$$(6.4) \qquad d_{Y/X}(\sigma(L) \circ w) = 0$$

where H is the function (3.16) and p^i_j is defined by (3.17), and $\sigma(L) \circ w$ is considered as a vertical T-valued linear differential form on Y, i.e. as a section of $T \otimes_Y T^*(Y/X)$.

Let $W \subset \pi_0^{-1} U$ be an open subset of $J_1(Y)$ and suppose that $\sigma(L)_{|W}$ is a diffeomorphism of W onto $\sigma(L)(W)$. Assume that $j_1(s) : \pi W \to W$. In terms of the coordinate system (x^i, y^j, p^i_j) on W, we suppose that

$$\omega = dx^1 \wedge \ldots \wedge dx^n$$

and we set

$$u^j(x) = y^j(s(x))$$

$$u^i_j(x) = p^i_j(j_1(s)(x)) \ .$$

We write $t = x^1$ and $x' = (x^2, \ldots, x^n)$, so that $x = (t, x')$.

Let $f^2, \ldots, f^n$ be functions defined on $\pi_0 W$ such that

$$(6.5) \qquad \frac{\partial f^i}{\partial y^j}(s(x)) = u^i_j(x)$$

for $i = 2, \ldots, n$, $j = 1, \ldots, m$. Such functions exist ; for example, we may take

$$f^i(x, y) = \sum_{j=1}^m y^j u^i_j(x) \quad , \quad i = 2, \ldots, n .$$

Let N be the submanifold of $J_1(Y)$ determined by

$$p_j^i = \frac{\partial f^i}{\partial y^j}, \quad i = 2, \ldots, n \ , \ j = 1, \ldots, m \ .$$

Then $j_1(s)(x) \in N$ for $x \in \pi W$.

We shall use (t, x', y^j, p_j^1) as coordinates on N. If ι is the injection of N into $J_1(Y)$, then

$$(6.6) \qquad \begin{aligned} \iota^* d\Theta = &- \sum_{j=1}^{m} \left\{ \frac{\partial H}{\partial y^j} + \sum_{\substack{1<i\leq n \\ 1\leq l\leq m}} \frac{\partial H}{\partial p_l^i} \frac{\partial^2 f^i}{\partial y^l \, \partial y^j} + \sum_{1<i\leq n} \frac{\partial^2 f^i}{\partial x^i \, \partial y^j} \right\} dy^j \wedge \omega \\ &- \sum_{j=1}^{m} \frac{\partial H}{\partial p_j^1} dp_j^1 \wedge \omega + \sum_{j=1}^{m} dp_j^1 \wedge dy^j \wedge dx^2 \wedge \ldots \wedge dx^n \ . \end{aligned}$$

LEMMA 6.1. – *There exists a unique vector field η on* N *such that*

i) *η is π-projectable and* $\pi_* \eta = \dfrac{\partial}{\partial t}$;

ii) $\eta \lrcorner \iota^* d\Theta = 0$.

Proof. – Condition i) implies that η is of the form

$$(6.7) \qquad \eta = \frac{\partial}{\partial t} + \sum_{j=1}^{m} a_j \frac{\partial}{\partial y^j} + \sum_{j=1}^{m} b_j \frac{\partial}{\partial p_j^1} \ .$$

From ii) and (6.6) one obtains

$$(6.8) \qquad \begin{aligned} a_j &= \frac{\partial H}{\partial p_j^1} \\ b_j &= -\left\{ \frac{\partial H}{\partial y^j} + \sum_{\substack{1<i\leq n \\ 1\leq l\leq m}} \frac{\partial H}{\partial p_l^i} \frac{\partial^2 f^i}{\partial y^l \, \partial y^j} + \sum_{1<i\leq n} \frac{\partial^2 f^i}{\partial x^i \, \partial y^j} \right\} \end{aligned}$$

$(j = 1, \ldots, m)$, proving the lemma.

LEMMA 6.2. – *If the section s is an extremal, then*

$$(6.9) \qquad \eta \circ j_1(s) = j_1(s)_* \frac{\partial}{\partial t} \ .$$

Proof. – We know that $j_1(s) : \pi W \to N$ so that equation (6.9) is equivalent to

$$\frac{\partial u^j}{\partial t}(x) = \frac{\partial H}{\partial p_j^1}(j_1(s)(x))$$

(6.10)

$$\frac{\partial u_j^1}{\partial t}(x) = -\Big(\frac{\partial H}{\partial y^j} + \sum_{\substack{1<i\leq n \\ 1\leq l\leq m}} \frac{\partial H}{\partial p_l^i}\frac{\partial^2 f^i}{\partial y^l\,\partial y^j} + \sum_{1<i\leq n}\frac{\partial^2 f^i}{\partial x^i\,\partial y^j}\Big)(j_1(s)(x))$$

($j = 1, \ldots, m$). The first of these equations holds according to (3.19). By (3.18) and (6.5), we have

$$\frac{\partial u_j^1}{\partial t}(x) = -\frac{\partial H}{\partial y^j}(j_1(s)(x)) - \sum_{i=2}^{n}\frac{\partial u_j^i}{\partial x^i}(x)$$

$$= -\Big(\frac{\partial H}{\partial y^j} + \sum_{i=2}^{n}\frac{\partial^2 f^i}{\partial x^i\,\partial y^j} + \sum_{\substack{1<i\leq n \\ 1\leq l\leq m}}\frac{\partial^2 f^i}{\partial y^j\,\partial y^l}\frac{\partial u^l}{\partial x^i}\Big)(j_1(s)(x))$$

from which the second equation of (6.10) follows according to (3.19).

LEMMA 6.3. – *Let $x_0 \in \pi W$ and let ψ_t be the flow generated by η defined on a neighborhood $\mathcal{V}$ of $j_1(s)(x_0)$ in N. Let N_0 be the submanifold of $\mathcal{V}$ defined by the equations $t = t(x_0)$ and*

$$p_j^1 = p_j^1(j_1(s)(x_0)) \ , \ j = 1, \ldots, m.$$

Then η is nowhere tangent to N_0. If N_1 is the submanifold of N swept out by N_0 under the flow ψ_t, then

i) $d\Theta_{|N_1} = 0$;

ii) $j_1(s)(x) \in N_1$, *for all x in a neighborhood of x_0* ;

iii) $\pi_0 : N_1 \to Y$ *is a diffeomorphism of some neighborhood of $j_1(s)(x_0)$ onto some neighborhood $\mathcal{U}$ of $s(x_0)$.*

Proof. – i) Since $\langle \eta, dt\rangle = 1$ and

$$dt_{|N_0} = dp^1_{j\,|N_0} = 0 \ ,$$

we see that η is nowhere tangent to N_0 and that $\iota^* d\Theta|_{N_0} = 0$, by (6.6). If $p \in N_1$, the tangent space $T_p(N_1)$ is spanned by

$$\psi_{t*}((T_{\psi_{-t}(p)}(N_0)) \quad \text{and } \eta(p) ,$$

where $\psi_{-t}(p) \in N_0$. It suffices to show that

$$\langle \eta_1 \wedge \ldots \wedge \eta_{n+1} , d\Theta \rangle = 0$$

and

$$\langle \eta \wedge \eta_2 \wedge \ldots \wedge \eta_{n+1} , d\Theta \rangle = 0$$

if $\eta_i = \psi_{t*} \xi_i$, with

$$\xi_i \in T_{\psi_{-t}(p)}(N_0) \quad , \quad i = 1, 2, \ldots, n+1 .$$

Since $\mathcal{L}_\eta \iota^* d\Theta = 0$, we have $\psi_t^* \iota^* d\Theta = \iota^* d\Theta$,

$$\begin{aligned} \langle \eta_1 \wedge \ldots \wedge \eta_{n+1} , d\Theta \rangle &= \langle \xi_1 \wedge \ldots \wedge \xi_{n+1} , \psi_t^* \iota^* d\Theta \rangle \\ &= \langle \xi_1 \wedge \ldots \wedge \xi_{n+1} , d\Theta \rangle \\ &= 0 , \end{aligned}$$

and

$$\langle \eta \wedge \eta_2 \wedge \ldots \wedge \eta_{n+1} , d\Theta \rangle = \langle \eta_2 \wedge \ldots \wedge \eta_{n+1} , \eta \lrcorner \iota^* d\Theta \rangle = 0 .$$

ii) From Lemma 6.2, it follows that

$$\psi_{t-t_0} j_1(s)(t_0 , x') = j_1(s)(t , x')$$

where $t_0 = t(x_0)$, for all $x = (t , x')$ in a neighborhood of x_0.

iii) It suffices to show that

$$\pi_{0*} : T_{j_1(s)(x_0)}(N_1) \to T_{s(x_0)}(Y) \tag{6.11}$$

is an isomorphism. Let $\mathcal{V}_0$ be the submanifold of $\pi_0 \mathcal{V}$ defined by the equation $t = t(x_0)$ and let $\mathcal{V}_1$ be the submanifold of $\pi_0 \mathcal{V}$ defined by $x' = x'(x_0)$ and $y^j = y^j(s(x_0))$, $j = 1, \ldots, m$. Then

$$\pi_{0*} : T_{j_1(s)(x_0)}(N_0) \to T_{s(x_0)}(\mathcal{V}_0)$$

is an isomorphism ; now

$$T_{j_1(s)(x_0)}(N_1)$$

is spanned by

$$T_{j_1(s)(x_0)}(N_0) \quad \text{and} \quad \eta(j_1(s)(x_0)).$$

Since

$$T_{s(x_0)}(Y) = T_{s(x_0)}(\mathcal{V}_0) \oplus T_{s(x_0)}(\mathcal{V}_1)$$

and

$$\pi_{0*}\, \eta(j_1(s)(x_0)) = \left(s_* \frac{\partial}{\partial t}\right)(s(x_0)) \, ,$$

it follows that (6.11) is an isomorphism.

If we define $w = \pi_0^{-1} : \mathcal{U} \to N_1$, then the initial condition (6.2) will be satisfied by ii) in a neighborhood of $s(x_0)$, and equation (6.1) follows from i). We have thus proved :

THEOREM 6.1. – *Assume that the map* $\sigma(L) : J_1(Y) \to T \otimes_Y V^*(Y)$ *is an immersion. Let s be an extremal. For any* $x_0 \in X$, *there exists a geodesic field w on a neighborhood of* $s(x_0)$ *in which s is embedded. In particular, if* L *is positive definite, then s, when restricted to a neighborhood of* x_0, *is a weak local minimum.*

7. The second variation.

Let A be a compact submanifold of X, with smooth boundary ∂A of dimension $n - 1$, and let σ be a section of Y over ∂A ; consider the set $\Sigma = \Sigma(A ; \sigma)$ of all sections of Y over A with $s_{|\partial A} = \sigma$. We shall think of Σ as an "infinite dimensional manifold". By the tangent space of Σ at the section s, we mean the vector space of all sections ν of $V_s(Y)$ over A with $\nu_{|\partial A} = 0$. We denote this space by $T\Sigma_s$.

A k-parameter variation of s is a function

$$\alpha : U \to \Sigma$$

defined on a neighborhood of 0 in $\mathbf{R}^k$ such that

i) $\alpha(0) = s$;

ii) the map

$$U \times A \to J_1(Y)$$

$$(t, x) \mapsto j_1(\alpha(t))(x)$$

is differentiable.

If α is a one-parameter variation of s, considered as a path in Σ, its tangent vector is the vector field ν along s given by

$$\nu(x) = \frac{d\alpha}{dt}(0)(x)$$

and is called the variation vector field of α.

Continuing the analogy with the finite dimensional case, if F is a real-valued function on Σ, we attempt to define

$$dF[s] : T\Sigma_s \to \mathbf{R}$$

by

$$\langle \nu , dF[s] \rangle = (\nu \cdot F)[s] = \frac{d}{dt} F[\alpha(t)]_{|t=0}$$

for $\nu \in T\Sigma_s$, where $\alpha(t)$ is a one-parameter variation whose variation vector field is ν. We shall say that s is a critical section for F if $dF[s] = 0$.

If we consider $F = I_A$, according to § 2, dI_A is well-defined, and s is a critical section for I_A if and only if it is an extremal for I_A. Indeed, from the first-variation formula

$$(7.1)\quad \int_A (\mathcal{L}[s]\,\nu)\,\omega = \int_A \langle \nu , \mathcal{E}[s]\,\omega \rangle + \int_{\partial A} (\sigma(\mathcal{L}[s])\,\nu) \lrcorner\, \omega$$

for $\nu \in C^\infty(V_s(Y))$, we deduce

$$\langle \nu , dI_A \rangle = \int_A (\mathcal{L}[s]\,\nu)\,\omega = \int_A \langle \nu , \mathcal{E}[s]\,\omega \rangle$$

for $\nu \in T\Sigma_s$.

We now wish to define in a similar fashion a bilinear map, the Hessian of I_A at s

$$H(I_A)[s] : T\Sigma_s \times T\Sigma_s \to \mathbf{R}$$

when s is an extremal. Given ν, $w \in T\Sigma_s$, choose sections ξ, η of V(Y) over Y of compact support such that

$$\xi_{|\pi^{-1}(\partial A)} = \eta_{|\pi^{-1}(\partial A)} = 0$$

and $\xi \circ s = \nu$, $\eta \circ s = w$. We think of ξ and η as vector fields on Σ. Then $\langle \eta , dI_A \rangle$ is a function on Σ and

$$H(I_A)[s](v, w) = (v \cdot \langle \eta , dI_A \rangle)[s] \cdot$$

Choose a 2-parameter variation $\alpha : U \to \Sigma$ defined on a neighborhood U of $(0, 0)$ in $\mathbf{R}^2$ such that

$$\alpha(0, 0) = s ,$$

$$\frac{\partial \alpha}{\partial t_1}(0, 0) = \xi(\alpha(0, 0)) = v ,$$

$$\frac{\partial \alpha}{\partial t_2}(t_1, 0) = \eta(\alpha(t_1, 0)) ,$$

if (t_1, t_2) are the coordinates on $\mathbf{R}^2$; for example, if φ_t, ψ_t are the flows on Y corresponding to ξ, η respectively, we could take

$$\alpha(t_1, t_2) = \psi_{t_2} \circ \varphi_{t_1} \circ s .$$

Then

$$H(I_A)[s](v, w) = \frac{\partial}{\partial t_1} \langle \eta , dI_A \rangle [\alpha(t_1, 0)]|_{t_1=0}$$

$$= \frac{\partial^2}{\partial t_1 \partial t_2} I_A[\alpha(t_1, t_2)]|_{t_1=t_2=0} \cdot$$

We now verify that $H(I_A)[s]$ is well-defined, that is, $H(I_A)[s](v, w)$ depends only on v, w and s.

If ξ is a section of V(Y) over Y, we denote, as usual, by ξ_1 the vertical vector field on $J_1(Y)$ induced by ξ. Then for any section s of Y

$$\xi_1 \circ j_1(s) = j_1(\xi \circ s) ,$$

where we have identified $V(J_1(Y))$ and $J_1(V(Y))$ according to (1.4) ; if $\mathfrak{L}[s]$ is the first-order linear differential operator on $V_s(Y)$ associated with the Lagrangian L defined in § 2, we have

$$\mathfrak{L}[s](\xi \circ s) = \langle \xi_1 \circ j_1(s), (d_{J_1(Y)/X} L) \circ j_1(s) \rangle$$

$$= (\xi_1 \cdot L) \cdot (j_1(s)) .$$

Define a Lagrangian L'_ξ on $J_1(Y)$ by

$$L'_\xi = \xi_1 \cdot L ;$$

if s is a section of Y, let $\mathfrak{L}'_\xi[s]$ be the first-order differential operator on $V_s(Y)$ associated with the Lagrangian L'_ξ.

LEMMA 7.1. – *If ξ, η are sections of* V(Y) *over* Y, *we have*

$$\mathcal{R}'_\eta[s](\xi \circ s) - \mathcal{R}'_\xi[s](\eta \circ s) = \mathcal{R}[s]([\xi, \eta] \circ s) \tag{7.2}$$

for all sections s *of* Y.

Proof. – By (1.11), we have

$$\begin{aligned} \mathcal{R}'_\eta[s](\xi \circ s) &= \langle \xi_1 , d_{J_1(Y)/X} L'_\eta \rangle (j_1(s)) \\ &= (\xi_1 \cdot (\eta_1 L))(j_1(s)) . \end{aligned}$$

Thus the left-hand side of (7.2) is

$$(\xi_1 \cdot (\eta_1 \cdot L) - \eta_1 \cdot (\xi_1 \cdot L))(j_1(s)) = ([\xi_1 , \eta_1] \cdot L)(j_1(s)) .$$

Since $[\xi_1 , \eta_1] = [\xi , \eta]_1$, we obtain (7.2).

LEMMA 7.2. – *Let ξ, η be sections of* V(Y) *over* Y *and let $\alpha(t_1 , t_2)$ be a 2-parameter family of sections of* Y *such that*

$$\alpha(0, 0) = s$$

$$\frac{\partial \alpha}{\partial t_1}(0, 0) = \xi \circ s$$

$$\frac{\partial \alpha}{\partial t_2}(t_1 , 0) = \eta(\alpha(t_1 , 0)) .$$

Then

$$\frac{\partial^2}{\partial t_1 \, \partial t_2} I_A[\alpha(t_1 , t_2)]_{|t_1 = t_2 = 0} = \int_A (\mathcal{R}'_\eta[s](\xi \circ s))\, \omega . \tag{7.3}$$

Proof. – According to the results of § 2,

$$\frac{\partial}{\partial t_2} I_A[\alpha(t_1 , t_2)]_{|t_2 = 0} = \int_A \left(\mathcal{R}[\alpha(t_1 , 0)] \frac{\partial \alpha}{\partial t_2}(t_1 , 0)\right) \omega .$$

Hence

$$\begin{aligned} \frac{\partial^2}{\partial t_1 \, \partial t_2} I_A[\alpha(t_1 , t_2)]_{|t_1 = t_2 = 0} &= \frac{\partial}{\partial t_1} \int_A (\mathcal{R}[\alpha(t_1 , 0)](\eta \circ \alpha(t_1 , 0)))\, \omega_{|t_1 = 0} \\ &= \int_A (\mathcal{R}'_\eta[s](\xi \circ s))\, \omega . \end{aligned}$$

PROPOSITION 7.1. – *The Hessian* $H(I_A)[s]$ *is a well-defined symmetric bilinear function* $T\Sigma_s \times T\Sigma_s \to \mathbf{R}$.

Proof. – If ξ, η are sections of $V(Y)$ over Y with $\xi \circ s, \eta \circ s \in T\Sigma_s$, then $[\xi, \eta] \circ s \in T\Sigma_s$, and

$$\int_A (\mathcal{E}[s]([\xi, \eta] \circ s))\, \omega = \langle [\xi, \eta] \circ s, dI_A \rangle = 0$$

since s is an extremal. Lemma 7.1 implies that

$$\int_A (\mathcal{E}'_\eta[s](\xi \circ s))\, \omega = \int_A (\mathcal{E}'_\xi[s](\eta \circ s))\, \omega$$

which shows, by Lemma 7.2, that the Hessian $H(I_A)[s]$ depends only on $\xi \circ s$ and $\eta \circ s$, is bilinear and symmetric.

The diagonal terms $H(I_A)[s](\nu, \nu)$ can be described in terms of one-parameter variations. If ξ is a section of $V(Y)$ over Y satisfying $\xi \circ s = \nu$ and $\xi_{|\pi^{-1}(\partial A)} = 0$, then

$$H(I_A)[s](\nu, \nu) = \frac{d^2}{dt^2} I_A[\alpha(t)]_{|t=0} \,,$$

where α is a one-parameter variation of s whose variation vector fields are

$$\frac{d\alpha}{dt}(t) = \xi(\alpha(t)) \,.$$

Indeed, define a 2-parameter variation α' by

$$\alpha'(t_1, t_2) = \alpha(t_1 + t_2) \ ;$$

then one easily verifies that

$$\frac{\partial \alpha'}{\partial t_1}(0, 0) = \frac{d\alpha}{dt}(0) = \nu \,,$$

$$\frac{\partial \alpha'}{\partial t_2}(t_1, 0) = \frac{d\alpha}{dt}(t_1) = \xi(\alpha(t_1)) \,,$$

and

$$\frac{\partial^2}{\partial t_1\, \partial t_2} I_A[\alpha'(t_1, t_2)]_{|t_1 = t_2 = 0} = \frac{d^2}{dt^2} I_A[\alpha(t)]_{|t=0} \,.$$

We thus obtain :

PROPOSITION 7.2. – *If s is a local minimum, the Hessian* $\mathrm{H}(\mathrm{I}_{\mathrm{A}})\,[s]$ *is positive semi-definite.*

Proof. – The inequality

$$\mathrm{I}_{\mathrm{A}}[\alpha(t)] \geqslant \mathrm{I}_{\mathrm{A}}[s] = \mathrm{I}_{\mathrm{A}}[\alpha(0)]$$

holds for all t in a neighborhood of 0, if $\alpha(t)$ is a one-parameter variation of s ; hence

$$\frac{d^2}{dt^2}\,\mathrm{I}_{\mathrm{A}}[\alpha(t)]|_{t=0} \geqslant 0 \cdot$$

We now discuss the Jacobi equation. If ξ is a section of $\mathrm{V}(\mathrm{Y})$ over Y, let

$$\mathcal{E}'_{\xi}[s] : \mathrm{C}^{\infty}(\Lambda^n \mathrm{T}^*) \to \mathrm{C}^{\infty}(\mathrm{V}_s^*(\mathrm{Y}) \otimes \Lambda^n \mathrm{T}^*)$$

be the adjoint of $\mathcal{L}'_{\xi}[s]$. We claim that $\mathcal{E}'_{\xi}[s]\,\omega$ depends only on the values of ξ along s. Let f be a real-valued function on Y satisfying $f \circ s = 1$. We must show that

$$\mathcal{E}'_{\xi}[s]\,\omega = \mathcal{E}'_{f\xi}[s]\,\omega \cdot \tag{7.4}$$

In fact, we shall prove that

$$\int_{\mathrm{A}} \langle w\,, \mathcal{E}'_{\xi}[s]\,\omega\rangle = \int_{\mathrm{A}} \langle w\,, \mathcal{E}'_{f\xi}[s]\,\omega\rangle \tag{7.5}$$

for all $w \in \mathrm{T}\Sigma_s$; this equation implies (7.4) on A and so (7.4) holds everywhere.

Now if $w \in \mathrm{T}\Sigma_s$, let η be a section of $\mathrm{V}(\mathrm{Y})$ over Y satisfying $\eta \circ s = w$. By Lemma 7.1 and Stokes' theorem

$$\int_{\mathrm{A}} \langle w\,, \mathcal{E}'_{\xi}[s]\,\omega\rangle = \int_{\mathrm{A}} (\mathcal{L}'_{\xi}[s]\,w)\,\omega$$

$$= \int_{\mathrm{A}} (\mathcal{L}'_{\eta}[s]\,(\xi \circ s))\,\omega + \int_{\mathrm{A}} (\mathcal{L}[s]\,([\xi\,,\eta] \circ s))\,\omega\ .$$

Hence (7.5) is equivalent to

$$\int_A (\mathcal{L}[s]([\xi,\eta]\circ s))\,\omega = \int_A (\mathcal{L}[s]([f\xi,\eta]\circ s))\,\omega\ .$$

To verify (7.5), it is therefore enough to show that

$$\int_A (\mathcal{L}[s](((\eta\cdot f)\,\xi)\circ s))\,\omega = 0\ .$$

This is indeed the case, since s is an extremal and because

$$((\eta\cdot f)\,\xi)\circ s\in \mathrm{T}\Sigma_s\,.$$

We have thus constructed a second-order linear differential operator

$$\mathcal{J}[s] : \mathrm{C}^\infty(\mathrm{V}_s(\mathrm{Y})) \to \mathrm{C}^\infty(\mathrm{V}_s^*(\mathrm{Y})\otimes\Lambda^n\,\mathrm{T}^*)$$

sending ν into $\mathcal{E}'_\xi[s]\,\omega$, where ξ is any section of $\mathrm{V(Y)}$ over Y which extends ν, i.e. $\xi\circ s=\nu$. The equation

$$\mathcal{J}[s]\,\nu = 0 \tag{7.6}$$

is known as *Jacobi's equation,* and a solution of this equation is called a *Jacobi field* along the extremal s.

From the above argument, we deduce :

THEOREM 7.1. – *The Hessian of* I_A *at the extremal* s *is given by*

$$\mathrm{H}(\mathrm{I}_A)\,[s]\,(\nu\,,w) = \int_A \langle \nu\,,\mathcal{J}[s]\,w\rangle \tag{7.7}$$

for all ν, $w\in \mathrm{T}\Sigma_s$.

Consider the integral $\mathrm{H}(\mathrm{I}_A)\,[s]\,(\nu\,,\nu)$ for $\nu\in\mathrm{T}\Sigma_s$. A section $\nu\in\mathrm{T}\Sigma_s$ is an extremal for this integral if

$$\frac{d}{dt}\,\mathrm{H}(\mathrm{I}_A)\,[s]\,(\nu+tw\,,\nu+tw)_{|t=0} = 0$$

for all $w\in\mathrm{T}\Sigma_s$. According to Theorem 7.1,

$$\mathrm{H}(\mathrm{I}_A)\,[s]\,(\nu+tw\,,\nu+tw) = \int_A \langle\nu\,,\mathcal{J}[s]\,\nu\rangle + 2t\int_A\langle w\,,\mathcal{J}[s]\,\nu\rangle + t^2\int_A\langle w\,,\mathcal{J}[s]\,w\rangle\,.$$

Hence the above condition on ν is

$$\int_{A} \langle w , \mathcal{J}[s] \nu \rangle = 0 \quad \text{for all } w \in T\Sigma_s ,$$

and so the Euler-Lagrange equation for this integral is the Jacobi equation on $\mathring{A}$.

The Jacobi equation (7.6) can also be obtained as the equation of variation of the Euler-Lagrange equation for I_A. In fact, let $\alpha(t)$ be a one-parameter family of sections of Y, with $\alpha(0) = s$. Then

$$\mathcal{E}[\alpha(t)]\, \omega = \beta(t) \otimes \omega$$

where $\beta(t)$ is a section of $V^*_{\alpha(t)}(Y)$ and $\beta(0) = 0$ since s is an extremal.

Let us now make the following digression. If E is a vector bundle over Y, the zero section 0 of E gives us a splitting $\lambda : V_{0(y)} \to E_y$ of the exact sequence

$$0 \to E_y \to V_{0(y)}(E) \to V_y(Y) \to 0$$

if $y \in Y$. Furthermore, let $\gamma(t)$, $w(t)$ be one-parameter families of sections of E, E* respectively over X whose projections onto Y give us the same one-parameter family $s(t)$ of sections of Y. If $w(0) = 0$, then

$$\frac{dw(t)}{dt}(x)_{|t=0}$$

belongs to $V_{0(s(0)(x))}(E)$ and

$$\frac{d}{dt}\langle \gamma(t), w(t) \rangle_{|t=0} = \left\langle \gamma(0), \lambda\left(\frac{dw(t)}{dt}_{|t=0}\right)\right\rangle .$$

We apply these remarks to $E = V^*(Y)$ and $w(t) = \beta(t)$. In fact,

$$\frac{d}{dt}\beta(t)(x)_{|t=0}$$

belongs to $V_{0(s(x))}(V^*(Y))$ and its projection in $V_{s(x)}(Y)$ is

$$\nu = \frac{d\alpha(t)}{dt}_{|t=0}$$

Thus

$$\lambda\left(\frac{d}{dt}\beta(t)_{|t=0}\right)$$

is a section of $V_s^*(Y)$ which clearly depends only on s and ν ; we claim that

$$(7.8)\quad \mathcal{J}[s]\,\nu = \lambda\left(\frac{d}{dt}\,\mathcal{E}[\alpha(t)]\,\omega_{|t=0}\right) = \lambda\left(\frac{d}{dt}\beta(t)_{|t=0}\right)\otimes\omega\,.$$

To verify this formula, we may assume that $\alpha_t = \varphi_t \circ s$, where φ_t is a one-parameter family of diffeomorphisms of Y whose infinitesimal generator ξ is a section of V(Y) over Y satisfying $\xi_{|\pi^{-1}(\partial A)} = 0$. If η is another section of V(Y) over Y satisfying the same condition as ξ and generating a flow ψ_t on Y, then

$$\frac{\partial}{\partial t_2} I_A[\psi_{t_2}\circ\varphi_{t_1}\circ s]_{|t_2=0} = \int_A (\mathcal{E}[\varphi_{t_1}\circ s]\,(\eta\circ\varphi_{t_1}\circ s))\,\omega$$

$$= \int_A \langle \eta\circ\varphi_{t_1}\circ s\,,\,\mathcal{E}[\varphi_{t_1}\circ s]\,\omega\rangle\,.$$

Hence

$$\frac{\partial^2}{\partial t_1\,\partial t_2} I_A[\psi_{t_2}\circ\varphi_{t_1}\circ s]_{|t_1=t_2=0}$$

$$= \int_A \left\langle \eta\circ s\,,\,\lambda\left(\frac{d}{dt_1}\,\mathcal{E}[\varphi_{t_1}\circ s]\,\omega_{|t_1=0}\right)\right\rangle$$

so that by Theorem 7.1

$$\int_A \left\langle w\,,\,\lambda\left(\frac{d}{dt}\,\mathcal{E}[\varphi_t\circ s]\,\omega_{|t=0}\right)\right\rangle = \int_A \langle w\,,\,\mathcal{J}[s]\,(\xi\circ s)\rangle$$

holds for all $w \in T\Sigma_s$ implying (7.8) on $\mathring{A}$ and hence everywhere.

The null-space of the Hessian $H(I_A)[s]$ is the subspace of $T\Sigma_s$ consisting of those elements ν of $T\Sigma_s$ satisfying

$$H(I_A)[s](\nu\,,w) = 0$$

for all $w \in T\Sigma_s$. The nullity of $H(I_A)[s]$ is the dimension of this space. We say that $H(I_A)[s]$ is degenerate if the null-space is non-zero.

PROPOSITION 7.3. – *A vector field* $\nu \in T\Sigma_s$ *along* s *belongs to the null space of* $H(I_A)[s]$ *if and only if* ν *is a Jacobi field.*

Proof. – If $v \in T\Sigma_s$ is a Jacobi field, according to (7.7), v belongs to the null-space of $H(I_A)[s]$. Conversely, let $v \in T\Sigma_s$ belong to the null-space of $H(I_A)[s]$. We have $\mathcal{J}[s]\, v = \beta \otimes \omega$, where β is a section of $V_s^*(Y)$. If f is a real-valued function on X whose support is contained in $\mathring{A}$, and w is any section of $V_s(Y)$, then $fw \in T\Sigma_s$ and therefore

$$H(I_A)[s](fw, v) = \int_A f < w, \beta > \omega = 0 .$$

Since this expression vanishes for all such functions f and sections w, we have $\beta = 0$ on $\mathring{A}$ and hence v is a Jacobi field.

PROPOSITION 7.4. – *Let $\alpha(t)$ be a one-parameter family of extremals, then*

$$\frac{d\alpha}{dt}\Big|_{t=0}$$

is a Jacobi field, along $\alpha(0)$.

Proof. – We have $\mathcal{E}[\alpha(t)]\,\omega = 0$ for all t. From (7.8), it follows that

$$\frac{d\alpha}{dt}\Big|_{t=0}$$

is a Jacobi field.

We now describe the above constructions in terms of coordinates. Let (x^i, y^j, y_i^j) be a local coordinate system on an open subset W of $J_1(Y)$ of the type considered in § 3, for which

$$\omega = dx^1 \wedge \ldots \wedge dx^n$$

on $U = \pi W$. Suppose that $j_1(s)(U) \subset W$ and that $A \subset U$. If $\tilde{\xi} \in V_p(J_1(Y))$, with $p \in W$, then

$$\tilde{\xi} = \sum_j v^j \frac{\partial}{\partial y^j} + \sum_{i,j} v_i^j \frac{\partial}{\partial y_i^j} .$$

Hence (x^i, v^j, v_i^j) are local coordinates on $V(J_1(Y))$. If

$$\xi = \sum_j \frac{\partial}{\partial y^j}$$

is a vertical vector field on $\pi_0 W \subset Y$, then the vector field ξ_1 on W is given by

$$\xi_1(p) = \sum_j \xi^j(y) \frac{\partial}{\partial y^j} + \sum_{i,j} \left(\frac{\partial \xi^j}{\partial x^i}(y) + \sum_l \frac{\partial \xi^j}{\partial y^l}(y) y_i^l\right) \frac{\partial}{\partial y_i^j}$$

for $p \in W$, if $\pi_0(p) = y$ and p has coordinates (x^i, y^j, y_i^j). If

$$\eta = \sum_j \eta^j \frac{\partial}{\partial y^j}$$

is another vertical vector field on $\pi_0 W$, then, setting $u = j_1(s)$,

$$\begin{aligned}
&\xi_1 \cdot (\eta_1 \cdot L)(u) \\
&= \sum_{j,l} \left(\frac{\partial^2 L}{\partial y^j \partial y^l} \circ u\right) \xi^j \eta^l \\
&+ \sum_{i,j,l} \left(\frac{\partial^2 L}{\partial y^l \partial y_i^j} \circ u\right) \left(\xi^l \frac{\partial(\eta \circ s)^j}{\partial x^i} + \eta^l \frac{\partial(\xi \circ s)^j}{\partial x^i}\right) \\
&+ \sum_{\substack{i,j \\ k,l}} \left(\frac{\partial^2 L}{\partial y_i^j \partial y_k^l} \circ u\right) \frac{\partial(\xi \circ s)^j}{\partial x^i} \frac{\partial(\eta \circ s)^l}{\partial x^k} + \sum_{j,l} \left(\frac{\partial L}{\partial y^j} \circ u\right) \frac{\partial \eta^j}{\partial y^l} \xi^l \\
&+ \sum_{i,j,l} \frac{\partial L}{\partial y_i^j} \left(\frac{\partial^2 \eta^j}{\partial y^l \partial x^i} + \sum_r \frac{\partial^2 \eta^j}{\partial y^l \partial y^r} \frac{\partial s^r}{\partial x^i}\right) \xi^l \\
&+ \sum_{i,j,l} \frac{\partial L}{\partial y_i^j} \frac{\partial \eta^j}{\partial y^l} \left(\frac{\partial \xi^l}{\partial x^i} + \sum_r \frac{\partial \xi^l}{\partial y^r} y_i^r\right).
\end{aligned}$$

If $v = \xi \circ s$, $w = \eta \circ s$ vanish on ∂A, then the integral over A of the sum of the last three terms of the right-hand side of the above equation vanishes by the Euler-Lagrange equation, and hence

$$\begin{aligned}
H(I_A)[s](v, w) = \int_A \Bigg\{ &\sum_{j,l} \left(\frac{\partial^2 L}{\partial y^j \partial y^l} \circ u\right) v^j w^l \\
&+ \sum_{j,l,k} \left(\frac{\partial^2 L}{\partial y^j \partial y_k^l} \circ u\right) \left(v^j \frac{\partial w^l}{\partial x^k} + w^j \frac{\partial v^l}{\partial x^k}\right) \\
&+ \sum_{\substack{i,k \\ j,l}} \left(\frac{\partial^2 L}{\partial y_i^j \partial y_k^l} \circ u\right) \frac{\partial v^j}{\partial x^i} \frac{\partial w^l}{\partial x^k} \Bigg\} \omega
\end{aligned}$$

for $v, w \in T\Sigma_s$. Integrating by parts, we obtain

$$\mathcal{J}[s]v = -\sum_j \Bigg\{ \sum_{\substack{i,k\\l}} \left(\frac{\partial^2 L}{\partial y_i^j \, \partial y_k^l} \circ u\right) \frac{\partial^2 v^l}{\partial x^i \, \partial x^k}$$

$$+ \sum_{k,l} \left(\frac{\partial^2 L}{\partial y^l \, \partial y_k^j} \circ u - \frac{\partial^2 L}{\partial y^j \, \partial y_k^l} \circ u + \sum_i \frac{\partial}{\partial x^i}\left(\frac{\partial^2 L}{\partial y_i^j \, \partial y_k^l} \circ u\right)\right) \frac{\partial w^l}{\partial x^k}$$

$$+ \sum_l \left(\sum_i \frac{\partial}{\partial x^i}\left(\frac{\partial^2 L}{\partial y^l \, \partial y_i^j} \circ u\right) - \frac{\partial^2 L}{\partial y^j \, \partial y^l} \circ u\right) w^l \Bigg\} \, dy^j \otimes \omega .$$

Identifying $J_1(V_s(Y))$ with $V_{j_1(s)}(J_1(Y))$ we obtain coordinates (x^i, v^j, v_i^j) on $J_1(V_s(Y))$; in fact, if

$$\xi = \sum_j \xi^j \frac{\partial}{\partial y^j}$$

is a vertical vector field on Y, then

$$\begin{aligned} v_i^j(j_1(\xi \circ s)) &= v_i^j(\xi_1 \circ j_1(s)) \\ &= \frac{\partial \xi^j}{\partial x^i} \circ s + \sum_l \left(\frac{\partial \xi^j}{\partial x^l} \circ s\right) \frac{\partial s^l}{\partial x^i} \\ &= \frac{\partial (\xi \circ s)^j}{\partial x^i} \end{aligned}$$

so that this coordinate system is of the type considered in § 1 on jet bundles. Define the Lagrangian $\widetilde{L} : J_1(V_s(Y))_{|U} \to \mathbf{R}$ by

$$\widetilde{L}(x, v^j, v_i^j) = \frac{1}{2} \sum_{j,l} \left(\frac{\partial^2 L}{\partial y^j \, \partial y^l} \circ u\right) v^j v^l$$

$$(7.9) \qquad + \sum_{k,j,l} \left(\frac{\partial^2 L}{\partial y^j \, \partial y_k^l} \circ u\right) v^j v_k^l$$

$$+ \frac{1}{2} \sum_{\substack{i,k\\j,l}} \left(\frac{\partial^2 L}{\partial y_i^j \, \partial y_k^l} \circ u\right) v_i^j v_k^l .$$

Then for $\nu \in T\Sigma_s$

$$H(I_A)(\nu, \nu) = \int_A \widetilde{L}(j_1(\nu))\,\omega$$

and Jacobi's equation is simply the Euler-Lagrange equation for the integral on the right-hand side of the above equation and can be written in the form

$$\frac{\partial \widetilde{L}}{\partial \nu^j}(j_1(\nu)) - \sum_i \frac{\partial}{\partial x^i}\left(\frac{\partial \widetilde{L}}{\partial \nu_i^j}(j_1(\nu))\right) = 0 \quad (j = 1, \ldots, m) .$$

Jacobi's equation gives us a morphism of vector bundles

$$\Psi : J_2(V_s(Y)) \to V_s^*(Y) \otimes \Lambda^n T^*$$

sending $j_2(\nu)(x)$ into $(\mathcal{J}[s]\,\nu)(x)$ if ν is a section of $V_s(Y)$ over a neighborhood of $x \in X$ whose symbol is determined by

$$\sigma'(\Psi) : S^2 T^* \otimes V_s(Y) \to V_s^*(Y)$$

where $\sigma'(\Psi) \otimes \omega = \sigma(\Psi)$. Identifying an element of $S^2 T^*$ with a symmetric bilinear form on T according to § 5, $\sigma'(\Psi)$ determines a map

$$\sigma'(\Psi) : T \otimes T \to V_s^*(Y) \otimes V_s^*(Y) .$$

Now $(d^2_{J_1(Y)/Y} L)(j_1(s))$ also determines a map

$$(d^2_{J_1(Y)/Y} L)(j_1(s)) : T \otimes T \to V_s^*(Y) \otimes V_s^*(Y) .$$

PROPOSITION 7.5. – *The symbol $\sigma(\Psi)$ of Ψ is determined by*

$$\sigma'(\Psi) = -(d^2_{J_1(Y)/Y} L)(j_1(s)) .$$

If L *is regular, the differential operator $\mathcal{J}[s]$ is strongly elliptic.*

The proposition follows directly from the above local computation of Jacobi's equation.

We now generalize the above considerations to variations of the section $j_1(s)$ among sections of $J_1(Y)$ to derive the Hamiltonian form of Jacobi's equation.

We shall proceed in exactly the same manner as above. Let u be a section of $J_1(Y)$ which is an extremal, in the sense of (3.7). Let S be the set of all sections u' of $J_1(Y)$ over A with $\pi_0 u'_{|\partial A} = \pi_0 u_{|\partial A}$. By TS_u, we denote the vector space of all sections ξ of $V_u(J_1(Y))$ over A with $\pi_{0*}\xi_{|\partial A} = 0$. We shall consider k-parameter variations of u

$$\alpha : U \to S$$

defined on a neighborhood of 0 in $\mathbf{R}^k$ with $\alpha(0) = u$, and, if $k = 1$, the corresponding variation vector field

$$\xi = \frac{d\alpha}{dt}(0) \in TS_u .$$

Let u_t be a one-parameter family of sections of $J_1(Y)$ over A and let

$$\xi = \frac{du_t}{dt}_{|t=0}$$

be the variation vector field along u_0. Recall the first variation formula

$$\text{(7.10)} \quad \frac{d}{dt}\int_A u_t^*\Theta_{\,|t=0} = \int_A u_0^*(\xi \lrcorner d\Theta) + \int_{\partial A} u_0^*(\xi \lrcorner \Theta) .$$

According to (3.5) if ξ belongs to TS_u, then $\xi \lrcorner \Theta = 0$ on $u(\partial A)$. A section of $J_1(Y)$ is therefore an extremal on $\mathring{A}$ if and only if it is a critical section for the functional defined on S by the left-hand side of (3.6). We now define a bilinear map, the Hessian,

$$H_A[u] : TS_u \times TS_u \to \mathbf{R} .$$

Given $\xi, \eta \in TS_u$, choose sections $\tilde{\xi}, \tilde{\eta}$ of $V(J_1(Y))$ over $J_1(Y)$ such that

$$\pi_{0*}\tilde{\xi}_{|\pi^{-1}(\partial A)} = \pi_{0*}\tilde{\eta}_{|\pi^{-1}(\partial A)} = 0$$

and $\tilde{\xi} \circ u = \xi$, $\tilde{\eta} \circ u = \eta$. Choose a 2-parameter variation $\alpha(t_1, t_2)$ of u such that

$$\frac{\partial\alpha}{\partial t_1}(0, 0) = \tilde{\xi}(\alpha(0, 0)) = \xi$$

$$\frac{\partial\alpha}{\partial t_2}(t_1, 0) = \tilde{\eta}(\alpha(t_1, 0))$$

and set

$$\mathrm{H}_{\mathrm{A}}[u](\xi, \eta) = \frac{\partial^2}{\partial t_1 \, \partial t_2} \int_{\mathrm{A}} \alpha(t_1, t_2)^* \Theta \Big|_{t_1 = t_2 = 0}$$

We now verify that $\mathrm{H}_{\mathrm{A}}[u]$ is well-defined.

LEMMA 7.3. – *Let* $\tilde{\xi}$, $\tilde{\eta}$ *be sections of* $\mathrm{V}(\mathrm{J}_1(\mathrm{Y}))$ *over* Y *and let* $\alpha(t_1, t_2)$ *be a 2-parameter family of sections of* $\mathrm{J}_1(\mathrm{Y})$ *such that*

$$\alpha(0, 0) = u'$$

$$\frac{\partial \alpha}{\partial t_1}(0, 0) = \tilde{\xi} \circ u'$$

$$\frac{\partial \alpha}{\partial t_2}(t_1, 0) = \tilde{\eta}(\alpha(t_1, 0)) \,.$$

Then

$$\text{(7.11)} \quad \frac{\partial^2}{\partial t_1 \, \partial t_2} \alpha(t_1, t_2)^* \Theta \Big|_{t_1 = t_2 = 0} = u'^* \, \mathcal{L}_{\tilde{\xi}} \mathcal{L}_{\tilde{\eta}} \Theta$$

$$= u'^*(d(\tilde{\xi} \lrcorner (\mathcal{L}_{\tilde{\eta}} \Theta)) + \tilde{\xi} \lrcorner d(\mathcal{L}_{\tilde{\eta}} \Theta))$$

and

$$\text{(7.12)} \quad \frac{\partial^2}{\partial t_1 \, \partial t_2} \int_{\mathrm{A}} \alpha(t_1, t_2)^* \Theta \Big|_{t_1 = t_2 = 0} = \int_{\mathrm{A}} u'^* \mathcal{L}_{\tilde{\xi}} \mathcal{L}_{\tilde{\eta}} \Theta$$

$$= \int_{\mathrm{A}} u'^*(\tilde{\xi} \lrcorner d(\tilde{\eta} \lrcorner d\Theta)) + \int_{\partial \mathrm{A}} u'^*(\tilde{\xi} \lrcorner \mathcal{L}_{\tilde{\eta}} \Theta) \,.$$

Proof. – (7.11) follows from (1.2) ; by applying Stokes' theorem to (7.11), we obtain (7.12).

PROPOSITION 7.6. – *The Hessian* $\mathrm{H}_{\mathrm{A}}[u]$ *is a well-defined symmetric bilinear function* $\mathrm{TS}_u \times \mathrm{TS}_u \to \mathbf{R}$.

Proof. – If $\tilde{\xi}$, $\tilde{\eta}$ are sections of $\mathrm{V}(\mathrm{J}_1(\mathrm{Y}))$ over $\mathrm{J}_1(\mathrm{Y})$ with $\tilde{\xi} \circ u$, $\tilde{\eta} \circ u \in \mathrm{TS}_u$, then $[\tilde{\xi}, \tilde{\eta}] \circ u$ belongs to TS_u. Since

$$\mathcal{L}_{\tilde{\xi}} \mathcal{L}_{\tilde{\eta}} - \mathcal{L}_{\tilde{\eta}} \mathcal{L}_{\tilde{\xi}} = \mathcal{L}_{[\tilde{\xi}, \tilde{\eta}]} \,,$$

we have by Stokes' theorem

$$\int_A u^* \mathcal{L}_{\tilde{\xi}} \mathcal{L}_{\tilde{\eta}} \Theta = \int_A u^* \mathcal{L}_{\tilde{\eta}} \mathcal{L}_{\tilde{\xi}} \Theta + \int_A u^*([\tilde{\xi}, \tilde{\eta}] \lrcorner d\Theta)$$

$$+ \int_{\partial A} u^*([\tilde{\xi}, \tilde{\eta}] \lrcorner \Theta) .$$

The second integral on the right-hand side of this equation vanishes since u is an extremal, and the third integral vanishes because $[\tilde{\xi}, \tilde{\eta}] \circ u \in TS_u$. Since $\tilde{\xi} \circ u$, $\tilde{\eta} \circ u$ belong to TS_u, we have

$$u^*(\tilde{\xi} \lrcorner \mathcal{L}_{\tilde{\eta}} \Theta) = 0$$

on ∂A. Indeed, since $\tilde{\xi} \circ u \in TS_u$ we may, in computing the above expression on ∂A, assume that $\pi_{0*} \tilde{\xi} = 0$. Then

$$\tilde{\xi} \lrcorner \mathcal{L}_{\tilde{\eta}} \Theta = \mathcal{L}_{\tilde{\eta}}(\tilde{\xi} \lrcorner \Theta) + [\tilde{\xi}, \tilde{\eta}] \lrcorner \Theta .$$

Now $\tilde{\xi} \lrcorner \Theta = 0$ by (3.5) and the second term of the right-hand side of the above equation vanishes on ∂A since $[\tilde{\xi}, \tilde{\eta}] \circ u \in TS_u$. Hence

$$\text{(7.13)} \qquad H_A[u](\tilde{\xi} \circ u, \tilde{\eta} \circ u) = \int_A u^*(\tilde{\xi} \lrcorner d(\tilde{\eta} \lrcorner d\Theta))$$

and this expression depends only on $\tilde{\xi} \circ u$ and $\tilde{\eta} \circ u$, is bilinear and symmetric.

If $\tilde{\xi}$ is a section of $V(J_1(Y))$ over $J_1(Y)$ satisfying

$$\pi_{0*} \xi_{|\pi^{-1}(\partial A)} = 0 ,$$

then

$$H_A[u](\tilde{\xi} \circ u, \tilde{\xi} \circ u) = \frac{d^2}{dt^2} \int_A \alpha(t)^* \Theta_{|t=0}$$

where α is a one-parameter variation of u whose variation vector fields are

$$\frac{d\alpha}{dt}(t) = \tilde{\xi}(\alpha(t)) .$$

PROPOSITION 7.7. – *If ξ, η are sections of* $V(Y)$ *over* Y *satisfying*

$$\xi_{|\pi^{-1}(\partial A)} = \eta_{|\pi^{-1}(\partial A)} = 0 \ ,$$

and if a section s of Y *over* X *is an extremal, then*

$$H(I_A)[s].(\xi \circ s, \eta \circ s) = H_A[j_1(s)](\xi_1 \circ j_1(s), \eta_1 \circ j_1(s)) \ .$$

Proof. – Set $u = j_1(s)$ and observe that $j_1(\xi \circ s) = \xi_1 \circ u$ and $j_1(\eta \circ s) = \eta_1 \circ u$ belong to TS_u. If $\alpha(t_1, t_2)$ is a 2-parameter variation of s satisfying the conditions of Lemma 7.2, then $j_1(\alpha(t_1, t_2))$ is a 2-parameter variation of u satisfying

$$\frac{\partial}{\partial t_1} j_1(\alpha(t_1, t_2))_{|t_1 = t_2 = 0} = \xi_1 \circ u$$

$$\frac{\partial}{\partial t_2} j_1(\alpha(t_1, t_2))_{|t_2 = 0} = \eta_1(j_1(\alpha(t_1, 0))) \ ;$$

the proposition follows directly from the definitions of the Hessians.

If $\tilde{\xi}$, $\tilde{\eta}$ are sections of $T(J_1(Y))$ over $J_1(Y)$, then if γ is the unique section of $\wedge^n T$ over X such that $\langle \gamma, \omega \rangle = 1$, we have

$$\begin{aligned} u^*(\tilde{\eta} \lrcorner d(\tilde{\xi} \lrcorner d\Theta)) &= (u_* \gamma \lrcorner \tilde{\eta} \lrcorner d(\tilde{\xi} \lrcorner d\Theta))\, \omega \\ &= (-1)^n \langle \tilde{\eta} \circ u, u_* \gamma \lrcorner d(\tilde{\xi} \lrcorner d\Theta) \rangle\, \omega \ . \end{aligned} \tag{7.14}$$

We claim that $u_* \gamma \lrcorner d(\tilde{\xi} \lrcorner d\Theta)$ depends only on the values of $\tilde{\xi}$ along u. Let f be a real-valued function on $J_1(Y)$ satisfying $f \circ u = 1$. Then

$$\begin{aligned} u_* \gamma \lrcorner d(f \tilde{\xi} \lrcorner d\Theta) &= u_* \gamma \lrcorner f d(\tilde{\xi} \lrcorner d\Theta) + u_* \gamma \lrcorner (df \wedge (\tilde{\xi} \lrcorner d\Theta)) \\ &= u_* \gamma \lrcorner d(\tilde{\xi} \lrcorner d\Theta) + (\tilde{\xi} \lrcorner u_* \gamma \lrcorner d\Theta)\, df \ , \\ &= u_* \gamma \lrcorner d(\tilde{\xi} \lrcorner d\Theta) \end{aligned}$$

since $u^* df = 0$ and u satisfies (3.12).

THEOREM 7.2.–*Suppose that the map* $\sigma(L) : J_1(Y) \to T \otimes_Y V^*(Y)$ *is an immersion. Let* $\tilde{\xi}$ *be a section of* $V(J_1(Y))$ *over* $J_1(Y)$. *The equation*

$$u_* \gamma \lrcorner d(\tilde{\xi} \lrcorner d\Theta) = 0 \tag{7.15}$$

is equivalent to the pair of equations

$$\tilde{\xi} \circ u = \xi_1 \circ u$$

and

$$\mathcal{J}[s](\xi \circ s) = 0$$

where $u = j_1(s)$ *and* ξ *is any section of* $V(Y)$ *over* Y *satisfying* $\xi \circ s = \pi_{0*}(\tilde{\xi} \circ u)$.

Equation (7.15) is Hamilton's form of Jacobi's equation and is equivalent to

$$u^*(\tilde{\eta} \lrcorner d(\tilde{\xi} \lrcorner d\Theta)) = 0 \tag{7.16}$$

for all $\tilde{\eta} \in C^\infty(T(J_1(Y)))$. In fact (7.15) holds for all $\tilde{\eta} \in C^\infty(T(J_1(Y)))$ if and only if it holds for all $\tilde{\eta} \in C^\infty(V(J_1(Y)))$. Indeed, if ζ is a vector field on X, then $u_* \zeta$ is a vector field along u and

$$u^*(u_* \zeta \lrcorner d(\tilde{\xi} \lrcorner d\Theta)) = \zeta \lrcorner u^* d(\tilde{\xi} \lrcorner d\Theta)) = 0$$

since $u^* d(\tilde{\xi} \lrcorner d\Theta)$ is an $(n + 1)$-form on an n-dimensional manifold. If $\tilde{\eta}$ is a vector field along u, then $\pi_* \tilde{\eta}$ is a well-defined vector field on X and $\tilde{\eta}$ can be written as

$$\tilde{\eta} = (\tilde{\eta} - u_* \pi_* \tilde{\eta}) + u_* \pi_* \tilde{\eta}$$

where $\tilde{\eta} - u_* \pi_* \tilde{\eta}$ is a vertical vector field ; the result follows from the above remarks.

As a first step in the proof of Theorem 7.2, we prove

LEMMA 7.4. – *Assume that* $\sigma(L)$ *is an immersion and let* $\tilde{\xi}$ *be a section of* $V(J_1(Y))$ *over* $J_1(Y)$. *Then*

$$\tilde{\xi} \circ u = \xi_1 \circ u$$

for some section ξ *of* $V(Y)$ *over* Y *if and only if*

$$u^*(\tilde{\eta} \lrcorner d(\tilde{\xi} \lrcorner d\Theta)) = 0$$

for all sections $\tilde{\eta}$ *of* $T(J_1(Y)/Y)$.

Proof. – Let $\tilde{\xi}$, $\tilde{\eta}$ be vector fields on $J_1(Y)$. Then, since u is an extremal

$$
\begin{aligned}
u^*(\tilde{\eta} \lrcorner d(\tilde{\xi} \lrcorner d\Theta)) &= u^*(\tilde{\eta} \lrcorner \mathcal{L}_{\tilde{\xi}}\, d\Theta) \\
&= u^*(\mathcal{L}_{\tilde{\xi}}(\tilde{\eta} \lrcorner d\Theta)) - u^*([\tilde{\xi}, \tilde{\eta}] \lrcorner d\Theta) \qquad (7.17)\\
&= u^* \mathcal{L}_{\tilde{\xi}}(\tilde{\eta} \lrcorner d\Theta) .
\end{aligned}
$$

Let ξ be a vertical vector field on Y ; if $y \in Y$, it is the infinitesimal generator of a flow φ_t on some neighborhood of y. Then $\varphi_{1,t}$ is a flow on a neighborhood of $\pi_0^{-1}(y)$ whose infinitesimal generator is ξ_1 . If $\tilde{\eta} \in C^\infty(T(J_1((Y)/Y)$, then by (7.17) on some neighborhood of $\pi(y)$

$$
\begin{aligned}
u^*(\tilde{\eta} \lrcorner d(\xi_1 \lrcorner d\Theta)) &= u^* \mathcal{L}_{\xi_1}(\tilde{\eta} \lrcorner d\Theta) \\
&= \frac{d}{dt}(\varphi_{1,t} \circ u)^* (\tilde{\eta} \lrcorner d\Theta)_{|t=0} \\
&= \frac{d}{dt} j_1(\varphi_t \circ s)^* (\tilde{\eta} \lrcorner d\Theta)_{|t=0}
\end{aligned}
$$

by Lemma 3.1. Conversely, if $\tilde{\xi}$ is a section of $V(J_1(Y))$ satisfying (7.16) for all $\tilde{\eta} \in C^\infty(T(J_1(Y)/Y))$ and $p \in J_1(Y)$, let $\tilde{\varphi}_t, |t| < \varepsilon$, be the corresponding flow on a neighborhood of p. On a neighborhood U of $\pi(p)$, by (1.2), we have

$$
\frac{d}{dt}(\tilde{\varphi}_t \circ u)^* (\tilde{\eta} \lrcorner d\Theta)_{|t=0} = u^* \mathcal{L}_{\tilde{\xi}}(\tilde{\eta} \lrcorner d\Theta) = 0
$$

for all $\tilde{\eta} \in C^\infty(T(J_1(Y)/Y))$. Hence on U,

$$
(\tilde{\varphi}_t \circ u)^* (\tilde{\eta} \lrcorner d\Theta) = O(t^2)
$$

for all $\tilde{\eta} \in C^\infty(T(J_1(Y)/Y))$. If we examine the proof of Proposition 1.2 and Lemma 3.1 we see that this implies that

$$
\tilde{\varphi}_t \circ u - j_1(\pi_0(\tilde{\varphi}_t \circ u)) = O(t^2) .
$$

Let ξ be a vertical vector field on Y such that $\xi \circ s = \pi_{0*}(\tilde{\xi} \circ u)$. Then

$$
\xi \circ s = \frac{d}{dt} \pi_0(\tilde{\varphi}_t \circ u)_{|t=0}
$$

and

$$\begin{aligned}\xi_1 \circ u = j_1(\xi \circ s) &= \frac{d}{dt} j_1(\pi_0(\tilde{\varphi}_t \circ u))_{|t=0} \\ &= \frac{d}{dt} \tilde{\varphi}_t \circ u_{|t=0} \\ &= \tilde{\xi} \circ u\end{aligned}$$

on U.

To complete the proof of Theorem 7.2, it suffices to show that for any vertical vector field ξ on Y, the equation $\mathcal{J}[s](\xi \circ s) = 0$ is equivalent to (7.15) with $\tilde{\xi} = \xi_1$ and $u = j_1(s)$. Let φ_t be the flow generated by ξ on a neighborhood of $s(x)$, where x is any given point of X. Let B be any compact neighborhood of x such that φ_t is defined on $s(\mathrm{B})$. If w is a vertical vector field on Y along s whose support is contained in $\mathring{\mathrm{B}}$, extend w to a vertical vector field η on Y of compact support contained in $\pi^{-1}(\mathring{\mathrm{B}})$ and let ψ_t be the corresponding flow on Y. Then according to Lemmas 7.1, 7.2, and 7.3, and the above computations

$$\frac{\partial^2}{\partial t_1\, \partial t_2} \mathrm{I}_{\mathrm{B}}[\psi_{t_2} \circ \varphi_{t_1} \circ s]_{|t_1=t_2=0} = \int_{\mathrm{B}} \langle \eta \circ s, \mathcal{J}[s](\xi \circ s)\rangle$$

since $\eta \circ s$ and $[\xi, \eta] \circ s$ vanish on $\partial\mathrm{B}$, and

$$\begin{aligned}\frac{\partial^2}{\partial t_1\, \partial t_2} \int_{\mathrm{B}} j_1(\psi_{t_2} \circ \varphi_{t_1} \circ s)^* \Theta_{|t_1=t_2=0} &= \frac{\partial^2}{\partial t_1\, \partial t_2} \int_{\mathrm{B}} (\psi_{1,t_2} \circ \varphi_{1,t_1} \circ u)^* \Theta_{|t_1=t_2=0} \\ &= \int_{\mathrm{B}} u^*(\eta_1 \lrcorner\, d(\xi_1 \lrcorner\, d\Theta))\end{aligned}$$

since $\eta_1 \circ u$ and $[\xi_1, \eta_1] \circ u$ vanish on $\partial\mathrm{B}$. The above integral depends only on $\eta_1 \circ u$ and hence only on $\eta \circ s$. Therefore, for any vertical vector field η on Y such that the support of $\eta \circ s$ is contained in $\mathring{\mathrm{B}}$

$$\int_{\mathrm{B}} \langle \eta \circ s, \mathcal{J}[s](\xi \circ s)\rangle = \int_{\mathrm{B}} u^*(\eta_1 \lrcorner\, d(\xi_1 \lrcorner\, d\Theta)) .$$

If (7.16) holds for all $\widetilde{\eta} \in C^\infty(T(J_1(Y)))$ with $\widetilde{\xi} = \xi_1$, then $\mathcal{J}[s]\,(\xi \circ s) = 0$ on $\overset{\circ}{B}$ and hence everywhere, since x was arbitrary. Conversely, if $\xi \circ s$ satisfies Jacobi's equation, then by (7.14) and the above equation

$$\int_B \langle \widetilde{\eta} \circ u \,, u_* \gamma \,\lrcorner\, d(\xi_1 \,\lrcorner\, d\Theta) \rangle = 0 \tag{7.18}$$

if $\widetilde{\eta} = \eta_1$, for all vertical vector fields η on Y such that the support of $\eta \circ s$ is contained in $\overset{\circ}{B}$. Equation (7.18) holds if $\widetilde{\eta}$ is a section of $T(J_1(Y)/Y)$ by Lemma 7.4 and if $\widetilde{\eta} \circ u = u_* \zeta$, for any vector field ζ on X. Hence (7.18) holds for all vector fields $\widetilde{\eta}$ on $J_1(Y)$ such that the support of $\widetilde{\eta} \circ u$ is contained in $\overset{\circ}{B}$. Thus Jacobi's equation holds on $\overset{\circ}{B}$ and hence everywhere.

If (x^i, y^j, y^j_i) is a local coordinate system on an open subset W of $J_1(Y)$ of the type considered in § 3 for which

$$\omega = dx^1 \wedge \ldots \wedge dx^n$$

on $U = \pi W$, such that $\sigma(L)$ is a diffeomorphism of W onto $\sigma(L)\,W$, we have coordinates (x^i, y^j, p^i_j) on W, where p^i_j is defined by (3.17), and a Hamiltonian H defined by (3.16). Let s be an extremal such that $j_1(s)(U) \subset W$. If $\widetilde{\xi}$ is an element of $V_{j_1(s)(x)}(J_1(Y))$, with $x \in U$,

$$\widetilde{\xi} = \sum_j v^j \frac{\partial}{\partial y^j} + \sum_{i,j} \beta^i_j \frac{\partial}{\partial p^i_j} \,. \tag{7.19}$$

Hence (v^j, β^i_j) are local coordinates on $J_1(V_s(Y))_{|U}$. In fact

$$\beta^i_j = \frac{\partial \widetilde{L}}{\partial v^j_i} \,,$$

where $\widetilde{L}$ is given by (7.9). Define the Hamiltonian $\widetilde{H} : J_1(V_s(Y))_{|U} \to \mathbf{R}$ by the formula

$$\widetilde{H} = \sum_{i,j} \beta^i_j v^j_i - \widetilde{L} \,.$$

Then

$$\widetilde{H}(\widetilde{\xi}) = \frac{1}{2} \sum_{j,l} \left(\frac{\partial^2 H}{\partial y^j \, \partial y^l} \circ u \right) v^j v^l + \sum_{j,k,l} \left(\frac{\partial^2 H}{\partial y^j \, \partial p^k_l} \circ u \right) v^j \beta^k_l$$
$$+ \frac{1}{2} \sum_{\substack{i,k \\ j,l}} \left(\frac{\partial^2 H}{\partial p^i_j \, \partial p^k_l} \circ u \right) \beta^i_j \beta^k_l$$

where $u = j_1(s)$ and $\tilde{\xi}$ is given by (7.19). Hamilton's form (7.15) of Jacobi's equation is equivalent to the equations

$$\frac{\partial \tilde{\mathrm{H}}}{\partial v^j} = - \sum_i \frac{\partial \beta^i_j}{\partial x^i}$$

($j = 1, \ldots, m$) and

$$\frac{\partial \tilde{\mathrm{H}}}{\partial \beta^i_j} = \frac{\partial v^j}{\partial x^i}$$

($i = 1, \ldots, n\,; j = 1, \ldots, m$).

We now show how Smale's generalization of the Morse index theorem [10] evaluating the index of a self-adjoint, strongly elliptic differential operator can be applied to the calculus of variations in several independent variables. Let $s \in \Sigma$ be an extremal.

Definition. – *The index of* $\mathrm{H}(\mathrm{I}_\mathrm{A})[s]$ *is the maximal dimension of a subspace of* $\mathrm{T}\Sigma_s$ *on which* $\mathrm{H}(\mathrm{I}_\mathrm{A})[s]$ *is negative definite.*

From Proposition 7.1 and Theorem 7.1, it follows that

$$\mathrm{H}(\mathrm{I}_\mathrm{A})[s](v, w) = \int_\mathrm{A} \langle v, \mathcal{J}[s]\, w\rangle = \int_\mathrm{A} \langle w, \mathcal{J}[s]\, v\rangle$$

for all $v, w \in \mathrm{T}\Sigma_s$, so that $\mathcal{J}[s]$ is *self-adjoint.* From standard facts about spectral theory for strongly elliptic differential operators and from Proposition 7.5, we obtain :

Proposition 7.8. – *Assume that* L *is regular. Then* $\mathcal{J}[s]$ *is a self-adjoint strongly elliptic differential operator of order* 2, *and there exist a sequence of real numbers*

$$\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_i \leqslant \cdots \to +\infty$$

and a sequence of elements $v_i \in \mathrm{T}\Sigma_s$ $(i = 1, 2, \ldots)$ *such that*

$$\int_\mathrm{A} \langle v_i, \mathcal{J}[s]\, v_j\rangle = \delta_{ij}\, \lambda_i$$

for all $i, j \geqslant 1$, and $\{v_i\}$ forms a basis for $\mathrm{T}\Sigma_s$.

It is easy to see that the index of $\mathrm{H}(\mathrm{I}_\mathrm{A})[s]$ is equal to the largest p such that $\lambda_p < 0$ and that the nullity of $\mathrm{H}(\mathrm{I}_\mathrm{A})[s]$ is equal to the number of λ_i's equal to 0. Hence, if L is regular, the index and the nullity of $H(I_A)[s]$ are finite.

Let $\{g_t\}$, $t_0 \leqslant t \leqslant t_1$, be a one-parameter family of diffeomorphisms of A with

$$g_{t_0} = \text{identity} .$$

Let A_t denote the image of A under g_t. We shall say that $\{g_t\}$ is a contraction of A if $\overline{A}_{t'} \subset A_t$ for $t' > t$; the contraction is of ε-type if

$$\int_{A_{t_1}} \omega < \varepsilon .$$

If $A' \subset A$ is an open submanifold of A whose boundary is a submanifold of A of codimension 1, we denote by $C_0^\infty(A', V_s(Y))$ the space of C^∞-sections of $V_s(Y)$ over A' which vanish on $\partial A'$; we have $C_0^\infty(A, V_s(Y)) = T\Sigma_s$. Given a contraction $\{g_t\}$, $t_0 \leqslant t \leqslant t_1$, of A, we say that ∂A_t is a conjugate boundary if a non-zero Jacobi field belongs to $C_0^\infty(A_t, V_s(Y))$; the multiplicity of a conjugate boundary ∂A is the dimension of the space of Jacobi fields belonging to $C_0^\infty(A_t, V_s(Y))$.

The following is an exposition of Smale's results in our context (cf. [10]).

LEMMA 7.5. – *Assume that* L *is regular. Then there exists* $\varepsilon > 0$ *such that for all open submanifolds* A' *of* A *with smooth boundary of co-dimension* 1 *satisfying*

$$\int_{A'} \omega < \varepsilon$$

the quadratic form $H(I_{A'})[s]$ *on* $C_0^\infty(A', V_s(Y))$ *is positive definite.*

THEOREM 7.3. – *Assume that* L *is regular and that* $\mathcal{J}[s]$ *has uniqueness in the Cauchy problem (that is, if* $v \in C^\infty(A, V_s(Y))$ *satisfies* $\mathcal{J}[s]\, v = 0$ *and* v *vanishes on some open subset of* A, *then* $v = 0$ *on* A). *Let* ε *be the positive number given by Lemma 7.5 and let* $\{g_t\}$, $t_0 \leqslant t \leqslant t_1$, *be a contraction of* A *of* ε*-type. Then there exist only a finite number of conjugate boundaries of* ∂A_t *and the index of* $H(I_A)[s]$ *is equal to the sum of the multiplicities of these boundaries for all* $t_0 \leqslant t \leqslant t_1$.

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