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SUBMANIFOLDS OF CODIMENSION TWO AND HOMOLOGY EQUIVALENT MANIFOLDS

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In this paper, we outline how new methods of classifying smooth, piecewise linear (Pl) or topological submanifolds are developed as consequences of a classification theory for manifolds that are homology equivalent, over various systems of coefficients [6] [7] [9]. These methods are particularily suitable for the placement problem for submanifolds of codimension two. The role of knot theory in this larger problem is studied systematically by the introduction of the local knot group of an arbitrary manifold. Computations of this group are used to determine when sufficently close embeddings in codimension two « differ » by a knot. A geometric periodicity is obtained for the knot cobordism groups. The homology surgery and its geometric applications described below theory also are used to obtain corresponding classification results for non-locally flat Pl and embeddings (see [10] and Jones [21].)

These methods can also be applied to get classification results on submanifolds invariant under group actions and of submanifolds fixed by group actions and to solve the general codimension two splitting problem. In particular, equivariant knot cobordism can be algebraically computed and geometric consequences derived. Other applications include a general solution of the surgery problem, given below, and corresponding results on smoothings of Poincaré embeddings in codimension two and « non-locally flat smoothings » of Poincaré embeddings.

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The proofs of many of the results use computations of new algebraic K-theory functors. A splitting theorem for odd-dimensional homology equivalent manifolds plays an important role in these computations.

In order to get significant classification results, the study of embeddings in codimension two, i.e. embeddings of a manifold M^n in W^{n+2}, has usually been restricted to the case in which M and W are spheres, i.e. knot theory. The peculiar difficulties in the study of codimension-two embeddings of M in W = are due to the fact the homomorphism $\pi_1(W - M) \to \pi_1(W)$ though always surjective, may not be an isomorphism. It is rather hard to conclude directly, from a knowledge of M, W, and the homotopy class of the embedding, enough information about $\pi_1(W - M)$ or, more generally, about the homotopy type of $W - M$ to give satisfactory geometric information about $W - M$. It is therefore natural to use instead the weaker information consisting of the homology groups of $W - M$ with coefficients in the local system of $\pi_1 W$. More precisely, we need a classification theory for manifolds that are only homotopy equivalent to $W - M$ in the weak sense defined by coefficients in $\pi_1 W$, i.e. manifolds *homology equivalent* to $W - M$ over the group ring $\mathbf{Z}[\pi_1 W]$. In knot theory ($M = S^n$, $W = S^{n+2}$) for example, while the homotopy type of the complement $S^{n+2} - S^n$ may be very complicated (2), it is in any case always a homology circle. Moreover, this fact together with other elementary data serves to characterize knot complements. In [9] we develop a general theory for classifying manifolds that are homology equivalent over a local coefficient system and perform calculations in this theory to get results on codimension two embeddings.

Even in the case of knot theory, the systematic study of homology equivalences leads to new understanding and new results. For example, we prove geometric periodicity theorem for high dimensional knot cobordism groups. Two embeddings of manifolds, $f_i : X \to Y$, $i = 0,1$, are said to be *concordant* if there is an embedding $F : X \times I \to Y \times I$ with $F(x, i) = f_i(x)$,

(2) In [Cl] it was shown that even the relative simple homotopy type of a knot complement does not uniquely determine a knot.

$i = 0,1$. (In the P.L. and topological case, unless there is an explicit statement to the contrary only locally that embeddings are under consideration.) Knots are said to be *cobordant* if they are equivalent to concordant knots [3]. This equivalence relation was introduced by Fox and Milnor [12], in the classical case $X = S^1$, $Y = S^3$. The cobordism classes of knots form a group, with addition defined by connected sum. Kervaire [13] studied analogous groups for higher dimensions and proved that they vanish in even dimensions and are very large in odd dimensions. Using our methods, we give in [9] a simple conceptual proof of the vanishing of the even dimensional knot cobordism groups, as a consequence of the vanishing of the obstruction group to odd dimensional surgery to abtain an (integral) homology equivalence. This proof extends to show the vanishing of the even dimensional equivariant knot cobordism groups.

Levine [14, 15] computed the odd dimensional P.L. or smooth groups (except in the classical case) and deduced an algebraic periodicity for high dimensional knot cobordism groups. In [8], it was shown that for topological knots periodicity applies all the way down to the case of S^3 in S^5. In our study of knots, we employ a new algebraic description of knot cobordism in terms of Hermitian or skew-Hermitian quadratic forms over $\mathbf{Z}[t, t^{-1}]$, the ring of finite Laurent series with integer coefficients, which become unimodular when one puts $t = 1$.

An especially simple formulation of geometric periodicity for knots is obtained by comparing, for a simply connected closed manifold M^k, the embeddings of $S^n \times M$ in $S^{n+2} \times M$ with the embeddings of S^n in S^{n+2} and S^{n+k} in S^{n+k+2}. A cobordism class x represented by $f : S^n \to S^{n+2}$ determines the cobordism class $\delta(M, n)(x)$ of embeddings of $S^n \times M$ in $S^{n+2} \times M$ represented by $f \times id_M$. A cobordism class y of embeddings of S^{n+k} in S^{n+k+2} determines, by connected sum with $f_0 \times id_M$, $f_0 : S^n \to S^{n+2}$ the usual inclusion, a cobordism class $\alpha(M,n)(y)$ of embeddings of $S^n \times M$ in $S^{n+2} \times M$. Let $G_n(M)$ be the cobordism classes of embeddings of $S^n \times M$ in $S^{n+2} \times M$ that are homotopic to $f_0 \times id_M$;

([3]) For P. L. and topological knots, cobordism implies concordance.

see [9] for the precise definition of cobordism in this context. Let $G_n = G_n(pt)$, the knot cobordism group.

THEOREM. — *Assume* $n \geqslant 2$ *and* M^k *is a simply-connected closed* P.L. *or topological manifold. Then* $\alpha(M, n) : G_{n+k} \to G_n(M^k)$ *is an one-to-one, onto map.*

In other words, up to cobordism every embeddings of G_n in $G_n(M^k)$ can be pushed into the usual embedding except in the neighborhood of a point, and this can be done in a unique way.

THEOREM. — *If* $k \equiv 0 \pmod 4$ *and* M *has index* ± 1, *and if* $n > 3$, *in the* P.L. *case, or* $n \geqslant 3$, *in the topological case, then* $\lambda(M, n) : G_n \to G_n(M^k)$ *is a one-to-one, onto map.*

In particular, let $M = CP^2$, the space of lines in complex three-space.

GEOMETRIE PERIODICITY THEOREM. — *The map*

$$\alpha(CP^2, n)^{-1} \circ \alpha(CP^2, n) : G_n \to G_{n+4}$$

in an isomorphism for $n > 3$, *in the* P.L. *case, and* $n \geqslant 3$, *in the topological case.*

Our methods also apply to the study of knots invariant under a free action of a cyclic group or fixed under a semi-free action. S. Lopez de Medrano [16], [17] proved a number of important results on knots invariant under free $\mathbf{Z}_2$-actions. His work and the ideas of our earlier work [8], [19] suggested the role that homology equivalences could play in codimension two. For free actions of cyclic groups, the classification of invariant codimension two spheres, in the neighborhood of which the actions behaves « nicely » (i.e. linearly on the fibres of a bundle neighborhood) is equivalent to the classification of embeddings of the quotient spaces. Our analysis of invariant spheres in codimension two breaks up naturally into two problems; first the determination of which actions can be obtained by the restriction of a given action to an invariant sphere, and then the classification up to equivariant cobordism od the equivariant embeddings of a given free action on a sphere in another.

On even dimensional spheres, only the cyclic group $\mathbf{Z}_2$ can act freely, and Medrano has determined which actions admit invariant spheres in codimension two; his result is reproved and interpreted in our context in [9]. The following result asserts that the even-dimensional equivariant knot cobordism groups vanish:

THEOREM. — *Let Σ^{2k+2}, $k \geqslant 3$, be a (homotopy) sphere equipped with a free $\mathbf{Z}_2$-action. Then any two invariant spheres of Σ^{2k+2} are equivariantly cobordant.*

On odd dimensional spheres, the results are slightly easier to state if the group has odd order.

THEOREM. — *Let Σ^{2k+1} and Σ^{2k-1} be spheres with free* P.L. *actions ρ and τ, respectively of Z_s, s odd. Then,*

i) *There is an equivariantly « nice » (i.e. smooth or equivariantly locally flat) embedding of Σ^{2k-1} in Σ^{2k+1} if and only if Σ^{2k-1}/τ is homotopy equivalent and normally cobordant to a desuspension of the homotopy lens space Σ^{2k+1}/ρ* [9],

ii) *there is an equivariant* P.L. *embedding of Σ^{2k-1} in Σ^{2k+1} if and only if Σ^{2k-1}/τ is homotopy equivalent to Σ^{2k+1}/ρ* [10].

As a consequence of similar criteria for invariant spheres of high codimension, one can prove the following:

COROLLARY. — *Let τ be a free action of $\mathbf{Z}_s$ on the (homotopy) sphere Σ^{2j+1} that is the restriction of the free action ρ on Σ^{2k+1} to an invariant sphere. Assume s is odd. Then Σ^{2j+1} is a characteristic sphere of ρ (i.e. Σ^{2j+1}/τ is a characteristic submanifold of Σ^{2k+1}/τ) if and only if there is a tower*

$$\Sigma^{2j+1} \subset \Sigma^{2j+3} \subset \dots \subset \Sigma^{2k-1} \subset \Sigma^{2k+1}$$

of invariant spheres.

Characteristic submanifolds are discussed in [16], [17], [2] and [9]. An obstruction theory computation shows that for $2j + 1 > \frac{1}{2}(2k + 1)$, the quotient space of an invariant sphere of dimension $(2j + 1)$ in an action on a $(2k + 1)$-sphere is a characteristic submanifold if and only if its normal bundle splits into is a sum of plane bundles.

The determination, for an action ρ of $\mathbf{Z}_s$, $s = 2p$, on

Σ^{2k+1}, of which actions appear as invariant spheres is somewhat more complicated. For $p = 1$, this was done by Medrano The general situation, (see [9], is that corresponding to each « homotopy desuspension » L of Σ^{2k+1}/ρ there is for k odd, precisely an entire normal cobordism class of homotopy lens spaces (or projective spaces) homotopy equivalent to L occurring as characteristic submanifolds of Σ^{2k+1}/ρ. For k even, there is an obstruction in $\mathbf{Z}_2$ to the existence of any characteristic submanifold homotopy equivalent to L, and if it vanishes, then an entire normal cobordism class occurs.

Our calculation of equivariant knot cobordism (see [9] is in terms of our new algebraic K-theoretic Γ-functors. A closely related computation of the knots fixed by semi-free actions yields the following:

THEOREM. — *For any interger* m *and any knot* $\varkappa : S^n \to S^{n+2}$, $n \geqslant 3$, $\varkappa \# \cdots \# \varkappa$ *(connected sum* m *times) is cobordant to a knot fixed under a semi-free action of* $\mathbf{Z}_m$. Moreover, *every element of* $\mathbf{Z}$, *for* $n \equiv -1 \pmod 4$, *or* $\mathbf{Z}_2$, *for* $n \equiv 1(4)$, *occurs as the index, or* Arf *invariant, of a knot* fixed *under a semi-free action of* $\mathbf{Z}_m$.

Of course, in the P.L. and topological cases only actions that are « nice » in the neighborhood of the fixed points are here under consideration. For $n = 3$, in the P.L. and smooth case one can only realize $16\mathbf{Z}$.

Of course in codimension two, close embeddings need not be isotopic; in [9], we study the question q when two sufficiently close embeddings f_0 and f_1 of M^n in W^{n+2} are concordant, at least up to taking connected sum with a knot. If f_1 is sufficiently close to f_0, it will lie in a bundle neighborhood. We therefore consider cobordism classes of embeddings of M, in the total space $E(\xi)$ of a 2-plane bundle ξ over M, homotopic to the zero-section. Such an embedding is called a *local knot* of M in ξ, and the set of cobordism classes is denoted $C(M, \xi)$, or just $C(M)$ if ξ is trivial. We also write $C_O(M; \xi)$, $C_{PL}(M; \xi)$, $C_{TOP}(M; \xi)$ to distinguish the various categories; in the last two, local flatness is understood. $C(M, \xi)$ is a monoid; the operation is called *composition* or *tunnel sum* and is defined as the composition $\bar{\iota}_1\iota_2$, where ι_1 and ι_2 are local knots of M in ξ and $\bar{\iota}_1$ is a thickening of ι_1

to an embedding of $E(\xi)$ in itself. We discuss only the case when M is closed; various relativizations exist.

THEOREM. — *For* $n = \dim M \geqslant 3$, $C(M, \xi)$ *is a group under composition, and for* $n \geqslant 4$ *it is abelian. For* $n \neq 1$, $C(S^n)$ *is isomorphic to the n-dimensional knot cobordism group. Connected sum with the zero-section defines a homomorphism* $\alpha : C(S^n) \to C(M, \xi)$. *In the* P.L. *or topological case,* α *is a monomorphism onto a direct summand provided* ξ *is trivial, and* $n \geqslant 4$.

We compute $C(M, \xi)$ in terms of an exact sequence involving the Γ-groups to be described below. In particular, for $\dim M \equiv 1 \pmod 2$, it is caught in an exact sequence of Wall surgery groups and hence tends to be fairly small. For example one has.

THEOREM. — *For* $n = \dim M \geqslant 4$ *even, there is an injection*

$$\bar{\rho} : C(M) \to L^h_{n+1}(\pi_1 M).$$

The map $\bar{\rho}$ has a geometric definition in terms of a surgery obstruction of a type of Seifert surface for local knots.

On the other hand, for M odd-dimensional, $C(M, \xi)$ is not, in general, finitely generated. For simply-connected M, the main result is the following:

THEOREM. — *Let* M *be a simply-connected closed n-manifold* $n \geqslant 4$, *and let* ξ *be a* 2*-plane bundle over* M. *Then* $\alpha : C(S^n) \to C(M, \xi)$ *is onto, and is an isomorphism for* ξ *trivial, in the* P.L. *and topological cases. For* n *even,* $C(M, \xi) = 0$.

We draw some consequences for the study of close embeddings.

THEOREM. — (See [9]. — *Let* $f_0 : M^n \to W^{n+2}$ *be an embedding (locally flat, of course) of the closed, simply-connected manifold* M *in the (not neccessarily compact) manifold* W. *Assume* $n \geqslant 5$. *Let* f *be another embedding, sufficiently close to* f_0 *in the* C_0 *topology. Then if the normal bundle* ξ *of* f *is trivial, or if* n *is even and the Euler class of* ξ *is not divisible by two, or if* $n \equiv 2 \pmod 4$ *and the Euler class of* ξ *is divisible only by two, then, after composition with a homeomorphism (or* *diffeomorphism or* P.L. *homeomorphism) of* M *homotopic to the identity,* f *is concordant to* f_0 *for* n *even and ti the connected sum of* f_0 *with a knot for* n *odd.*

Of course if M is 2-connected, the normal bundle ξ must be trivial. The importance of simple connectivity of M is demonstrated by the next result.

THEOREM. — *Let* $T^n = S^1 \times \cdots \times S^1$, $n \geqslant 4$. *Then, in the* P.L. *category,*

$$C(T^n) = C(S^n) \oplus [\Sigma(T^n - pt);\ G/PL],$$

and every element of $C(T^n)$ *can be represented by an embedding arbitrarily close to the zero-section* $T^n \subset T^n \times D^2$. $C(T^n)$ *is generated by products with various* $T^{n-i} \subset T^n$ *of the connected sum of* $T^i \subset T^i \times D^2$ *with knots of dimension* i.

If n is even, this follows from the injectivity of $\bar{\rho}$, as quoted above, and known results in ordinary surgery theory. For n odd, it requires a splitting theorem for homology equivalences and Γ-groups, to be discussed shortly.

The second summand in $C(T^n)$ is a direct sum of copies of $\mathbf{Z}$ and $\mathbf{Z}_2$ corresponding to the index or Arf invariant of knots sitting along various subtori of non-zero codimension, as described in the theorem. This knots with vanishing index or Arf invariant — a huge supply of them exists — disappear from sight when placed along a subtorus. In another paper [10], we will show how exactly these knots reappear in the classification of *non-locally flat* cobordism classes of non-locally flat embeddings of T^n in $T^n \times D^2$.

Given a manifold or even a Poincaré complex Y^{n+2} and a submanifold or Poincaré complex X^k, the problem of making a homotopy equivalence $f: W \to Y$ transverse regular to X, with $f|f^{-1}X : f^{-1}X \to X$ a homotopy equivalence, is called the ambient surgery problem. There is always an abstract surgery obstruction, an element of $L_R(\pi_1 X)$, to solving this problem. We solve the codimension two surgery problem, i.e. the case $k = n$, using the methods of surgery to obtain homology equivalences. The odd dimensional result resembles the results of higher codimension [1]; if the abstract surgery obstruction vanishes the problem can be solved and all the manifolds homotopy equivalent to X in one normal cobordism class occur as $f^{-1}X$. In even dimensions, there is an additional obstruction to this problem, defined in terms of the Γ-functors. Note that in codimension two, even if $f|f^{-1}X : f^{-1}X \to X$ is a homotopy equivalence, $f|W - f^{-1}X : W - f^{-1}X \to Y - X$ need only be a $\mathbf{Z}[\pi_1 Y]$-homology equivalence.

As an application of our codimension two surgery, one can study the problem of finding locally flat spines in codimension two. This problem has been studied by Kato and Matsumoto, using different methods of codimension two surgery in a special case. In another paper [10], we apply our methods to the classification of non-locally flat embeddings and, in particular to « non-locally » flat smoothings of Poincaré embeddings. One consequence of this is the following.

THEOREM. — [10]. *Let* W^{n+2} *and* M^n, *n odd or* $\pi_1 M = 0$, *be* P.L. *manifolds with* M *closed and* $i : M \to W$ *a* P.L. *embedding. If* $f : N \to M$ *is a homotopy equivalence of closed pl manifolds, then* $f_i : N \to W$ *is homotopy equivalent to a piece-wise-linear, but not necessarily locally-flat, embedding.*

Chapter I of [9] develops the theory of homology surgery. Let $Z[\pi]$ be the integral group ring of the group π, with a usual involution determined by a homomorphism $\omega : \pi \to \{\pm 1\}$, and let $\mathcal{F} : Z[\pi] \to \Lambda$ be a homomorphism of rings with unit and involution. Below we make the convenient, though not always essential, assumption that $\mathcal{F}$ is surjective. If $\mathcal{F}$ is induced by a map of fundamental groups $\pi_1(W - M) \to \pi_1(W)$, for M a codimension two submanifold of W and $\pi = \pi_1(W - M)$ $\Lambda = Z[\pi_1(W)]$, then $\mathcal{F}$ is, of course, onto.

THEOREM. — *Let* $(Y^n, \partial Y)$ *be a manifold pair (or even just a Poincaré pair over* Λ*) with* $(\pi_1 Y, w^1 Y) = (\pi, w)$, $n \geqslant 5$. *A normal map* (f, b), $f : (X, \partial X) \to (Y, \partial Y)$, *of degree one, inducing a homology equivalence over* Λ *of boundaries, determines an element* $\sigma(f, b)$ *of an algebraically defined abelian group* $\Gamma_n^h(\mathcal{F})$. *The element* $\sigma(f, b)$ *vanishes if and only if* (f, b) *is normally cobordant, relative the boundary, to a homology equivalence over* Λ.

This result, together with a realization theorem for elements of $\Gamma_n^h(\mathscr{F})$ and a special study of homology surgery for manifolds $(Y, \partial Y)$ with $\pi_1 \partial Y = \pi_1 Y$ leads, by a procedure analogous to [20, § 9] to a general relative theory for homology surgery. In this theory a periodicity theorem that asserts that $\sigma((f, b) \times \mathrm{id}_{CP^2}) = \sigma(f, b)$ plays an important role. An analogous theory for simple homology equivalences, with absolute groups $\Gamma_n^s(\mathscr{F}) = \Gamma_n(\mathscr{F})$, is also developed in [9].

If $\mathscr{F}$ is the identity of $Z[\pi]$, then $\Gamma_n(\mathscr{F})$ is the Wall group $L_n(\pi)$, and $\sigma(f, b)$ is the usual surgery obstruction of Wall [20].

For $n = 2k$, $\Gamma_n(\mathscr{F})$ is defined as a Grothendieck group of $(-1)^k$-symmetric Hermitian forms over $Z\pi$ that become non-singular forms on stably free modules when tensored into Λ. An element of this group which is represented by a module P equipped with a bilinear pairing and inner product which restrict to 0 on a submodules Q, with $Q \otimes_{Z\pi} \Lambda$ being a free module of half the rank of $P \otimes_{Z\pi} \Lambda$, represents 0 element of $\Gamma_{2k}(\mathscr{F})$. Using the surjectivity of $\mathscr{F}$, or even a weaker condition on $\mathscr{F}$, it is straightforward to construct for each element in $\Gamma_{2k}(\mathscr{F})$ a representative for which the underlying module of the Hermitian form is already free over $Z[\pi]$. If $\mathscr{F}$ is onto, so is the natural map $\Gamma_{2k}(\mathscr{F}) \to L_{2k}(\Lambda)$. For $\mathscr{F}$ onto, we show directly that $\Gamma_{2k+1}(\mathscr{F})$ is a subgroup of $L_{2k+1}(\Lambda)$. Geometrically, this implies that in odd dimensions the vanishing of an obstruction in a Wall surgery group is enough to permit completion of homology surgery. Using the splitting theorem for homology equivalences, Theorem 15.1 of [9] we show that if $\Lambda = Z\pi'$ and $\mathscr{F}$ is induced by a group homomorphism $\pi \to \pi'$, then $\Gamma_{2k+1}(\mathscr{F}) = L_{2k+1}(\pi')$.

In [9] we prove the splitting theorem for Γ-groups alluled to earlier. Assume still that $\mathscr{F}$ is induced by a group homomorphism.

THEOREM. — *Let $\mathscr{F} \times Z$ denote the natural map from $Z[\pi \times Z]$ to $\Lambda \times Z$ induced by $\mathscr{F}$. Then*

$$\Gamma_n^s(\mathscr{F} \times Z) = \Gamma_n^s(\mathscr{F}) \oplus L_{n-1}^h(\mathscr{F}).$$

This is analogous to [18, 5.1] (see also [20, § 13]) for Wall groups, and is in fact the same result for n odd or $\mathscr{F} = id_{Z\pi}$ The result is proven using a geometric splitting theorem for.

homology equivalences in odd dimensions, analogous the splitting theorem of Farrell-Hsiang [11] for homotopy equivalences. This homology splitting theorem is proven by adapting the methods of [4], [5]. It is definitely false for even dimensions.

In terms of Γ-groups one has a calculation of $C(M; \xi)$.

THEOREM. — *If* $n = \dim M \geqslant 4$, *then there is an exact sequence*

$$0 \to C_H(M, \xi) \xrightarrow{\Sigma} \Gamma_{n+3}(\varnothing) \xrightarrow{\rho} \operatorname{coker} S_H,$$

$H = 0$, PL, TOP.

Here $\varnothing$ is the diagram

$$\begin{array}{ccc} \mathbf{Z}[\pi_1(\partial E)] & \xrightarrow{\text{id}} & \mathbf{Z}[\pi_1(\partial E)] \\ \downarrow{\scriptstyle \text{id}} & & \downarrow{\scriptstyle P_*} \\ \mathbf{Z}[\pi_1(\partial E)] & \xrightarrow{P_*} & \mathbf{Z}[\pi_1 M], \end{array}$$

P the projection of ξ, and $S_H : [\Sigma E; G/H] \to L^s_{n+3}(P_*)$ is the usual map, $E = E(\xi)$. This Γ-group is calculated by an exact sequence involving absolute groups as the other terms. Our results on equivariant knot cobordism [7] [9] are also obtained by using Γ-groups.

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