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Catastrophes and partial differential equations

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CATASTROPHES
AND PARTIAL DIFFERENTIAL EQUATIONS
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Introduction.

We reformulate here certain aspects of the theory of first
order partial differential equations whose origins, date at
least from Darboux [2]. Our aim is to describe the singularities
of solutions of first order partial differential equations in such
a way that the singularity theory of mappings can be applied
to describe the generic singularities which arise in the solution
of partial differential equations. The formalism we adopt is
based upon the concept of *lagrangean manifold*, introduced by
Arnold [1] and employed in contexts related to ours by
Hörmander and Weinstein [4, 12].

In his forthcoming book [9], Thom observes that the singu-
larities developed by wavefronts are related to the unfoldings
of singularities of real valued functions. The propogation of
wavefronts is determined by first order differential operators.
Porteous has described this situation in detail for the propo-
gation of waves in Euclidean 4-space corresponding to the
wave equation [8]. Neither Thom nor Porteous elaborates
the way in which the theorems of singularity theory can be
explicitly applied to the situation at hand to describe those
singularities which are generic. They approach the subject from
the viewpoint of variational principles and initial value pro-
blems. This partially masks the local nature of the problem.
By working in the context of lagrangean manifolds of a cotan-
gent bundle, we clarify this situation somewhat.

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We also point out here that this connection between catastrophes and partial differential equations allows one, in some instances, to connect catastrophes more directly with physical and biological phenomena than is possible with Thom's « metabolic model ». Physical laws are often stated in terms of partial differential equations. Hörmander and Duistermaat have recently proved that for linear operators of principal type, the singularities of solutions propogate along bicharacteristics [5]. The bicharacteristics in turn are determined by the characteristic equation of the operator, and this is of first order. This allows our theory to be applied to solutions of these operators. This description of discontinuous physical phenomena avoids certain technical difficulties which arise in using a model based upon the bifurcation theory of vector fields [3].

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1. First Order P.D.E.'s.

Classically, a first order partial differential equation is written in one of the two forms

$$H(x, \xi) = 0 \tag{1}$$
$$H(x, u, \xi) = 0 \tag{1'}$$

Here, x and ξ are variables in $\mathbf{R}^n$ and u is a variable in $\mathbf{R}$. A solution of either equation is a function $f: \mathbf{R}^n \to \mathbf{R}$ such that if $u = f(x)$ and $\xi_i = \dfrac{\partial f}{\partial x_i}$, then $H(x, \xi)$ or $H(x, u, \xi)$ is identically zero.

We can view equation (1) in a coordinate-free manner in the setting of manifolds by regarding x as an element of a smooth n-dimensional manifold M and ξ as an element of the cotangent space of M at x. Then $H: T^*M \to \mathbf{R}$ is a function on the cotangent bundle of M. A solution of $H = 0$ is a function $f: M \to \mathbf{R}$ such that the graph of df in T^*M lies in the hypersurface of zeros of H.

One can also interpret equation (1') in a coordinate-free manner. Recall the definition of an r-jet. Two smooth functions f, $g : M \to R$ are said to be r-equivalent at x if the Taylor series of $f - g$ at x begins with terms of degree $r + 1$ in some coordinate system. An r-jet is an equivalence class with respect to r-equivalence at x. The r-jets of functions form a vector bundle $J^r(M, R)$ over M. Given a function $f : M \to R$, there is a natural map $J^r f : M \to J^r(M, R)$, the r-jet extension of f, defined by $J^r f(x) =$ the r-jet of f at x. The left hand side of equation (1') can be interpreted as a function defined on $J^1(M, R)$. Then a solution of (1') is a function $f : M \to R$ such that $J^1 f$ lies in the hypersurface of zeros of H.

$J^1(M, R)$ is isomorphic as a vector bundle over M to $T^*M \times R$. We can use this isomorphism to relate the study of equations (1) and (1'). Let $N = M \times R$. The cotangent bundle T^*N is the product $T^*M \times T^*R$. Let (x, u) be local coordinates on $N = M \times R$ and (ξ, η) the conjugate coordinates in T^*N. Define $H' : T^*N \to R$ by

$$H'(x, u, \xi, \eta) = H\left(x, u, \frac{\xi}{\eta}\right) \quad \text{if} \quad \eta \neq 0.$$

Note that we identify the fibers of T^*R with R in writing this equation. Let f be a solution of $H' = 0$ such that $f(x_0, u_0) = 0$ and $\frac{\partial f}{\partial u}(x_0, u_0) \neq 0$. On the hypersurface $f = 0$ in $M \times R$, we can then implicitly solve for u as a function of x near (x_0, u_0). That is, there is a function $g : M \to R$ such that $f(x, g(x))$ vanishes in a neighborhood of x_0. We calculate

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} \bigg/ \frac{\partial f}{\partial u}.$$

It follows that

$$H\left(x, g(x), \frac{\partial g}{\partial x}\right) = H'\left(x, u, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial u}\right) \qquad (2)$$

on the hypersurface $f = 0$ of N in a neighborhood of (x_0, u_0). Thus the restriction of a solution f of $H' = 0$ to the hypersurface $f = 0$ in $M \times R$ leads to a solution of (1') if

$\dfrac{\partial f}{\partial u} \neq 0$. We shall make further use of this interpretation of equation (1') in describing the singularities of its solutions in section 5.

If we try to find global solutions of equations (1) or (1') on a manifold M, certain pathologies may occur. In particular, it may not be possible to describe a solution as a single valued function. To avoid this difficulty, we give a geometric characterization of solutions. This requires a brief digression concerning symplectic geometry.

There is a canonical two form Ω on T*M which one may define in local coordinates as follows. If $(x_1, \ldots, x_n)$ are coordinates on an open set $U \subset M$ and $p \in U$, then we choose coordinates in T_p^*M by setting $(\xi_1, \ldots, \xi_n)$ to be the coordinates of the covector $\sum_{i=1}^{n} \xi_i \, dx_i(p)$. Since $(x_1, \ldots, x_n)$ are coordinates on $U \subset M$, $dx_1(p), \ldots, dx_n(p)$ are linearly independent covectors for each $p \in U$. Hence $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ define local coordinates in T_U^*M. ξ_i is the conjugate coordinate of x_i. In terms of these local coordinates,

$$\Omega(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} d\xi_i \wedge dx_i. \qquad (3)$$

Note that we regard dx_i here as a one form on T*M and not on M. The canonical two form Ω can also be defined intrinsically via the following diagram:

$$
\begin{array}{ccc}
 & T(T^*M) & \\
{\scriptstyle \pi_3}\swarrow & & \searrow{\scriptstyle d\pi_1} \\
T^*M & & T(M) \\
{\scriptstyle \pi_1}\searrow & & \swarrow{\scriptstyle \pi_2} \\
 & M &
\end{array}
$$

The maps π_1, π_2, π_3 are the vector bundle projections. If X is a vector field on T*M, define

$$\omega(X) = \pi_3(X) \, (d\pi_1(X)).$$

The expression of ω in the local coordinates defined above is

$$\omega(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n) = \sum_{i=1}^{n} \xi_i \, dx_i \qquad (4)$$

We shall call ω the canonical one form on T^*M. From equations (3) and (4) it follows that $\Omega = d\omega$. Ω has maximal rank and consequently defines a bundle isomorphism between $T(T^*M)$ and $T^*(T^*M)$.

PROPOSITION. — *Let θ be a closed one form on M and let i:* graph $\theta \to T^*M$ *be the inclusion. Then $i^*\Omega \equiv 0$. Conversely, if $j : \lambda \to T^*M$ is a submanifold of T^*M transverse to the fibers of T^*M and if $j^*\Omega \equiv 0$, then $j(\lambda)$ is locally the graph of a closed one form.*

COROLLARY. — *A submanifold λ of T^*M is locally of the form graph df for some function $f : M \to \mathbf{R}$ if and only if*

1) δ *is transverse to the fibers of M, and*

2) Ω *pulls back to zero on λ.*

Proof. — The implicit function theorem implies that a necessary and sufficient condition for a submanifold λ of T^*M to be locally the graph of a one form is that $d\pi|_\lambda$ be an isomorphism. Here $\pi : T^*M \to M$ is the projection. This implies that λ is transverse to the fibers of T^*M. The maximal dimension of a subspace of $T_p(T^*M)$ on which Ω vanishes is n, the dimension of M. Therefore, we may assume that the dimension of λ is n if it is to satisfy the hypotheses of the second part of the proposition. This, together with the transversality hypothesis, implies that λ is locally the graph of a one form θ. Suppose θ can be written in local coordinates

$$\theta(x_1, \ldots, x_n) = \sum_{i=1}^{n} a_i(x)\, dx_i$$

then

$$d\theta(x_1, \ldots, x_n) = \sum_{i,j} \frac{\partial a_i}{\partial x_j}(x)\, dx_j \wedge dx_i.$$

It follows that $d\theta = 0$ if and only if

$$\frac{\partial a_i}{\partial x_j} = \frac{\partial a_j}{\partial x_i}.$$

The graph of θ is the set of points of the form $(x_1, \ldots, x_n, a_1(x), \ldots, a_n(x))$. A basis for the tangent space of graph θ

is given by

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial a_j}{\partial x_i} \frac{\partial}{\partial \xi_j}, \quad i = 1, \ldots, n.$$

We see that $\Omega(X_i, X_k) = 0$ if and only if $\dfrac{\partial a_i}{\partial x_k} = \dfrac{\partial a_k}{\partial x_i}$. Hence, θ is closed if and only if Ω vanishes on the tangent space θ. This proves the proposition.

The corollary follows from the proposition by the Poincaré lemma. If θ is a closed one form, $\int_{\gamma} \theta$ depends only on the homotopy class of the path γ. Therefore $f(x) = \int_{x_0}^{x} \theta$ is locally well defined and gives a function f such that $\theta = df$.

The proposition and corollary allow us to characterize the solutions of equation (1) geometrically: if M is simply connected, a solution of equation (1) is an n-dimensional submanifold $j : \lambda \to T^*M$ such that

  i) λ lies in the hypersurface of zeros of H,

  ii) $j^*(\Omega) \equiv 0$, and

  iii) λ is transverse to the fibers of T^*M.

If M is not simply connected, then we need the additional hypothesis that λ is the graph of an exact one form. However, we propose to eliminate condition (iii) and make the following definition.

DEFINITION. — *A solution of equation* (1) *is an n-dimensional submanifold* λ *of* T^*M *which satisfies conditions* (i) *and* (ii) *above. A singularity of* λ *is a point* $x \in \lambda$ *such that* λ *intersects the fiber* $T^*_{\pi(x)}M$ *non-transversely at* x.

The singularities are precisely the points at which there is an obstruction to defining λ as the graph of an exact one form.

We can treat solutions of equation (1') in a similar manner. Observe that if f is a solution of the equation $H' = 0$ defined by (2) and if $c \neq 0$, then f and cf lead to the same solution of $H = 0$. Conversely, two solutions of $H' = 0$ which lead to the same solution of $H = 0$ are scalar multiples of one another. The graph of $d(cf)$ is obtained from the graph of df by multiplying the fiber coordinates by c. In $T^*(M \times \mathbf{R})$ this gives us a cone λ of dimension $n + 1$. It is easy to check that Ω pulls back to zero on λ. The generator of the

cone is the restriction of graph df to the hypersurface of zeros of f in $M \times \mathbf{R}$. We already have proved that Ω pulls back to zero on graph df, so it certainly pulls back to zero on a submanifold of graph df. All that remains to prove is that the radial tangent vector $\left(\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i} \text{ in local coordinates} \right)$ is Ω-orthogonal to the tangent space of graph $df|_{\{f=0\}}$. The tangent space of graph df has as basis

$$X_i = \frac{\partial}{\partial x_i} + \sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_i \, \partial x_j} \frac{\partial}{\partial \xi_j} \quad j = 1, \ldots, n$$

in local coordinates. $\sum_{i=1}^{n} a_i X_i$ is tangent to graph $df|_{\{f=0\}}$ if

$$df \left(\sum_{i=1}^{n} a_i \frac{\partial}{\partial x_i} \right) = \sum_{i=1}^{n} a_i \frac{\partial f}{\partial x_i} = 0.$$

Now

$$\Omega \left(\sum_{i=1}^{n} \xi_i \frac{\partial}{\partial \xi_i}, \sum_{i=1}^{n} a_i X_i \right) = \sum_{i=1}^{n} a_i \xi_i.$$

On graph df, $\xi_i = \frac{\partial f}{\partial x_i}$. Hence $\sum_{i=1}^{n} a_i \xi_i = 0$ and Ω pulls back to zero on λ.

Thus we may identify the classical solutions of (1′) with $n + 1$ dimensional submanifolds $\lambda \subset T^*(M \times \mathbf{R}) - \{0 \text{ section}\}$ satisfying

i′) λ lies in the hypersurface of zeros of H' [H' is defined by equation (2)],

ii′) Ω pulls back to zero on λ,

iii′) λ is homogeneous in the fiber coordinates,

iv′) λ is transverse to the fibers of the vector bundle $T^*(M \times \mathbf{R}) \to M$.

Once again we generalize the definition of *solution* by dropping condition (iv′). The *singularities* of a solution are the set of points where (iv′) fails to hold.

This context is a special case of one which arises in studying the characteristic equation of a linear partial differential operator. There one obtains a first order equation H: $T^*M - \{0 \text{ section}\} \to \mathbf{R}$ such that H is homogeneous in the

fiber coordinates. In this context also, we may define a solution to be a submanifold of T*M satisfying the conditions i), ii), and iii'). The singularities of a solution λ are then the points at which corank $\pi|_\lambda > 1$.

The primary goal of the rest of the paper is to describe the local structure of the singularities of a generic set of solutions of (1) and (1').

2. Lagrangean Manifolds.

In this section we describe the set of solutions of the equation $H(x, \xi) = 0$ and give this set the structure of a topological space. Arnold introduced the concept of a lagrangean submanifold of a symplectic manifold which is basic to our viewpoint. Let (P^{2n}, Ω) be a symplectic manifold of dimension $2n$. Recall that this means that Ω is a closed two form of maximal rank on P.

DEFINITION. — A *lagrangean submanifold of* P *is a submanifold* $i : \lambda \to P$ *such that*

i) $\dim \lambda = n = \dfrac{1}{2} \dim P$,

ii) $i^*(\Omega) \equiv 0$.

Thus the solutions of $H = 0$, $H : T^*M \to \mathbf{R}$, are lagrangean submanifolds of T*M which lie in the hypersurface of zeros of H.

We wish to give the set of lagrangean submanifolds of P a topology so that we may speak about perturbations of solutions of $H = 0$. Fix a manifold N of dimension n and consider the lagrangean submanifolds of P diffeomorphic to N as a subset of $C^\infty(N, P)$. Interpreted this way, each lagrangean manifold carries with it a specific parametrization. Different embeddings of N with the same image are regarded as different submanifolds. If we wish to regard a submanifold as a subset of P and not as an embedding, then we must divide $C^\infty(N, P)$ by the group $D(N)$ of C^∞ diffeomorphisms of N acting by composition on the right. It is clear that the action of $D(N)$ perserves the set of lagrangean submanifolds in $C^\infty(N, P)$. Denote the quotient of the set of lagrangean

submanifolds by the action of $D(N)$ by $\Lambda(N)$. $\Lambda(N)$ inherits a topology as a subset of $C^\infty(N, P)/D(N)$.

To understand more clearly the topological structure of $\Lambda(N)$, we begin by describing the set of germs of lagrangean submanifolds of $T^*\mathbf{R}^n \approx \mathbf{R}^{2n}$. The set of linear lagrangean subspaces of $\mathbf{R}^{2n}$ is a closed subset of the Grassmannian $G_{n,n}$ of n-planes in $\mathbf{R}^{2n}$. $G_{n,n}$ is the homogeneous space $O(n) \times O(n)\backslash O(2n)$. If the linear transformation A preserves the quadratic form

$$Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix};$$

i.e., $A^t Q A = Q$, then $A^* \Omega = \Omega$. If A is also orthogonal, then $A^t = A^{-1}$ and A preserves Ω if and only if it commutes with Q. If we write A in block form

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$$

where each block is an $n \times n$ matrix, then the condition for A to commute with Q is that

$$\begin{pmatrix} A_{21} & A_{22} \\ -A_{11} & -A_{12} \end{pmatrix} = \begin{pmatrix} -A_{12} & A_{11} \\ -A_{22} & A_{21} \end{pmatrix}$$

Therefore A is of the form

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

If we embed $Gl(n, \mathbf{C}) \to Gl(2n, \mathbf{R})$ by

$$B + iC \longmapsto \begin{pmatrix} B & -C \\ C & B \end{pmatrix},$$

then the orthogonal matrices which preserve Ω are identified with $U(n)$. It follows that the set of lagrangean planes can be identified with $O(n)\backslash U(n)$ where $O(n)$ is embedded in $Gl(2n, \mathbf{R})$ as matrices of the form

$$\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}$$

$O(n)$ is the stabilizer of the lagrangean plane represented by the identity matrix.

In the last section we noted that the lagrangean planes which are transverse to the plane determined by Q can be represented in the form graph df. In terms of the Grassmannian representation above, the transversality condition for the plane determined by

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

is that B be non-singular. In this case $B^{-1}C$ is a symmetric matrix since

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}\begin{pmatrix} B^t & C^t \\ -C^t & B^t \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$

implies $CB^t - BC^t = 0$. $B^{-1}C$ is independent of the particular choice of matrix in $U(n)$ determining a given lagrangean plane since $D \in O(n)$ implies

$$(DB)^{-1}DC = B^{-1}C.$$

The lagrangean plane determined by

$$\begin{pmatrix} B & -C \\ C & B \end{pmatrix}$$

is the span of the first n-column vectors. This is the same as the span of

$$\begin{pmatrix} I \\ B^{-1}C \end{pmatrix}$$

if B is invertible. This span is graph df where $f: \mathbf{R}^n \to \mathbf{R}$ is the quadratic function

$$f(x) = \frac{1}{2}(x^t B^{-1} C x).$$

Now we study the set of germs of lagrangean manifolds through a given point with a given tangent plane. Choose local coordinates so that the point is $(0, 0) \in T^*\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ and the tangent plane is $\mathbf{R}^n \times \{0\}$. If λ is a lagrangean manifold through $(0, 0)$ with this tangent plane, we can write $\lambda = $ graph df near $(0, 0)$ for some $f: \mathbf{R}^n \to \mathbf{R}$. f is determined only up to a constant, so we may assume that $f(0) = 0$. Furthermore, $df(0) = d^2f(0) = 0$ since graph df passes

through $(0, 0) \in T^*R^n$ and has a horizontal tangent plane there. Thus we may identify the cube of the maximal ideal of the ring of germs of functions on R^n with the set of germs of (unparametrized) lagrangean manifolds through a given point with a given tangent plane. Denoting by $\tilde{\Lambda}_0$ the set of germs of lagrangean manifolds through $(0, 0) \in T^*R^n$, we obtain the fibration

$$\tilde{\Lambda}_0 \to O(n)\backslash U(n),$$

with fiber m_n^3. Here m_n is the maximal ideal of the set of germs of C^∞ functions defined on R^n at the origin. $O(n)\backslash U(n)$ is a smooth manifold of dimension $n(n + 1)/2$.

Next, we describe the set of germs of solutions of the first order partial differential equation $H(x, \xi) = 0$. Choose coordinates so that $H(0, 0) = 0$. If $(0, 0)$ is not a critical point of H, we may choose local canonical coordinates so that H is a coordinate function, say $H(x, \xi) = \xi_n$. These coordinates usually cannot be chosen to preserve the fibration of T^*M as well as the canonical two form.

The equation $H(x, \xi) = \xi_n$ can be solved explicitly. The classical solutions of this equation are functions $f: R^n \to R$ such that $\dfrac{\partial f}{\partial x_n} = 0$. By continuity, it follows that the germs of solutions of $H = 0$ through $(0, 0)$ form a subset $\tilde{\Lambda}_{0,H}$ of $\tilde{\Lambda}_0$ corresponding to those lagrangean manifolds comprising families of lines parallel to the x_n-axis. This gives us the fibration

$$\tilde{\Lambda}_{0,H} \to O(n - 1)\backslash U(n - 1)$$

with fiber m_{n-1}^3. The set of germs of solutions of $H = 0$ through a given regular point of H can therefore be identified with germs of lagrangean manifolds of T^*R^{n-1}.

Return now to the study of lagrangean submanifolds of P which are diffeomorphic to N. The proper C^∞ embeddings of N in P form an open subset $E \subset C^\infty(N, P)$. The lagrangean proper embeddings form a closed subset $L \subset E$ because L is defined as the locus of zeros of equations which are defined on the 1-jets of elements of E. We have the following:

THEOREM (Weinstein [13]). — L *is a Frechet manifold modeled on* $Z(N) \oplus \chi(N)$. *Here* $Z(N)$ *is the vector space of*

closed C^∞ *one forms on* N *and* $\chi(N)$ *is the vector space of* C_∞ *vector fields on* N.

We briefly indicate the proof of this theorem. Let λ : N → P be a lagrangean manifold of P. Weinstein proves that there is a tubular neighborhood T of N in P and a symplectic diffeomorphism h of T into T^*N so that $h \circ \lambda$: N → T^*N is the inclusion of the zero section. Thus h maps a neighborhood of $\lambda \in L(P)$ into a neighborhood of 0-section $\in L(T^*N)$. The unparametrized lagrangean submanifolds close to the 0-section in $L(T^*N)$ are represented by the graphs of small, closed one forms. The different parametrizations of a given parametrized submanifold form a space isomorphic to $D(N)$. A neighborhood of the identity in $D(N)$ is isomorphic to a neighborhood of $0 \in \chi(N)$. Putting these facts together, we obtain a coordinate chart centered at λ diffeomorphic to an open subset of $Z(N) \oplus \chi(N)$.

$D(N)$ acts freely on $L(N)$, yielding the following corollary : let $\Lambda_{pr}(N) \subset \Lambda(N)$ denote the set of embedded proper lagrangean submanifolds of P.

COROLLARY. — $\Lambda_{pr}(N)$ *is a Frechet manifold modeled on the vector space* $Z(N)$.

If N is not compact, then there is a choice to be made among C^∞-topologies. The appropriate topology to use for the study of singularities is the Whitney topology. Therefore the C^∞-topology will mean the Whitney C^∞-topology in this paper [7].

If H : T^*M → **R** is a differential operator, then the proper solutions of H = 0 clearly form a subvariety of Λ_{pr}. If 0 is not a critical value of H, then our local analysis of germs of solution can be used to prove that the set of proper solutions of H = 0 is a submanifold of Λ_{pr}.

3. Equivalence and Stability of Lagrangean Manifolds.

We have defined lagrangean manifolds of T^*M. Let us now formally state the following :

DEFINITION. — *If* $\lambda \subset T^*M$ *is a lagrangean manifold, then the singular set of* λ, *denoted* $S(\lambda)$, *is* $\{x \in \lambda | \lambda$ *is not*

transverse to $T^*_{\pi(x)}M$ *at* x}. $\pi : T^*M \to M$ *is the projection. The caustic set of* λ *is* $\pi(S(\lambda))$.

This definition of caustic agrees with the definition of caustic in geometric optics when the lagrangean manifold λ is a solution of the (inhomogeneous) characteristic equation of the wave equation.

There are a number of different equivalence relations defined on the set of lagrangean manifolds which give different interpretations to the statement : lagrangean manifolds λ and λ' have equivalent singular sets. We state three of these :

I. Lagrangean manifolds λ, $\lambda' \subset T^*M$ are equivalent if there is a diffeomorphism $h : M \to M$ which maps the caustic set of λ onto the caustic set of λ'.

II. Lagrangean manifolds λ, $\lambda' \subset T^*M$ are equivalent if there is a fiber preserving diffeomorphism $H : T^*M \to T^*M$ defined in a neighborhood of λ such that $H(\lambda) = \lambda'$.

In order to define III-equivalence, we need to discuss the relationship between families of mappings and lagrangean manifolds.

DEFINITION. — *Let* M, N *and* P *be manifolds. A family* F *of maps from* N *to* P *parametrized by* M *is a commutative diagram*

$$N \times M \xrightarrow{F} P \times M$$
$$\pi_2 \searrow \quad \swarrow \pi_2$$
$$M$$

π_2 *is the projection onto the second factor.*

PROPOSITION (Weinstein-Hörmander [4, 12]). — *Let*

$$N \times M \xrightarrow{F} R \times M$$
$$\searrow \quad \swarrow$$
$$M$$

be a family of functions parametrized by M. *Then*

(1) *The critical set* $\Sigma(F)$ *of* F *is the set of points* $(x, t) \in N \times M$ *such that* $\dfrac{\partial F_1}{\partial x}(x, t) = 0$. *Here* $F_1 = \pi_1 \circ F$; $\pi_1 : R \times M \to R$ *is projection.*

(2) *For a generic set of families,* $\Sigma(F)$ *is a manifold. There is a map* $\sigma : \Sigma(F) \to T^*M$ *defined by*

$$\sigma(x, t) = dF_1(x, t) = \frac{\partial F_1}{\partial t}(x, t)\, dt.$$

(3) $\lambda = \sigma(\Sigma(F))$ *is a lagrangean submanifold of* T^*M.

This proposition gives a canonical map C from generic families of functions parametrized by M to lagrangean submanifolds of T^*M.

$$C(F) = \sigma(\Sigma(F)).$$

III. If $\lambda, \lambda' \subset T^*M$ are lagrangean manifolds in the image of C, then λ and λ' are equivalent if there exist families of functions F, F' parametrized by M such that $\lambda = C(F)$, $\lambda' = C(F')$, and F is equivalent to F'.

The last condition means the following:

DEFINITION. — *Let*

$$N \times M \xrightarrow{F} P \times M \qquad and \qquad N' \times M \xrightarrow{F'} P' \times M$$

with M at the base of each.

be families of maps. F *and* F' *are equivalent if there is a commutative diagram*

$$
\begin{array}{ccc}
N \times M & \xrightarrow{F} & P \times M \\
\downarrow & M & \downarrow \\
N' \times M & \xrightarrow{F'} & P' \times M
\end{array}
$$

with M at the base of each.

such that all of the vertical arrows are diffeomorphisms.

The three equivalence relations defined above are ordered in the sense that λ, λ' III-equivalent implies λ, λ' II-equivalent, and λ, λ' II-equivalent implies λ, λ' I-equivalent. I-equivalence is the intuitive concept which is most immediate for describing the singularities of wave propagation, but the

slightly stronger II-equivalence seems technically much easier to work with. III-equivalence is excessively strong but lends itself easily to applications of the theory of singularities of maps.

Corresponding to each of these definitions of equivalence is a corresponding definition of local equivalence for germs of lagrangean manifolds. In this connection, one has the following:

PROPOSITION (Weinstein-Hörmander [4, 12]). — *Let* M *be an n-dimensional manifold. The map* C *defined above is a surjective map from germs of families of functions on* $\mathbf{R}^n$ *to germs of lagrangean manifolds of* T^*M.

Proof. — Let $\lambda \subset T^*M$ be a lagrangean manifold and let $p \in \lambda$. Following Hörmander, we can choose coordinates $(x_1, \ldots, x_n)$ for M in a neighborhood of $\pi(p)$ so that in the corresponding canonical coordinates for T^*M at p, λ is transverse to the constant section of $T^*\mathbf{R}^n$ through p. Therefore λ is of the form graph df near p, where f: $(\mathbf{R}^n)^* \to \mathbf{R}$ and we identify $\mathbf{R}^n$ with $(\mathbf{R}^n)^{**}$. Define

$$\mathbf{R}^{n*} \times \mathbf{R}^n \xrightarrow{\ F\ } \mathbf{R} \times \mathbf{R}^n$$
$$\searrow \quad \swarrow$$
$$\mathbf{R}^n$$

by

$$F(\xi_1, \ldots, \xi_n, x_1, \ldots, x_n) = \left(- f(\xi_1, \ldots, \xi_n) + \sum_{i=1}^n x_i \xi_i, x_1, \ldots, x_n\right).$$

We have

$$\Sigma(F) = \left\{(\xi, x) \Big| \frac{\partial f}{\partial \xi_i} = x_i; i = 1, \ldots, n\right\}.$$

If $(\xi, x) \in \Sigma(F)$, then $\sigma(\xi, x) = \left(x, \frac{\partial F_1}{\partial x} = (x, \xi)\right)$. Therefore σ is the identity on $(\mathbf{R}^n)^* \times \mathbf{R}^n = T^*\mathbf{R}^n$. Furthermore $\Sigma(F) = $ graph $df = \lambda$, proving the proposition.

For each of the definitions of equivalence for (germs of) lagrangean manifolds, there is a definition of stability relative to a topology on the space of (germs of) lagrangean manifolds. Recall the definition of stability: If X is a topological space

and $\sim$ is an equivalence relation on X, then $x \in X$ is $\sim$-stable if it is an interior point of its $\sim$ equivalence class.

As John Mather pointed out to me, a theorem of Latour [6] applied to the family constructed in the proof of the proposition allows us to state a necessary and sufficient condition for III-stability of germs of lagrangean manifolds.

Let $\lambda \subset T^*\mathbf{R}^n$ be a lagrangean manifold passing through $(0, 0)$ so that λ is transverse to the zero section $\mathbf{R}^n \times \{0\}$ of $T^*\mathbf{R}^n$ at $(0, 0)$. Let $f: (\mathbf{R}^n)^* \to \mathbf{R}$ be a function such that $f(0) = 0$ and $\lambda = \text{graph } df$ in a neighborhood of the origin. (Here we identify $(\mathbf{R}^n)^{**}$ with $\mathbf{R}^n$ as above.) Let $\tilde{C}_0^\infty(\mathbf{R}^{n*})$ be the maximal ideal of the ring of germs of C^∞ functions at 0, and let J be the ideal generated by the germs of $\left\{\dfrac{\partial f}{\partial \xi_i}; \ i = 1, \ldots, n\right\}$. $(\xi_1, \ldots, \xi_n)$ are the coordinates of $\mathbf{R}^{n*}$. Then we have.

THEOREM (Criterion for Stability). — *The germ of λ at 0 is III-stable if and only if the germs of $\xi_1, \ldots, \xi_n$ (mod J) span $\tilde{C}(^\infty_0\mathbf{R}^{n*})/J$. λ is III-stable if and only if each germ of λ is stable.*

The proof of this theorem is an immediate application of the theorem of Latour [6], applied to the family

$$\mathbf{R}^{n*} \times \mathbf{R} \xrightarrow{\ F\ } \mathbf{R} \times \mathbf{R}^n$$
$$\searrow \quad \swarrow$$
$$\mathbf{R}^n$$

defined by $F_1(\xi, x) = -f(\xi) + \sum\limits_{i=1}^{n} \xi_i x_i$.

We next demonstrate how the criterion for stability can be used to locally identify the caustic set of a lagrangean manifold with the catastrophe set of the unfolding of a singularity. Recall the framework of catastrophes: Consider the ideal $C_0^\infty(\mathbf{R}^k)$ of C^∞ functions on $\mathbf{R}^k$ vanishing at the origin. The group of diffeomorphisms of $\mathbf{R}^k$ fixing the origin acts on $C_0^\infty(\mathbf{R}^k)$ by composition on the right. This induces an action of the group of germs of diffeomorphisms fixing the origin on the germs of functions $\tilde{C}_0^\infty(\mathbf{R}^k)$ A germ $f \in \tilde{C}_0^\infty(\mathbf{R}^k)$ has *codimension* n if there is an n-dimensional complement to

the orbit of f through f. Let $\Phi : \mathbf{R}^n \to \tilde{C}_0^\infty(\mathbf{R}^k)$ be a map transversal to the orbit of f at $\Phi(0) = f$. From Φ we construct an n-parameter family of functions

$$\mathbf{R}^k \times \mathbf{R}^n \xrightarrow{\;F\;} \mathbf{R} \times \mathbf{R}^n$$
$$\searrow \quad \swarrow$$
$$\mathbf{R}^n$$

defined by $F_1(x, t) = \Phi(t)(x)$.

F is a *universal unfolding* of f. It is universal in the sense that every other family of functions through f maps into F, and F is a minimal dimensional stable family passing through f. The *catastrophe* set of F is the set of $t \in \mathbf{R}^n$ such that $\Phi(t)$ is not stable. The catastrophe set will consist of those t satisfying either

(1) $\Phi(t)$ has a degenerate critical point, or

(2) $\Phi(t)$ has two critical points with the same critical value. The caustic set of a lagrangean manifold which we have defined corresponds to the first condition; the second condition corresponds to shock phenomena. For the purposes of this paper, we do not wish to consider condition (2) as pathological, so we modify the definition of catastrophe set.

DEFINITION. — *The c-catastrophe set of Φ is the set of $t \in \mathbf{R}^n$ such that the germ of $\Phi(t)$ is not stable for some $x \in \mathbf{R}^k$. This definition of c-catastrophe set includes precisely those points which satisfy condition* (1) *above.*

The codimension of a germ $f \in \tilde{C}_0^\infty(\mathbf{R}^k)$ can be calculated. Let J be the Jacobian ideal generated by the first partial derivatives $\left\{ \dfrac{\partial f}{\partial x_1}, \ldots, \dfrac{\partial f}{\partial x_k} \right\}$ of f. Then the codimension of f is the dimension of $\tilde{C}_0^\infty(\mathbf{R}^k)/J$ as a real vector space. Assume f has codimension n. Then a universal unfolding of f can be constructed by selecting germs $v_1, \ldots, v_n$ such that $v_1, \ldots, v_n$ (mod J) form a basis of $\tilde{C}_0^\infty(\mathbf{R}^k)/J$. The universal unfolding $F : \mathbf{R}^k \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$ is then defined by

$$F(x, t) = \left(f(x) + \sum_{i=1}^n t_i v_i(x),\ t \right).$$

Note that we may choose polynomial representatives for the ν_i.

Every catastrophe set can be constructed from the unfolding of a function $f: \mathbf{R}^k \to \mathbf{R}$ such that $f(0) = df(0) = d^2f(0) = 0$, for the proper choice of k. In this case, we may choose $\nu_1, \ldots, \nu_n$ so that $\nu_1 = x_1, \ldots, \nu_k = x_k$ in constructing the unfolding of f since $\{x_1, \ldots, x_k\}$ is linearly independent (mod J). In order to apply the criterion of stability, we want to replace f by a function $g: \mathbf{R}^n \to \mathbf{R}$ so that the universal unfolding G of g is given by $G(x, t) = (g(x) - \Sigma x_i t_i, t)$ and $C(G) = C(F)$. Then the stability of G implies that the germ of the lagrangean manifold $C(G)$ is stable.

Define $g: \mathbf{R}^n \to \mathbf{R}$ by

$$g(x_1, \ldots, x_n) = f(x_1, \ldots, x_k) - \frac{1}{2} \sum_{l=k+1}^{n} (x_l - \nu_l(x_1, \ldots, x_k))^2.$$

The local algebra $\tilde{C}_0^\infty(\mathbf{R}^n)/J(g)$ of g is isomorphic to the local algebra $\tilde{C}_0^\infty(\mathbf{R}^k)/J(f)$ of f. We shall prove that an unfolding of g is given by $G: \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R} \times \mathbf{R}^n$ defined by $G(x, t) = \left(g(x) - \sum_{i=1}^{n} x_i t_i, t \right)$, that $C(F)$ and $C(G)$ are equivalent by a fiber preserving symplectic diffeomorphism of $T^*\mathbf{R}^n$, that the catastrophe sets of F and G are the same, and that the caustic set of $C(G)$ is the catastrophe set of G.

First we prove that G is the unfolding of g. The ideal $J(g)$ is generated by $\left\{ \dfrac{\partial g}{\partial x_i}; i = 1, \ldots, n \right\}$. If $i \leqslant k$

$$\frac{\partial g}{\partial x_i} = \frac{\partial f}{\partial x_i} + \sum_{l=k+1}^{n} \frac{\partial \nu_l}{\partial x_i} (x_l - \nu_l).$$

If $i > k$,

$$\frac{\partial g}{\partial x_i} = - (x_i - \nu_i).$$

Therefore, the ideal $J(g)$ is also generated by

$$\left\{ \frac{\partial f}{\partial x_1}, \ldots, \frac{\partial f}{\partial x_k}, x_{k+1} - \nu_{k+1}, \ldots, x_n - \nu_n \right\}.$$

Recalling that $\nu_i = x_i$ if $i \leqslant k$, it is clear that $\nu_1, \ldots, \nu_n$

$(\mathrm{mod}\ J(f))$ span $\tilde{C}_0^\infty(\mathbf{R}^k)/J(f)$ if and only if $x_1,\ \ldots,\ x_n$ $(\mathrm{mod}\ J(g))$ span $\tilde{C}_0^\infty(\mathbf{R}^n)/J(g)$. Since

$$F(x,\ t) = \left(f(x) + \sum_{i=1}^n t_i\nu_i(x),\ t \right)$$

is the universal unfolding of f, G is the universal unfolding of g.

The c-catastrophe set of F is the set of t for which $f_t(x) = f(x) - \sum_{i=1}^n t_i\nu_i(x)$ has a degenerate critical point. The critical points of f_t are given by the common zeros of

$$\frac{\partial f}{\partial x_j}(x) - \sum_{i=1}^n t_i \frac{\partial \nu_i}{\partial x_j}(x), \quad j = 1,\ \ldots,\ k. \tag{5}$$

A critical point is degenerate if

$$\det\left(\frac{\partial^2 f}{\partial x_{j_1}\,\partial x_{j_2}} - \sum_{i=1}^n t_i \frac{\partial^2 \nu_i}{\partial x_{j_1}\,\partial x_{j_2}} \right)_{j_1,\,j_2} = 0 \tag{6}$$

Thus the set of t for which the $k + 1$ equations (5) and (6) have a common solution is the c-catastrophe set of F.

The c-catastrophe set of G is given similarly as the set of t for which the following $n + 1$ equations have a common solution:

$$\frac{\partial g_t}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{l=k+1}^n \frac{\partial \nu_l}{\partial x_j}(x_l - \nu_l) - t_j = 0; \quad j = 1,\ \ldots,\ k \tag{7}$$

$$\frac{\partial g_t}{\partial x_j} = -(x_j - \nu_j) - t_j; \quad j = k+1,\ \ldots,\ n \tag{8}$$

$$\det\left(\frac{\partial g_t}{\partial x_{j_1}\,\partial x_{j_2}} \right)$$

$$= \left(\frac{\dfrac{\partial f}{\partial x_{j_1}\,\partial x_{j_2}} + \sum\limits_{l=k+1}^n \left[\dfrac{\partial \nu_l}{\partial x_{j_1}} \dfrac{\partial \nu_l}{\partial x_{j_2}} + \dfrac{\partial^2 \nu_l}{\partial x_{j_1}\,\partial x_{j_2}}(x_l - \nu_l) \right] \dfrac{\partial \nu_l}{\partial x_{j_1}}}{\dfrac{\partial \nu_l}{\partial x_{j_2}}} \quad \begin{vmatrix} \dfrac{\partial \nu_l}{\partial x_{j_1}} \\[4pt] 1- \end{vmatrix} \right) = 0 \tag{9}$$

Using (8) with $t_j = -(x_j - \nu_j)$ and recalling that $\nu_j = x_j$ for $j \leqslant k$, we see that (7) and (8) together are equivalent to (5) as equations for t to satisfy. Again, substituting

$t_j = - (x_j - \nu_j)$ for $j > k$ gives

$$\left(\frac{\partial g_t}{\partial x_{j_1} \partial x_{j_2}}\right) = \left(\begin{array}{c|c} \dfrac{\partial f_t}{\partial x_{j_1} \partial x_{j_2}} & \dfrac{\partial \nu_l}{\partial x_{j_1}} \\ \hline 0 & 1 \end{array}\right) \left(\begin{array}{c|c} 1 & 0 \\ \hline \dfrac{\partial \nu_l}{\partial x_{j_2}} & -1 \end{array}\right) \qquad (10)$$

Thus (8) implies that $\det\left(\dfrac{\partial g_t}{\partial x_{j_1} \partial x_{j_2}}\right) = 0$ if and only if $\det\left(\dfrac{\partial f_t}{\partial x_{j_1} \partial x_{j_2}}\right) = 0$. We conclude that F and G have the same c-catastrophe sets.

The next step is to calculate $C(G)$ and $C(F)$. We have already calculated the value of C for a family of the form of G. $C(G)$ is the set of $(t, x) \in T^* \mathbf{R}^n$ which satisfy equations (7) and (8). To calculate $C(F)$, let $(x, t) \in \mathbf{R}^k \times \mathbf{R}^n$ be a point satisfying (5). Then

$$\sigma(x, t) = (t, - \nu_1(x), \ldots, - \nu_n(x)) \in T^* \mathbf{R}^n$$

Let $H : T^* \mathbf{R}^n \to T^* \mathbf{R}^n$ be the map defined by

$$H(t, x) = (t_1, \ldots, t_n, x_1, \ldots, x_k, - t_{k+1} + x_{k+1}, \ldots, - t_n + x_n).$$

H is a fiber preserving symplectic diffeomorphism. Comparing equations (5) with (7) and (8), we find that $H(C(F)) = C(G)$.

Finally, the caustic set of $C(G)$ is the set of points in $\mathbf{R}^n$ for which $C(G)$ is not transverse to the fibers of $T^* \mathbf{R}^n$. We have previously seen that this is the set of t for which

$$\det\left(\frac{\partial g_t}{\partial x_{j_1} \partial x_{j_2}} (x)\right) = 0$$

for some $(x, t) \in C(G)$. This is precisely the catastrophe set of F.

We summarize this discussion by stating

THEOREM. — *Let $f \in C_0^\infty(\mathbf{R}^k)$ have a single critical point at 0. Assume codimension $f = n$. Then the germ of the c-catastrophe set of a universal unfolding of f occurs as the germ of a caustic set of a III-stable lagrangean manifold of $T^* \mathbf{R}^n$.*

We end this section with some final remarks concerning II-stability. There are many interesting questions concerning this concept. Perhaps the most fundamental of these is to

give practical necessary and sufficient conditions for a lagrangean manifold of T^*M to be II-stable. In low dimensions (in particular, for dimensions less than or equal four) moduli of singularities do not arise. This raises the question whether II-stability implies III-stability for these dimensions. When does I-stability imply II-stability?

4. Solutions of P.D.E.'s.

We would like to apply the results of the last section to the solutions of a first order partial differential operator H: $T^*M \to \mathbf{R}$. Consider two examples.

Example 1. — $H: T^*\mathbf{R}^n \to \mathbf{R}$ is defined by $H(x, \xi) = \xi_n$. The classical solutions of $H = 0$ are given by functions $f(x_1, \ldots, x_n)$ which do not depend upon x_n. In other words, every solution consists of an $n - 1$ dimensional family of lines parallel to the x_n axis. Consequently, the c-catastrophe set of a universal unfolding of a singularity of codimension n will not occur as the caustic set of a solution. The caustic sets of stable solutions of $H = 0$ will be of the form $\mathbf{R} \times \Sigma$ where Σ is the caustic set of a stable lagrangean manifold of dimension $n - 1$.

Underlying this example is a theorem of Weinstein [13] which states that a fiber-preserving symplectic diffeomorphism of T^*M is affine on each fiber. In this example, the linearity of H on each fiber is an intrinsic property which is independent of the coordinate system in $\mathbf{R}^n$. The linearity of H on the fibers is highly non-generic.

Example 2. — $H(x, \xi) = x_n$. No solution of this equation is transverse to the fibers of $T^*\mathbf{R}^n$ at any of its points. More generally, if $\{H(x, \xi) = 0\}$ is not transverse to the fibers of $T^*\mathbf{R}^n$ at the point (x_0, ξ_0), then no solution passing through (x_0, ξ_0) can be transverse to the fibers of $T^*\mathbf{R}^n$ through x_0. These non-transversal points cannot, in general, be avoided as they have codimension n in $\{H = 0\}$ and our solutions have dimension n.

These two examples demonstrate that it is necessary to make assumptions about H in order to make statements

about the generic caustics of solutions of H. The first hypo-thesis we make is that 0 is not a critical value of H. We assume this throughout the remainder of this section. Note that this hypothesis, like others to be made, leaves us with a generic set of H.

The theorem which we want to prove should state that for generic H, the stable caustics of solutions of H = 0 are the stable caustics of lagrangean manifolds. The proof of such a statement is to be an application of a transversality theorem. Let us proceed to develop an appropriate setting for this problem.

Denote by L the set of parametrized proper lagrangean submanifolds of T*M diffeomorphic to an n-dimensional manifold N. There is a map

$$L \times C^{\infty}(T^*M) \xrightarrow{S} C^{\infty}(N) \qquad (11)$$

defined by $S(\lambda, H) = H \circ \lambda$. $S^{-1}(0) \cap L \times \{H\}$ is the set of solutions of H = 0. Roughly speaking, we would like to say that $S^{-1}(0)$ is transversal to the stratification of L for almost all $H \in C^{\infty}(T^*M)$. One encounters here the difficulty that all the spaces involved are infinite dimensional Frechet manifolds. However, we are primarily interested in stable n-dimensional lagrangean manifolds and these are determined by their $n + 2$ jets.

If we work locally, the quotient map $L \to \Lambda$ (see section 2) has a section. Upon restriction to Λ, the map (11) induces a map S_r

$$J^r(\Lambda) \times_{T^*M} J^r(T^*M) \xrightarrow{S_r} J^r(\mathbf{R}^n)$$
$$\downarrow$$
$$T^*M$$

The domain of S_r is the fiber product of the two factors as bundles over T*M. Now $S_r^{-1}(0) = \{(\lambda^r, H^r) | \lambda^r$ is the r-jet of a solution λ of an equation H whose r-jet is $H^r\}$. $J^r(\Lambda)$ has a stratification induced by III-equivalence, and the asser-tion we want to prove is the following :

THEOREM. — *Let* S_r *be the map defined by* (12). *Then for almost all* $H^r \in J^r(T^*M)$, $S_r^{-1}(0) \cap J^r(\Lambda) \times \{H^r\}$ *is transversal to the stratification of* $J^r(\Lambda)$ *induced by* III-*equivalence.*

Before indicating the proof of this theorem, let us compute some dimensions. Fix a point $p \in M$, and consider the map S_r restricted to T_p^*M. As a bundle over M, the dimension of the fiber of $J^r(\Lambda)$ is the same as the dimension of $J_0^{r+1}(\mathbf{R}^n) = \binom{n+r+1}{r+1} - 1$. This follows from associating the $r+1$ jet of $f \in C_0^\infty(\mathbf{R}^n)$ with the r-jet of graph df. The dimension of $J^r(\mathbf{R}^n)$ is $\binom{n+r}{r}$. Therefore, we expect that for generic fixed H, $S_r^{-1}(0) \cap J^r(\Lambda) \times \{H\}$ should have dimension $\binom{n+r+1}{r+1} - \binom{n+r}{r} - 1 = \binom{n+r-1}{r+1} - 1$ = the dimension of $J_0^{r+1}(\mathbf{R}^{n-1})$. This agrees with our prior observation that if 0 is not a critical value of H, then the space of solutions of H is a manifold modeled on $C_0^\infty(\mathbf{R}^{n-1})$.

Under our genericity hypothesis on H,

$$S_r^{-1}(0) \cap J^r(\Lambda) \times \{H\}$$

is a submanifold $S(H)$ of $J^r(\Lambda)$ of dimension $\binom{n+r-1}{r+1} - 1$. To prove the theorem, we must show that for generic H, $S(H)$ is transverse to the stratification of $J^r(\Lambda)$. We indicate two ways of proving this.

The simpler is to use a suitable version of the Thom transversality theorem. For example, one can use Lemma 3.2 of Mather [7, V]. If 0 is not a critical value of H, then the map S_r is of maximal rank at every point of $J^r(\Lambda) \times \{H\}$. The inverse image of 0 is transverse to the stratification induced by $J^r(\Lambda)$ on $J^r(\Lambda) \times T^*M$ at H. Consequently, Mather's Lemma 3.2 implies that for generic $H \in T^*M$, $S(H)$ is transverse to the stratification of $J^r(\Lambda)$.

A more interesting argument which yields more information is based upon the facts that the map S_r is a real algebraic map and that the stratification of $J^r(\Lambda)$ is real algebraic. Transversality of $S(H)$ to the stratification of $J^r(\Lambda)$ is an open algebraic condition. Thus, the set of H in $J^r(T^*M)$ for which transversality is satisfied is the complement of a closed algebraic subset of $J^r(T^*M)$. To conclude that this set is dense we need only verify that for one particular H the transversality condition is satisfied. A particularly nice choice of H

to use for this verification is the inhomogeneous characteristic
equation of the wave equation in $\mathbf{R}^n$. This is defined by

$$H(x, \xi) = |\xi|^2 - 1$$

where $|\xi|^2 = \sum_{i=1}^{n} \xi_i^2$ is the Euclidean norm. The analysis of the
solutions of $H = 0$ then becomes a study of the Riemannian
geometry of manifolds embedded in Euclidean space. A large
part of this analysis is carried out in the paper of Porteous [8]
where references to the classical literature may be found.

Summarizing, choosing $r \geqslant n + 2$ yields the following:

THEOREM. — *For generic first order differential operators*
$H : T^*M \to \mathbf{R}$, *the caustics of* III-*stable solutions of the equation*
$H = 0$ *are the c-catastrophe sets of n-dimensional, stable unfol-*
dings of functions of codimension $\leqslant n$.

It would be interesting to write down explicitly the generic
set of H for which this theorem is true.

5. The Homogeneous Case.

In this section we examine the modifications necessary to
apply the theory developed thus far to studying the singula-
rities of conic solutions of homogeneous partial differential
operators. Let Q^{n+1} be a manifold of dimension $n + 1$.
Denote the complement of the zero section in T^*Q by
$\dot{T}^*Q$.

DEFINITION. — *A lagrangean manifold* $\lambda \subset \dot{T}^*Q$ *is conic if*
$(x, \xi) \in \lambda$ *implies* $(x, c\xi) \in \lambda$ *for all* $c > 0$. *A differential*
operator $H : \dot{T}^*Q \to \mathbf{R}$ *is homogeneous if* H *is a homogeneous*
function on each fiber of $\dot{T}^*Q$. *If* $H : \dot{T}^*Q \to \mathbf{R}$ *is a homoge-*
neous differential operator, then a solution of $H = 0$ *is a conic*
lagrangean manifold lying in the hypersurface of zeros of H.

We consider the problem of describing the generic singula-
rities of solutions of homogeneous first order partial differential
operators. The first task is to determine the structure of the
space of conic lagrangean manifolds. We do this at the level of
germs. If $\lambda \subset \dot{T}^*Q$ is a conic lagrangean manifold, then we

can find fiber preserving canonical coordinates for T^*Q at $p \in \lambda$ so that λ is transverse to the constant sections of $T^*\mathbf{R}^{n+1}$ at p. If they are constructed from a coordinate system on Q, these coordinates preserve the linear structure of the fibers of T^*Q. Locally, λ can be represented as graph df where $f: (\mathbf{R}^{n+1})^* \to \mathbf{R}$ is homogeneous of degree 1. Here we identify $(\mathbf{R}^{n+1})^{**}$ and $\mathbf{R}^{n+1}$.

Conic lagrangean manifolds close to λ can also be represented locally in the form graph dg with $g: (\mathbf{R}^{n+1})^* \to \mathbf{R}$ homogeneous of degree 1. As in section 2, this leads to a representation for the set of germs of conic lagrangean manifolds at $p \in \dot{T}^*Q$ as a fibration with base $O(n)\backslash U(n)$ and fiber the cube of the maximal ideal of $\tilde{C}^\infty(\mathbf{R}^n)$.

To pass from germs to global conic lagrangean manifolds we make use of Euler's equation:

PROPOSITION. — *Let* $i : \lambda \hookrightarrow \dot{T}^*Q$ *be a lagrangean manifold. Let* ω *be the fundamental one form on* T^*Q. *Then* λ *is contained in a conic lagrangean manifold if and only if* $i^*(\omega) = 0$.

We verify this proposition in local coordinates. Assume $\lambda = \text{graph } df$ with $f: (\mathbf{R}^{n+1})^* \to \mathbf{R}$ homogeneous of degree 1. Then λ is the set of points of the form $(df(\xi), \xi)$. In the local coordinates, $\omega(x, \xi) = \sum_{i=1}^{n+1} \xi_i \, dx_i$ the tangent space of λ has a basis

$$X_i = \frac{\partial}{\partial \xi_i} + \sum_{j=1}^{n+1} \frac{\partial^2 f}{\partial \xi_i \, \partial \xi_j} \frac{\partial}{\partial x_j} \quad i = 1, \ldots, n+1.$$

We have

$$\omega(X_i) = \sum_{j=1}^{n+1} \xi_j \frac{\partial^2 f}{\partial \xi_i \, \partial \xi_j}.$$

On $\lambda, \dfrac{\partial f}{\partial \xi_i} = x_i$ and therefore λ is homogeneous if and only if $\omega(X_i) = 0$ for $i = 1, \ldots, n+1$.

The proposition implies that the conic lagrangean submanifolds of $\dot{T}^*Q$ form a submanifold of the space Λ of proper lagrangean submanifolds of $\dot{T}^*Q$. In order to make sense out of the concept of perturbation of a conic lagrangean manifold, we must be a bit careful about the topologies involved since conic lagrangean manifolds are non-compact. A conic lagran-

gean manifold induces a corresponding submanifold of the unit sphere bundle on Q. We give the space of conic lagrangean manifolds the topology induced on it as a subspace of the space of embeddings into the sphere bundle on Q.

There are at least two different interpretations of the meaning of caustic of a conic lagrangean manifold $\lambda \subset \dot{T}^*Q$. If we regard λ as a lagrangean manifold of T^*Q, then it is nowhere transverse to the fibers of T^*Q and therefore the entire projection of λ onto Q is its caustic set. On the other hand, we can define the singularities of λ to be the set of points at which the corank of the projection of λ onto Q is greater than 1. The projection of the singularities onto Q then become the caustic set. The second interpretation is more appropriate when working with homogeneous operators which arise from equation (1'). There one projects $\pi(\lambda)$ onto a submanifold M of Q of codimension 1 in such a way that the projection is a local diffeomorphism when restricted to the regular points of $\pi(\lambda)$.

The local structure of the caustics of generic solutions of a generic homogeneous first order differential equation remains unchanged from the theory developed in section 4. In particular, the caustics of III-stable solutions of generic equations are the catastrophe sets of stable unfoldings of singularities.

6. Bifurcation Theory of Gradient Vector Fields.

One can use our techniques to study a problem in the theory of bifurcation of singular points of gradient dynamical systems. Thom makes the statement that the unfolding of a potential function corresponds to the bifurcation of the corresponding gradient dynamical system [9]. Here we explore the nature of this correspondence.

Recall the relevant definitions. Let M be a smooth n-dimensional manifold with a Riemannian metric and let $f: M \to \mathbf{R}$ be a smooth function. Then grad f is the vector field defined by setting grad $f(x)$ to be the tangent vector v at x such that if $w \in T_xM$, then $df(w) = w(f) = \langle v, w \rangle$. Thus grad : $C^\infty(M) \to \chi(M)$ where $\chi(M)$ is the space of C^∞ vector fields on M.

The natural concept of geometric equivalence for vector fields is topological conjugacy. Vector fields $X, Y \in \chi(M)$ are *topologically conjugate* if there is a homeomorphism of M mapping the integral curves of X to integral curves of Y. If X is an interior point of its topological conjugacy class (usually with respect to the C' topology), then X is *structurally stable*.

For our purposes, topological conjugacy is too strong an equivalence relation [3]. Smale has defined Ω-conjugacy which is a weaker equivalence relation than topological conjugacy. For gradient vector fields with a finite number of critical points, Ω-conjugacy can be defined very simply: grad f and grad g are Ω-*conjugate* if f and g have the same number of critical points.

Bifurcation theory is the study of qualitative changes in the structure of a vector field under deformation. One approaches the subject in the following setting: A k-parameter *family* of vector fields is a map $\Phi : \mathbf{R}^k \to \chi(M)$. Given an equivalence relation $\sim$ on $\chi(M)$, $\Phi, \Psi : \mathbf{R}^k \to \chi(M)$ are said to be $\sim$ equivalent if there is a diffeomorphism (homeomorphism?) $h : \mathbf{R}^k \to \mathbf{R}^k$ such that $\Phi(h(p)) \sim \Psi(p)$. One then wishes to study a generic set of Φ and those properties which are $\sim$ stable under perturbation of Φ.

Bifurcation theory in this sense was first considered by Poincaré. Even now, however, the theory is in a primitive state. Sotomayor has studied 1-parameter families of vector fields on two-dimensional manifolds and Brunovsky has studied 1-parameter families of periodic orbits of diffeomorphisms. The variation of periodic orbits of Hamiltonian vector fields on energy surfaces has been considered by Robinson and Meyer. Little is known about higher dimensional bifurcation theory.

We deal here with a more restrictive situation and make only a few remarks about general bifurcation theory. Specifically, fix a Riemannian metric on the compact manifold M and consider the set $\chi_g(M)$ of gradient vector fields on M. As we remarked earlier grad $: C^\infty(M) \to \chi_g(M)$ is continuous and has the constant functions as kernel. Let $\Phi : \mathbf{R}^k \to \chi_g$ be a k-parameter family of gradient vector fields. We can lift Φ to a map $\hat{\Phi} : \mathbf{R}^k \to C^\infty(M)$ so that grad $\hat{\Phi} = \Phi$. $\hat{\Phi}(x)$ is determined up to a constant. The main result of this section

is the following:

PROPOSITION. — Φ is an Ω-stable family of gradient vector fields if and only if $C(\hat{\Phi})$ is a II-stable lagrangean manifold of T^*R^k.

C is the map defined in section 3.

By stable here, we mean stable within the class of families of gradient vector fields. One could ask whether such a stable Φ is stable in the larger class of families of vector fields. The following example shows that this will not always be true.

Consider the function $f : R^2 \to R$ defined by

$$f(x, y) = x^4 + y^4.$$

This presents a degenerate minimum at 0 and hence a weak source at 0 for grad f. It is an immediate consequence of the singularity theory of maps that f can be embedded in a stable family of functions. The proposition then implies that grad f can be embedded in a gradient family stable within the spase of families of gradient dynamical systems.

We have grad $f(x, y) = x^3 \dfrac{\partial}{\partial x} + y^3 \dfrac{\partial}{\partial y}$. Consider the perturbation

$$X_{\delta, \varepsilon} = (x^3 + \delta x + \varepsilon y) \frac{\partial}{\partial x} + (y^3 + \delta y - \varepsilon x) \frac{\partial}{\partial y}.$$

For $\delta = 0$, we still have a weak source of $X_{0, \varepsilon}$ at 0, but if $\varepsilon \neq 0$, it is now of codimension 1 in the space of all vector fields and belongs to the class Sotomayor calls a center-node. For $\delta < 0$ and very small, $X_{\delta, \varepsilon}$ has a small limit cycle surrounding the origin. Thus there are perturbations of grad f which show oscillatory behaviour and cannot be Ω-conjugate to any element of a family of gradient vector fields. This shows that the bifurcation theory of gradient vector fields in the class of gradient vector fields is much different from the bifurcation theory of gradient vector fields in the class of all vector fields.

We sketch the proof of the proposition. Given a generic family of functions $F : R^k \times M \to R$, define $f_y : M \to R$ by $f_y(x) = F(y, x)$. There is the map σ sending the critical points of f_y into T^*R^k by $\sigma(y, x) = (y, dF(y, x))$. The assertion that grad F and grad G are Ω-equivalent families means that

there is a diffeomorphism $h : \mathbf{R}^k \to \mathbf{R}^k$ such that grad $f_{h(y)}$ is Ω-equivalent to grad g_y. This means that there is a fiber preserving map of $C(F)$ to $C(G)$ in $T^*\mathbf{R}^k$. This is the content of the proposition.

BIBLIOGRAPHY

[1] V. I. ARNOLD, Characteristic class entering in quantization conditions, *Functional Anal. Appl.*, 1 (1967), 1-13.

[2] DARBOUX, Mémoire sur les solutions singulières des équations aux dérivées partielles du premier ordre, Mémoires de l'Institut Sav. Étrangers, (1883).

[3] J. GUCKENHEIMER, Bifurcation and catastrophe, to appear.

[4] L. HÖRMANDER, Fourier integral operators I (especially section 3. 1), Acta Mathematica, v. 127, (1971), 79-183.

[5] L. HÖRMANDER and DUISTERMAAT, Fourier integral operators II, Acta Mathematica, v. 128, (1972), 183-270.

[6] F. LATOUR, Stabilité des champs d'applications différentiables; généralisation d'un théorème de J. Mather, *C.R. Acad. Sci. Paris*, v. 268, (1969), 1331-1334.

[7] J. MATHER, Stability of mappings I - VI
    I. Annals of Mathematics, v. 87, (1968), 89-104.
    II. Annals of Mathematics, v. 89, (1969), 254-291.
    III. Publ. Math. IHES, n° 35, (1968), 127-156.
    IV. Publ. Math. IHES, n° 37, (1969), 223-248.
    V. Advances in Mathematics, v. 4, (1970), 301-336.
    VI. Proceedings of Liverpool Singularities Symposium, Lecture Notes in Math., v. 192, pp. 207-253.

[8] I. PORTEOUS, Normal singularities of submanifolds, *J. Diff. Geo.*, v. 5, (1971), 543-564.

[9] R. THOM, Stabilité Structurelle et Morphogenèse, to appear.

[10] R. THOM and H. LEVINE, Lecture notes on singularities, Proceedings of Liverpool Singularities Symposium, *Lecture Notes in Math.*, v. 192.

[11] C. T. C. WALL, Lectures on C^∞-stability and classification, Proceedings of Liverpool Singularities Symposium, *Lecture Notes in Math.*, v. 192, pp. 178-206.

[12] A. WEINSTEIN, Singularities of families of functions, Berichte aus den Mathematischen Forschungsinstitut, Band 4, (1971), 323-330.

[13] A. WEINSTEIN, Lagrangean manifolds, Advances in Math., v. 6, (1971), 329-346.

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