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J. L. DOOB

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BOUNDARY APPROACH FILTERS FOR ANALYTIC FUNCTIONS

by J. L. DOOB

1. Introduction.

Let D be the unit disk of the complex plane, with boundary C. Let H^∞ be the class of bounded holomorphic functions on D. The purpose of this paper is to discuss the filters along which the members of H^∞ have limits at C. If A is a subset of D with accumulation point 1, discussing the limits of members of H^∞ at the point 1 along A is equivalent to discussing the limits of these functions along the filter generated by the traces on A of neighborhoods of 1. It is essential to treat filters with limit 1 which are not of this type, however, for example the filter $\boldsymbol{\Gamma}_A$ corresponding to nontangential approach to 1 and the filter $\boldsymbol{\Gamma}_F$ corresponding to approach to 1 in the fine topology relative to D.

Local problem. – For what filters $\boldsymbol{\Gamma}$ of subsets of D, with limit 1, is it true that there is no coarser filter $\boldsymbol{\Gamma}_1$ with the property that every f in H^∞ with limit along $\boldsymbol{\Gamma}$ also has this limit along $\boldsymbol{\Gamma}_1$? Sufficient conditions for this kind of maximality are given in Section 19.

Global problem. – If $\boldsymbol{\Gamma}$ is a filter of subsets of D, with limit 1, and if $z \in C$, let $z\boldsymbol{\Gamma}$ be the image of $\boldsymbol{\Gamma}$ under the rotation about the center of D taking 1 into z. According to Fatou's theorem each f in H^∞ has a nontangential limit $f^*(z) = \lim_{z\Gamma_A} f$ for (Lebesgue) almost every z on C. What other filters $\boldsymbol{\Gamma}$ with limit 1 have the property that $\lim_{z\Gamma} f = f^*(z)$ for almost every z on C ? The coarser such a "Fatou" filter the better the Fatou type theorem. It is shown in Sections 22-30 that there are many Fatou filters coarser than Γ_A, in fact that in a natural topological sense nonFatou ultrafilters are exceptional. The Fatou filters, partially ordered by coarseness, form a directed set but there is no coarsest Fatou filter, that is no "best" Fatou theorem.

Some aspects of the discussion are interpreted probabilistically in Sections 20 and 21 but the problem seems to be essentially topological and algebraic rather than measure theoretic.

The work is carried through in the context of the minimal compactification $\overline{D}$ of D under which members of D have continuous extensions, that is, the maximal ideal space of the algebra H^∞. Thus on the one hand the extensive theory of this algebra is available and on the other hand the results add to the known properties of this maximal ideal space. It would however perhaps be somewhat more natural analytically to choose the minimal compactification for which positive superharmonic functions have continuous extensions.

The Maximal Ideal Space

2. Compactification of the unit disk.

The maximal ideal compactification $\overline{D}$ of D is the (unique up to homeomorphisms) Hausdorff space with the following properties :

a) $\overline{D}$ is compact ;

b) D is an open dense subset of $\overline{D}$;

c) each function f in H^∞ has a continuous extension to $\overline{D}$ and the class of these extensions separates $\overline{D}$.

See [3] for this space and its properties described in this and the next section.

If u is a harmonic function on D, bounded from above or below, with conjugate function v, either the function $\exp(u + iv)$ or its reciprocal is in H^∞, and it follows that u has a continuous extended real valued extension to $\overline{D}$.

If f is a function on D with a continuous extension to $\overline{D}$, the extension will be denoted by $\hat{f}$. If $A \subset \overline{D}$, $\overline{A}$ will denote the closure of A in $\overline{D}$ and A$'$ will denote the set $\overline{A} \cap (\overline{D} - D)$. In particular

$D' = \overline{D} - D$. The identity function on D has a continuous extension to $\overline{D}$ and the compact subset of D' on which the value of this function is 1 will be denoted by D_1. The rotation about the center of D taking 1 into z has a continuous extension taking $\overline{D}$ onto itself and the image of a set A [filter $\mathbf{\Gamma}$] will be denoted by zA [$z\mathbf{\Gamma}$].

Let f be a function from D into a first countable Hausdorff space and suppose that $\hat{f}$ exists. Then the set $A = \{z \in D_1 : \hat{f}(z) = \alpha\}$ is perfect.

3. Harmonic measure and the Šilov boundary.

The following is an outline of the material on harmonic measure and the Šilov boundary needed later. The space $L^\infty(C)$ using complex valued functions and Lebesgue measure is an algebra under ordinary multiplication. The maximal ideal space X of this algebra can be identified with the Šilov boundary of H^∞, a compact extremally disconnected subset of D'. The space $L^\infty(C)$ is algebraically and metrically isomorphic to the space C(X) of continuous complex valued functions on X with sup norm, under the following map. If $u^* \in L^\infty(C)$, the Poisson integral of u^* defines a bounded complex valued function u on D. The map $u^* \to u^s$, where u^s here and below is the restriction of $\hat{u}$ to X, is the stated map. In particular if $u^* = 1_A$ is the indicator function of a Lebesgue measurable set A, u^s is the indicator function of a subset of X, denoted by A^s, clopen relative to X. In this case, $u = \mu(\cdot\,, A)$ is the harmonic measure of A. In the general case u is given by

$$u(z) = \int_C u^*(\xi)\, \mu(z\,, d\xi) \quad , \quad z \in D. \tag{3.1}$$

If $(u^*\,, u\,, u^s)$ is a triple as just described and if $z \in \overline{D}$, the map $u^s \mapsto \hat{u}(z)$ is a linear functional on C(X) which determines a probability measure $\nu(z\,,.\,)$ on X,

$$\hat{u}(z) = \int_X u^s(\xi)\, \nu(z\,, d\xi) = \int_X \hat{u}(\xi)\, \nu(z\,, d\xi) \quad , \quad z \in \overline{D}. \tag{3.2}$$

Instead of determining the triple (u^*, u, u^s) starting from u^* one can start from any complex bounded harmonic function u on D and define u^* as its (nontangential limit) Fatou boundary function. In particular, (3.2) is valid for every u in H^∞. Moreover $\nu(z, .)$ is uniquely determined by (3.2) for u in H^∞. The measure $\nu(z, .)$ varies continuously with z (vague topology of measures on X). If $A^s \subset X$ is clopen, $\nu(., A^s)|_D = \mu(., A)$ is harmonic and has the continuous extension $\nu(., A^s) = \hat{\mu}(., A)$ to $\overline{D}$.

The classical Jensen inequality

$$\log |f(z)| \leqslant \int_C \log |f^*(\xi)| \, \mu(z, d\xi), \ z \in D, \tag{3.3}$$

for f in H^∞, yields

$$\log |\hat{f}(z)| \leqslant \int_X \log |\hat{f}(\xi)| \, \nu(z, d\xi), \ z \in \overline{D}. \tag{3.4}$$

According to (3.4), $\hat{f}(z) = 0$ if $\hat{f}$ vanishes on a subset of X having strictly positive $\nu(z, .)$ measure.

Let $S(z)$ be the compact support of the measure $\nu(z, .)$. Then $S(z) \subset X \cap wD_1$ if $z \in wD_1$ and in particular $S(z) = \{z\}$ if $z \in X$. If $z \in D' - X$, $S(z)$ is perfect, and nowhere dense in X.

Cluster Sets

4.

Let $\boldsymbol{\Gamma}$ be a filter of subsets of D, with Euclidean limit 1. Let f be a function from D into a compact Hausdorff space, and let $f(A)^-$ be the closure of $f(A)$. Then $\bigcap_{A \in \Gamma} f(A)^-$ will be called the set of $\boldsymbol{\Gamma}$ cluster values of f at 1. In particular if the range space of f is $\overline{D}$ and if f is the identity function from D onto D, the set of cluster values is a compact subset of D_1 which will be denoted by $\boldsymbol{\Gamma}'$ and the points of $\boldsymbol{\Gamma}'$ will be called $\boldsymbol{\Gamma}$ points. A point z_0 is a $\boldsymbol{\Gamma}$ point if and only if every $\overline{D}$ neighborhood of z_0 meets every member of $\boldsymbol{\Gamma}$, that is if and only if for every finite subset $f_1, \ldots, f_k$ of H^∞

and every $\varepsilon > 0$ the set $\{z \in D : |f_j(z) - \hat{f}_j(z_0)| < \varepsilon, j \leqslant k\}$ meets every member of $\mathbf{\Gamma}$, equivalently if and only if the vector function $[f_1, \ldots, f_k]$ has $\mathbf{\Gamma}$ cluster value $[\hat{f}_1(z_0), \ldots, \hat{f}_k(z_0)]$. If f is a function from D into a compact Hausdorff space, and if f has a continuous extension to $\overline{D}$, f has $\mathbf{\Gamma}$ cluster value α at 1 if and only if $\alpha \in \hat{f}(\mathbf{\Gamma}')$, and $\lim_{\Gamma} f = \alpha$ if and only if the restriction of $\hat{f}$ to $\mathbf{\Gamma}'$ is identically α.

If B is a closed subset of D_1 we denote by $\mathbf{\Gamma}(B)$ the filter of traces on D of $\overline{D}$ neighborhoods of B. Then $B = \mathbf{\Gamma}(B)'$. If $\mathbf{\Gamma}$ is a filter of subsets of D with Euclidean limit 1, $\mathbf{\Gamma}(\mathbf{\Gamma}') \subset \mathbf{\Gamma}$, and the inclusion may be strict. Thus it may be a stronger statement to say that a function on D has a limit along $\mathbf{\Gamma}(\mathbf{\Gamma}')$ than that the function has a limit along $\mathbf{\Gamma}$. If the range space of the function is a compact Hausdorff space and if the function has a continuous extension to $\overline{D}$ the assertions are equivalent, however, because both assert that the extension is constant on $\mathbf{\Gamma}'$.

5. Convergence stable filters.

A filter $\mathbf{\Gamma}$ of subsets of D with Euclidean limit 1 will be called convergence stable if whenever $\{A_n, n \geqslant 1\}$ is a sequence of sets in Γ there is a sequence $\{G_n, n \geqslant 1\}$ of disks of center 1, so small that $A = \bigcap_n \{A_n \cup (D \setminus G_n)\} \in \mathbf{\Gamma}$.

Proposition. – *Let $\mathbf{\Gamma}$ be a filter of subsets of* D *with Euclidean limit 1. Then $\mathbf{\Gamma}$ is convergence stable if and only if whenever f is a function from* D *into a first countable Hausdorff space, the existence of* $\lim_{\Gamma} f = \alpha$ *implies the existence of a member* A *of $\mathbf{\Gamma}$ for which f has limit α at 1 along A.*

Suppose that $\mathbf{\Gamma}$ is convergence stable and that $\lim_{\Gamma} f = \alpha$. Let $\{G_n^*, n \geqslant 1\}$ be a basis for the neighborhoods of α, and define $A_n = f^{-1}(G_n^*)$. Then $A_n \in \mathbf{\Gamma}$ and if $\{G_n, n \geqslant 1\}$ satisfies the conditions of the convergence stable definition, $f(A \cap G_n) \subset G_n^*$. Hence f has limit α at 1 along A. Conversely suppose that $A_n \in \mathbf{\Gamma}, n \geqslant 1$. We wish to find G_n with the properties stated in the convergence stable definition (and we shall need only numerical valued functions). Define

$$B_n = \bigcap_1^n A_k - \bigcap_1^{n+1} A_k \quad \text{and} \quad f = \sum_1^\infty 2^{-n} 1_{B_n} + 1_{D-A_1}.$$

Then $$\{z : f(z) \leqslant 2^{-n}\} = \bigcap_1^n A_k \in \boldsymbol{\Gamma}$$

so $\lim\limits_{\Gamma} f = 0$ and under the hypotheses of the converse f has limit 0 along some set A in $\boldsymbol{\Gamma}$. If G_n is a disk with center 1, so small that $f \leqslant 2^{-n}$ on $A \cap G_n$ then

$$A \cap G_n \subset \bigcap_1^n A_k \subset A_n \quad \text{and} \quad A \subset \bigcap_1^\infty \{A_n \cup (D - G_n)\}.$$

Hence the intersection on the right is in $\boldsymbol{\Gamma}$, as was to be proved.

6. Cluster stable filters.

A filter of subsets of D with Euclidean limit 1 will be called cluster stable if whenever $A_1 \supset A_2 \supset \ldots$ are subsets of D, each of which meets every member of $\boldsymbol{\Gamma}$, there is a sequence $\{G_n, n \geqslant 1\}$ of disks of center 1 so small that $\bigcap\limits_n \{A_n \cup (D - G_n)\}$ meets every member of $\boldsymbol{\Gamma}$.

PROPOSITION. – *Let* $\boldsymbol{\Gamma}$ *be a filter of subsets of* D *with Euclidean limit 1. Then* $\boldsymbol{\Gamma}$ *is cluster stable if and only if whenever f is a function from* D *into a first countable Hausdorff space which has* $\boldsymbol{\Gamma}$ *cluster value* α, *there is a subset* A *of* D, *meeting every member of* $\boldsymbol{\Gamma}$, *along which f has limit* α *at 1.*

If $\boldsymbol{\Gamma}$ is cluster stable and f has $\boldsymbol{\Gamma}$ cluster value α let $\{G_n^*, n \geqslant 1\}$ be a basis for the neighborhoods of α, with $G_1^* \supset G_2^* \supset \ldots$ and define $A_n = f^{-1}(G_n^*)$. Then $A_1 \supset A_2 \supset \ldots$, A_n meets every member of $\boldsymbol{\Gamma}$, and if $G_1, G_2, \ldots$ satisfy the conditions of the cluster stable definition, $f(A \cap G_k) \subset G_k^*$. Thus f has limit α at 1 along A. Conversely suppose that A_n is a subset of D which meets every element of $\boldsymbol{\Gamma}$ and that $A_1 \supset A_2 \supset \ldots$. Define

$$f = \sum_{1}^{\infty} 2^{-n} 1_{A_n - A_{n+1}} + 1_{A - A_1},$$

so that f has $\boldsymbol{\Gamma}$ cluster value 0 at 1. Under the hypotheses of the converse there is a set $A \subset D$ which meets every member of $\boldsymbol{\Gamma}$ along which f has limit 0 at the point 1. If G_n is a disk with center 1, so small that $f \leqslant 2^{-n}$ on $A \cap G_n$ then

$$A \cap G_n \subset A_n \quad \text{and} \quad A \subset \bigcap_{1}^{\infty} \{A_n \cup (D - G_n)\}$$

so the intersection on the right meets every member of $\boldsymbol{\Gamma}$, as was to be proved. As in Proposition 6, the converse proof only involved numerically valued functions.

7. A separation theorem.

If u is a harmonic function on D, it is the Poisson integral of a (signed) measure if and only if it is the difference between two positive harmonic functions. We suppose in the following theorems that the Poisson measure on C for u assigns measure 0 to the point 1, but it is not clear whether or not this hypothesis is necessary. The hypothesis is satisfied if u is bounded because the measure is then absolutely continuous with respect to Lebesgue measure on C.

THEOREM. — *Let u be a harmonic function on D, with*

$$a = \inf_{D} u > -\infty,$$

whose Poisson measure on C *assigns measure* 0 *to the point* 1. *Suppose* B *is a subset of* D, *with accumulation point* 1, *and that u has limit a at* 1 *along* B. *If* $A = \{z \in D_1 : \hat{u}(z) > a\}^-$ *then* $A \cap B' = \emptyset$ *and there is a positive harmonic function v on* D *with* $\hat{v}(B') = 0$, $\underset{\wedge}{v}(A) = +\infty$.

It is sufficient to prove that there is a function v as described in the theorem. We can assume that $a = 0$. Let $G_1 \supset G_2 \ldots$ be open disks of center 1 so small that $u < 2^{-n}$ on $B \cap G_n$. The function u is the Poisson integral of a measure. Let u_n $(\leqslant u)$ be the Poisson integral of the part of this measure in a neighborhood of the point 1, choosing this neighborhood so small that $u_n < 2^{-n}$ outside G_n. Define $\nu = \sum_1^\infty u_n$. Then ν is either harmonic or identically $+\infty$. On

$$B \cap (G_k - G_{k+1}) \quad , \quad \nu \leqslant k\,2^{-k} + \sum_{k+1}^{\infty} 2^{-n} \ ,$$

so that ν is harmonic, has limit 0 at the point 1 along B, and $\hat{\nu}(B') = 0$. Since the restrictions to D_1 of $\hat{u}$ and $\hat{u}_n$ are identical,

$$\hat{\nu}(z) \geqslant \sum_1^k \hat{u}_n(z) = k\hat{u}(z)$$

for z in A. Thus $\hat{\nu}(\overline{A}) = \hat{\nu}(A) = +\infty$ and the proof is complete.

Nontangential and Tangential Cluster Values

8. The nontangential filter $\boldsymbol{\Gamma}_A$.

If $w \in C$, the open connected subset of D cut off by two rays into D from W will be called a Stolz angle with vertex w. A subset of D which, for each Stolz angle A with vertex w, contains the part of A in a sufficiently small disk with center w, will be called a deleted nontangential neighborhood of w. The filter of these deleted nontangential neighborhoods will be denoted by $w\,\boldsymbol{\Gamma}_A$, or by $\boldsymbol{\Gamma}_A$ if $w = 1$. From now on it will be convenient to phrase all definitions and theorems for $w = 1$. It is trivial that a function f from D into a topological space has a limit along $\boldsymbol{\Gamma}_A$ if and only if f has that limit at 1 along every Stolz angle with vertex 1.

A subset A of D which meets every member of $\boldsymbol{\Gamma}_A$ must contain a sequence converging to 1 in a Stolz angle (or $D - A \in \boldsymbol{\Gamma}_A$ even though A does not meet $D - A$).

The filter $\boldsymbol{\Gamma}_A$ is convergence stable. In fact if $A_1, A_2, \ldots$ are in $\boldsymbol{\Gamma}_A$, let K_n be the complement of an open disk G_n with center the point 1 and radius so small that $A_n \cup K_n$ contains the symmetric Stolz angle with vertex 1 and angular measure $\pi - n^{-1}$. Then

$$\bigcap_n (A_n \cup K_n) \in \boldsymbol{\Gamma}_A .$$

The filter $\boldsymbol{\Gamma}_A$ is not cluster stable. To see this let L_n be the ray from the point 1 into D making the angle $\pi/2 - 1/n$ with the radius to 1 and let L_n° be the set of points on L_n at distances 1, 1/2, 1/3, ... from the point 1. Define $A_n = \bigcup_{k \geq n} L_k^\circ$. Then $A_1 \supset A_2 \supset \ldots$ and A_n meets every member of $\boldsymbol{\Gamma}_A$. If G_k is an open disk with center the point 1 and if

$$A = \bigcap_n \{A_n \cup (D \setminus G_n)\},$$

then $A \cap G_n \subset A_n \cap G_n$. We conclude that A contains no sequence converging to the point 1 in a Stolz angle, that therefore A does not meet every member of $\boldsymbol{\Gamma}_A$, as it would have to for proper choice of $\{G_n\}$ if $\boldsymbol{\Gamma}_A$ were cluster stable.

9. Stolz and nontangential points of D_1.

If A is a Stolz angle with vertex the point 1 and if $\boldsymbol{\Gamma}$ is the filter of traces on A of Euclidean neighborhoods of the point 1, the $\boldsymbol{\Gamma}$ cluster set of a function f will be called the cluster set of f at 1 along A and the union of these sets for all A will be called the Stolz cluster set of f at 1. In particular if f is the identity function from D into $\overline{D}$ the Stolz cluster set at 1 will be called the set of Stolz points of D_1.

Define the harmonic function u on D by $u(z) = \arg(1 - z)$ (branch with values between $-\pi/2$ and $\pi/2$). If z_0 is a Stolz point of D_1 the set $\{z \in D_1 \ : \ |u(z) - u(z_0)| < \varepsilon\}$ is an open (relative to D_1) set of Stolz points of D_1, for small ε. Thus the set of Stolz points of D_1 is open relative to D_1.

The $\boldsymbol{\Gamma}_A$ cluster values of a function f on D will be called the nontangential cluster values of f at 1, and the points of $\boldsymbol{\Gamma}'_A$ will be called the nontangential points of D_1.

PROPOSITION. – *If f is a function from* D *to a compact Hausdorff space, the set of nontangential cluster values of f at the point 1 is the closure of the set of Stolz cluster values there. In particular* Γ'_A *is the closure of the set of Stolz points of* D_1.

In fact if α is a nontangential cluster value of f at 1 but is not in the closure of the set of Stolz cluster values there, some closed $\overline{\mathrm{D}}$ neighborhood G of α contains no Stolz cluster value of f. Hence the intersection of $f^{-1}(\mathrm{G})$ with a Stolz angle of vertex 1 contains no point sufficiently near 1. That is $\mathrm{D} - f^{-1}(\mathrm{G})$ is a deleted nontangential neighborhood of 1, and α cannot be a nontangential cluster value of f at 1, contrary to hypothesis.

10. Tangential points.

Let u be defined as in Section 9. A subset B of D will be said to be tangent to C at the point 1 if $|u|$ has limit $\pi/2$ at 1 along B, equivalently if $\mathrm{D} - \mathrm{B}$ is a deleted nontangential neighborhood of the point 1. A point of D_1 will be called tangential if it is in B' for some set B tangent to C at 1. Trivially $|\hat{u}|$ has the value $\pi/2$ at every tangential point of D_1, but also at every nontangential point which is not a Stolz point. The following proposition justifies the nomenclature.

PROPOSITION. – *Each point of* D_1 *is either nontangential or tangential, never both.*

If a point z of D_1 is not in the nontangential set $\boldsymbol{\Gamma}'_{\mathrm{A}}$, some closed $\overline{\mathrm{D}}$ neighborhood B of z does not meet $\boldsymbol{\Gamma}'_{\mathrm{A}}$, that is $\mathrm{D} - \mathrm{B}$ is a deleted nontangential neighborhood of the point 1. But then B is tangent to C at 1, and z is tangential. There remains the proof that no tangential point z of D_1 is in $\boldsymbol{\Gamma}'_{\mathrm{A}}$. Suppose that z is tangential, that is that z is in B' for some subset B of D, tangent to C at the point 1. We can suppose that B is tangent only on one side, say from above, so that, defining u as in Section 9, u has limit $-(\pi/2) = \inf\limits_{\mathrm{D}} u$ at 1 along B. If Theorem 7 is applied we find, using the notation of that theorem, that B' and $\boldsymbol{\Gamma}'_{\mathrm{A}} \subset \overline{\mathrm{A}}$ are disjoint. Hence z is not in Γ'_{A}, as was to be proved.

This proof shows what is obvious from simpler considerations, that no point of $\boldsymbol{\Gamma}'_{\mathrm{A}}$ is a $\overline{\mathrm{D}}$ accumulation point of $\mathrm{D}' - \mathrm{D}_1$.

THEOREM. – *If* A *is a deleted nontangential neighborhood of the point* 1, $A \cup D_1$ *is a* $\bar{D}$ *neighborhood of* $\mathbf{\Gamma}'_A$. *Hence* $\mathbf{\Gamma}_A = \mathbf{\Gamma}(\mathbf{\Gamma}'_A)$.

The theorem is an immediate consequence of the fact that $(D - A)'$ is tangential so does not meet $\mathbf{\Gamma}'_A$ and, as remarked above, $D' - D_1$ has no nontangential accumulation point.

This theorem, together with the fact that $\mathbf{\Gamma}_A$ is convergence stable, implies for example that if f in H^∞ has a nontangential limit c at the point 1, that is if $\hat{f}(\mathbf{\Gamma}'_A) = c$, then f has limit c along the trace on D of a $\bar{D}$ neighborhood of $\mathbf{\Gamma}'_A$ and thus that $\hat{f}$ is identically c on a neighborhood relative to D_1 of $\mathbf{\Gamma}'_A$.

Fine Cluster Values

12. Fine points.

The filter of deleted fine neighborhoods of the point 1 relative to D, that is of complements in D of subsets of D thin at 1 relative to D (sometimes called "minimally thin" at 1) will be denoted by $\mathbf{\Gamma}_F$. The $w\mathbf{\Gamma}_F$ cluster values of a function defined on D will be called fine cluster values of the function at w and the points of $w\mathbf{\Gamma}'_F$ will be called fine points of wD_1. From now on we always take $w = 1$.

Define $f = \exp(z + 1)/(z - 1)$. Then $h = -\log|f|$ is a minimal harmonic function for D corresponding to the boundary point 1. It is trivial that h has Euclidean limit 0 on C except at the point 1, and it is well known that h has fine limit ∞ at that point. Then f has Fatou nontangential boundary function f^* of modulus 1 (almost everywhere) but has fine limit 0 at the point 1. Now if z is a Šilov point in D_1 it is known [3] that f^* is arbitrarily close to $\hat{f}(z)$ on a subset of C whose trace on each neighborhood of the point 1 has strictly positive Lebesgue measure. Hence no fine point of D_1 is on the Šilov boundary.

It is a standard fact of classical potential theory that $\mathbf{\Gamma}_F$ is both convergence and cluster stable.

The relation $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'_{\mathrm{F}}) \subset \boldsymbol{\Gamma}_{\mathrm{F}}$ is strict. For example the complement in D of a countable dense set is in Γ_{F} but not in $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'_{\mathrm{F}})$.

13.

PROPOSITION. – *The sets* $\boldsymbol{\Gamma}'_{\mathrm{F}}$ *and* $\boldsymbol{\Gamma}'_{\mathrm{A}}$ *are connected.*

We give the proof for $\boldsymbol{\Gamma}'_{\mathrm{F}}$. The proof for $\boldsymbol{\Gamma}'_{\mathrm{A}}$ is similar but simpler. Let B be a $\overline{\mathrm{D}}$ open superset of $\boldsymbol{\Gamma}'_{\mathrm{F}}$. Then $\boldsymbol{\Gamma}'_{\mathrm{F}} = \bigcap_{\mathrm{B}} \overline{\mathrm{B}}$. Now $\mathrm{B} \cap \mathrm{D}$ is a deleted fine neighborhood of the point 1, so by a theorem of Naïm-Lumer [5] some (unique) open component $\mathrm{B}_0(\mathrm{B})$ of B is also a deleted fine neighborhood of the point 1. Then $\mathrm{B}_1 = \bigcap_{\mathrm{B}} \overline{\mathrm{B}_0(\mathrm{B})} \subset \Gamma'_{\mathrm{F}}$. Since $\overline{\mathrm{B}}_0$ is connected and since the class of sets $\overline{\mathrm{B}}_0$ when ordered by inclusion is a directed decreasing class, B_1 is connected. The set $\boldsymbol{\Gamma}'_{\mathrm{F}}$ was defined as the intersection of the $\overline{\mathrm{D}}$ closures of the deleted fine neighborhoods of the point 1. Hence $\boldsymbol{\Gamma}'_{\mathrm{F}} \subset \mathrm{B}_1$. We have already proved the reverse inclusion, so $\mathrm{B}_1 = \boldsymbol{\Gamma}'_{\mathrm{F}}$ and $\boldsymbol{\Gamma}'_{\mathrm{F}}$ is connected.

14.

THEOREM. – $\boldsymbol{\Gamma}'_{\mathrm{A}} \subset \boldsymbol{\Gamma}'_{\mathrm{F}}$ *and the inclusion is strict.*

Since $\boldsymbol{\Gamma}'_{\mathrm{A}}$ is the closure of the set of Stolz points in D_1, to prove the stated inclusion it is sufficient to prove that every Stolz point of D_1 is a fine point. If z is a Stolz point for the Stolz angle S, and if $f_1, \ldots, f_k$ are in H^∞, there is a sequence $\{z_n, n \geqslant 1\}$ in S along which $[f_1, \ldots, f_k]$ has the limit $[\hat{f}_1(\hat{z}), \ldots, \hat{f}_k(\hat{z})]$. Let c be a strictly positive constant. If G_n is a disk of radius $c\,|z_n - 1|$ and center z_n, f_j is uniformly (as n varies) near $f_j(z_n)$ in G_n when c is small, by a standard normal family argument. Since for each $c > 0$ every deleted fine neighborhood of the point 1 meets $\bigcup_n \mathrm{G}_n$ [1], $[f_1, \ldots, f_k]$ has fine cluster value $[\hat{f}_1(z), \ldots, \hat{f}_k(z)]$ at the point 1 and according to Section 4 this condition implies that z is a fine point. We have now proved that $\boldsymbol{\Gamma}'_{\mathrm{A}} \subset \boldsymbol{\Gamma}'_{\mathrm{F}}$. To prove that the conclusion is strict we need merely remark that $\boldsymbol{\Gamma}'_{\mathrm{F}}$ contains tangential points. This fact can be seen for example as follows. We must show that there is a set B in D tangent to C at the point 1, with $B' \cap \boldsymbol{\Gamma}'_F \neq \emptyset$. It is convenient to replace D by the upper half plane. Then [1] if ϕ is monotone increasing on $(0, 1)$, with $\phi(0+) = 0$, the set $\{[x, \phi(x)], 0 < x < 1\}$ is thin at the origin or, equivalently, the set $\{(x, y) : 0 < x < 1, y \leqslant \phi(x)\}$ is thin at the origin, if and only if $\int_0^1 \phi(x) x^{-2}\, dx < \infty$. We need only define $\phi(x) = x/|\log x|$ say, to find a set which is tangent to the boundary at the origin and which has fine cluster points.

It is easy to check that if $\boldsymbol{\Gamma}_{AF} = \boldsymbol{\Gamma}_A \cap \boldsymbol{\Gamma}_F$ then $\boldsymbol{\Gamma}'_{AF} = \boldsymbol{\Gamma}'_F$ and the inclusion relations $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'_F) \subset \boldsymbol{\Gamma}_{AF} \subset \boldsymbol{\Gamma}_A$ are strict. Thus the assertion that a function on D have at the point 1 a nontangential limit, both a nontangential and fine limit, a limit along $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'_F)$ are successively strictly stronger, but the latter two are actually equivalent for functions with continuous extensions to $\overline{D}$.

15. Example.

We shall give an example of a function in H^∞ which has a nontangential but not a fine limit at a point of C. This example provides a second proof that $\boldsymbol{\Gamma}'_A \neq \boldsymbol{\Gamma}'_F$. The existence of such an example was announced in [1]. It is sufficient to exhibit a positive harmonic function u with nontangential limit ∞ at a boundary point but not with a fine limit there. In fact if v is a conjugate function of u the function $\exp(-u - iv)$ is the required element of H^∞. Going to the upper half plane we shall exhibit a positive harmonic function u there with nontangential limit ∞ at the origin but with limit 0 along a continuous arc A to the origin, where A is not thin at the origin. The arc A is in the first quadrant, with equation

$$y = x(-\log x)^{-1}, \quad 0 < x < 1, \tag{15.1}$$

and u is defined in (15.2) as the Poisson integral of a function with limit $+\infty$ at the origin from the left, vanishing on the right, so that u has nontangential limit $+\infty$ at the origin,

$$u(x, y) = y \int_{-1}^{0} \frac{(-\log|t|)^{1/2}\, dt}{(x-t)^2 + y^2}. \tag{15.2}$$

If $(x\ ,\ y) \in \mathrm{A}$,

$$u(x\ ,\ y) \leqslant (-\ x \log x)^{-1} \int_0^x (-\log t)^{1/2}\, dt$$
$$+\ x(-\log x)^{-1/2} \int_x^1 t^{-2}\, dt \leqslant 2\,(-\log x)^{-1/2}.$$

Hence u has limit 0 along A. Finally A is not thin at the origin according to the criterion already used in Section 14.

16. Thin sets.

Even if a subset A of D is thin at the point 1 the cluster set A′ may contain fine points of D_1. For example a sequence of points of D converging nontangentially to the point 1 is thin at 1 but its cluster set in D_1 consists of points in $\Gamma'_{\mathrm{A}} \subset \Gamma'_{\mathrm{F}}$. However if A is sufficiently strongly tangential to C at the point 1, A′ will not contain any fine points of D_1. We shall state conditions that A′ contain fine points using the upper halfplane instead of the disk. We suppose that D has been mapped onto the upper halfplane by a linear transformation taking 1 into 0, and it should cause no misunderstanding if we keep the notation used for D wherever possible. That is, in the halfplane context Γ'_{F} is the set of fine points over the Euclidean boundary point 0 and so on.

THEOREM. – *Let ϕ be a function from an interval* $(0\ ,\ \delta)$ *to the positive reals, monotone increasing, with* $\phi(0\ +) = 0$ *and*

$$\lim_{x \to 0} \phi(x)/x = 0.$$

Let G *be the graph* $\{(x\ ,y)\ :\ y = \phi(x)\ , 0 < x < \delta\}$, *and define* $\mathrm{A} = \{(x\ ,\ y)\ :\ 0 < x < \delta\ ,\ y \leqslant \phi(x)\}$.

a) *If* $\int_0^\delta \phi(x)\, x^{-2}\ dx\ = \infty$, G *is not thin at the origin (relative to the upper halfplane) and, if ϕ is continuous,* G′ *contains both fine and nonfine points.*

b) *If* $\int_0^\delta \phi(x)\, x^{-2} < \infty$, A *is thin at the origin (relative to the upper halfplane).*

The only assertion not covered by the already used criterion for thinness is the assertion about G' in a). If the integral is infinite, G is not thin at the origin, so G' must contain some fine points. On the other hand we shall show that G' cannot be a subset of $\mathbf{\Gamma}'_F$. In fact, translating the context back to the disk, if $G' \subset \mathbf{\Gamma}'_F$ a function f in H^∞ has a limit at w on C along wG whenever the function has a fine limit at w. Since f has a fine limit at w for Lebesgue almost every w (fine topology Fatou theorem), f has a limit at w along wG for almost every w. According to a theorem of Littlewood [8] this is impossible for a continuous curve G, tangent to C at the point 1, so G' must contain nonfine points.

It seems plausible that A' in the theorem contains no fine points over the origin when A is thin but the author was able to prove this result only under additional restrictions on ϕ, as indicated in the following theorem.

17.

THEOREM. – *Let ϕ be a function from an interval $(0, \delta)$ to the positive reals, satisfying the following conditions.*

a) *ϕ and $x \mapsto \phi(x)/x$ are continuous and monotone increasing, with limit* 0 *at the origin.*

b) *For some $\varepsilon > 0$, $x \mapsto \phi(x)\, x^{\varepsilon - 2}$ is monotone decreasing, with limit ∞ at the origin.*

c) $\int_0^\delta \phi(x)\, x^{-2}\, dx < \infty$.

Then the set $A = \{(x, y) : 0 < x < \delta,\ 0 < y \leqslant \phi(x)\}$ is thin at the origin relative to the upper halfplane and A' contains no fine point over the origin.

According to this theorem, the closure in the Gelfand compactification of the complement of A in the open halfplane is a neighborhood of Γ'_F. We need prove only that A' contains no fine point over the origin. Note that hypotheses a) and b) imply that $\varepsilon < 1$. Let u be the Poisson integral for the upper halfplane of the function $t \mapsto -t/\phi(-t)$ on $(-\delta, 0)$ (vanishing elsewhere),

$$u(x,y)=\int_0^{\delta} \frac{tydt}{\phi(t)\,[(x+t)^2+y^2]}. \tag{17.1}$$

If $(x,y) \in A$ we use a) to find

$$u(x,y) \leqslant \phi(x)\int_0^x \frac{t\,dt}{\phi(t)\,x^2} + x\int_x^{\delta} \frac{dt}{(x+t)^2} \tag{17.2}$$

and then using b)

$$u(x,y) \leqslant 1/\varepsilon + 1/2 < 2/\varepsilon. \tag{17.3}$$

Define $B_b = \{(x,y) : 0<x<\delta,\ y>b\phi(x)\}$, where b is a strictly positive number to be chosen later. If $y = b\phi(x)$,

$$u(x,y) \geqslant bx\int_0^x \frac{dt}{(x+t)^2+b^2\phi(x)^2} > b/2 \tag{17.4}$$

if x is near 0. Now choose $b > 4/\varepsilon$, and let u_1 be the restriction of u to the following domain B :

$$B = \{(x,y) : x \leqslant 0,\ y>0\} \cup \{(x,y) : x>0,\ y>b\phi(x)\}.$$

The function u_1 is positive, harmonic, and has a continuous boundary function in a deleted neighborhood of the origin. At the origin this boundary function has limit $+\infty$ along the half axis and, by (17.4), limit inferior $\geqslant b/2$ along the arc on the other side. Then u_1 and u have inferior limit $\geqslant b/2 > 2/\varepsilon$ at the origin along B. Moreover, by the criterion already used, B is a deleted fine neighborhood of the origin, relative to the upper halfplane. Hence $\hat{u} > 2/\varepsilon$ on the set of fine points over the origin. On the other hand, by (17.3), $\hat{u} \leqslant 2/\varepsilon$ at the points of A′, and we conclude that A′ contains no fine points.

18. Example.

Define $\log_n y = \log \ldots \log y$ as the *nth* iterated logarithm of y for y so large, say $y > 1/\delta_n$, that the logarithms involved are all well defined and positive. Define

$$A(n,\varepsilon) = \{(x,y) : 0<x<\delta_n,\ 0<y\leqslant x\,[(\log 1/x)\ldots (\log_{n-1} 1/x)]^{-1}(\log_n 1/x)^{-1-\varepsilon}\}.$$

If $\varepsilon > 0$, $A(n, \varepsilon)$ is thin at the origin relative to the upper halfplane and $A(n, \varepsilon)$ contains no fine point over the origin. But $A(n, 0)$ is not thin at the origin and $A(n, \varepsilon)'$ contains fine points over the origin.

Extreme Filters and Sets

19. L minimal filters ; L maximal sets.

Let $\mathbf{\Gamma}$ be a filter of subsets of D, with Euclidean limit 1. Then $\mathbf{\Gamma}$ will be called L minimal if there is no strictly coarser filter $\mathbf{\Gamma}_1$ with Euclidean limit 1 for which $\lim_{\Gamma_1} f$ exists whenever $f \in H^\infty$ and $\lim_{\Gamma} f$ exists. Since f in H^∞ has a limit along Γ if and only if f has a limit along $\mathbf{\Gamma}(\Gamma')$, minimality of $\mathbf{\Gamma}$ implies that $\Gamma(\Gamma') = \mathbf{\Gamma}$.

A subset A of D_1 will be called L maximal if there is no strictly larger subset A_1 of D_1 (equivalently of $\overline{D}$) for which f is identically constant on A_1 whenever $f \in H^\infty$ and f is identically constant on A. If A is L maximal it is closed. A filter $\mathbf{\Gamma}$ of subsets of D, converging to 1, is L minimal if and only if $\mathbf{\Gamma} = \mathbf{\Gamma}(\mathbf{\Gamma}')$ and $\mathbf{\Gamma}'$ is L maximal. Since the extensions to $\overline{D}$ of the members of H^∞ separate $\overline{D}$ each singleton subset of D_1 is L maximal.

Let A be any subset of D_1 and consider the set A^L of all points z in D_1 such that, whenever f is in H^∞, $\hat{f}(A) = 0$ implies that $\hat{f}(z) = 0$. Then A^L is the smallest L maximal superset of A. If f in H^∞ is identically constant on $A \subset D_1$ then f is even identically constant on A^L.

The intersection of an arbitrary number (finite or infinite) of L maximal sets is trivially L maximal. The union of a finite number of L maximal sets is L maximal. In fact if $A_1, \ldots, A_n$ are L maximal and if z is not in their union there is a function f_j in H^∞ for which $\hat{f}_j(A_j) = 0$ but $\hat{f}_j(z) \neq 0$. The function $\prod_j f_j = g$ has the property that $\hat{g}(\bigcup_j A_j) = 0$ but $\hat{g}(z) \neq 0$ from which it follows that $\bigcup_j A_j$ is L maximal. In particular, finite subsets of D_1 are L maximal.

THEOREM. – *Let u be a bounded harmonic function on* D, *with* $a = \inf u < b = \sup u$. *Then each of the sets*

$\{z \in D_1 : a < \hat{u}(z)\}^-$, $\{z \in D_1 : \hat{u}(z) < b\}^-$, $\{z \in D_1 : a < \hat{u}(z) < b\}^-$

is L *maximal or empty.*

It is sufficient to prove that the first set is L maximal, because the second set is reduced to one like the first if u is replaced by $-u$, and the third set is the intersection of the first two. If the first set, denoted by A, is D_1 itself, there is nothing to prove. Otherwise let z be in $D_1 - A$. It will be sufficient to prove that there is a member of H^∞ vanishing identically on A but not vanishing at z. Since $\hat{u}(z) = a$ Theorem 7, with B the trace on D of a closed neighborhood of z not meeting A, can be applied to yield a positive harmonic function ν with $\hat{\nu}(z) = 0$, $\hat{\nu}(A) = \infty$. If ν_1 is conjugate to ν, $\exp(-\nu - i\nu_1)$ is the desired function in H^∞.

This same argument shows that if u is harmonic on D with finite infimum a, and if u is the Poisson integral of a measure on C assigning the value 0 to the singleton $\{1\}$ then the first of the three sets in the theorem is L maximal or empty.

Rosenfeld and Weiss [7] proved a theorem equivalent to Theorem 19 with u the harmonic measure of a subset of C.

If $u = \arg(1-z)$ (branch with values in $(-\pi, \pi)$) the theorem implies that $\mathbf{\Gamma}'_A$ is L maximal, that is $\mathbf{\Gamma}_A$ is L minimal. If f in H^∞ has a radial limit at 1 it is classical that f has a nontangential limit at that point. Thus the cluster set in D_1 of the radius to 1 is not L maximal.

If A is an L maximal set and if z is a point of D_1 with $\nu(z, A) > 0$ then $z \in A$ because (see Section 3) if, for f in H^∞, $\hat{f}$ vanishes identically on A the function also vanishes at z. The Šilov set $X \cap D_1$ is therefore not L maximal.

Brownian Paths

20. Brownian cluster points.

We recall that (conditional) Brownian paths from a point of D to the point 1 are continuous paths and that a subset B of D is thin at 1 relative to D if and only if there is a Borel superset B_0 of B with the property that almost no Brownian path from a point of D to 1 hits B_0 sufficiently near 1. This property is independent of the initial path point. The cluster set in D_1 of a Brownian path from a point of D to the point 1 is compact and connected. Since almost every such path hits every ray from 1 into D arbitrarily near 1, the cluster set in D_1 contains many Stolz points. It will be shown below that this cluster set also contains fine tangential points.

LEMMA. – *A point z of D_1 is in Γ'_F if and only if whenever $z_0 \in D$ and $f_1, \ldots, f_k$ is a finite subset of H^∞ the vector function $[f_1, \ldots, f_k]$ has cluster value $[\hat{f}_1(z), \ldots, \hat{f}_k(z)]$ on almost every conditional Brownian path from z_0 to 1. If $f_1, \ldots, f_k$ are specified, the condition is satisfied either for every z_0 or no z_0.*

The last assertion is true because if h is minimal harmonic in D corresponding to the point 1, the probability that a conditional Brownian path from z_0 to 1 hits a specified open subset of D arbitrarily near the path lifetime defines an h-harmonic function of z_0, bounded and therefore identically constant. To prove the lemma recall that $z \in \Gamma'_F$ if and only if whenever $f_1, \ldots f_k$ is a finite subset of H^∞ and $\varepsilon > 0$ the $\overline{D}$ neighborhood of z

$$A = \{w \in \overline{D} : |\hat{f}_j(w) - \hat{f}_j(z)| < \varepsilon , j \leqslant k\}$$

meets every set in Γ_F, that is $D - A$ is not to be thin at 1 ; equivalently almost every conditional path from a point of D to 1 is to meet A at times arbitrarily near the path lifetime. The lemma is now obvious.

21.

In the following theorem "Brownian path to the point 1" refers to conditional Brownian paths from a point of D. The assertions are true for every initial point, using the reasoning of the proof of Lemma 20.

THEOREM. – *If $A \subset D_1$ and if A is a neighborhood of Γ'_F relative to D_1, the cluster set of almost every Brownian path to the point 1* *is a subset of* A. *If* $A \subset D_1$ *and if* A *is a neighborhood relative to* D_1 *of some fine point then almost every Brownian path to the point* 1 *has a cluster point in* A.

In proving the first assertion we can assume that A is the trace on D_1 of a closed $\overline{D}$ neighborhood B of $\boldsymbol{\Gamma}'_F$. Then $B \cap D$ is a deleted fine neighborhood of the point 1 and the cluster set of almost every Brownian path to that point must be a subset of $B \cap D_1 = A$, as was to be proved. To prove the second assertion let z be a fine point in A and assume as we can that A is the trace on D_1 of a closed $\overline{D}$ neighborhood B of z. According to Lemma 20 almost every Brownian path to the point 1 meets B arbitrarily near (Euclidean topology) the point 1, and it follows that almost every such path has a cluster set meeting $B \cap D_1 = A$, as was to be proved.

Boundary Limit Theorems

22. Fatou filters.

An otherwise unspecified measure of subsets of C is to be understood to be Lebesgue measure. Let $\boldsymbol{\Gamma}$ be a filter of subsets of D with Euclidean limit the point 1. The filter will be called a Fatou filter if, for each f in H^∞, $\lim_{z\,\Gamma} f$ exists and is equal to the nontangential limit of f at z for almost every z on C for which the latter limit exists. According to Fatou's theorem the latter limit exists almost everywhere on C. If $\boldsymbol{\Gamma}$ is a Fatou filter, every finer filter is also a Fatou filter, as is the (possibly coarser) filter $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}')$. The intersection of two Fatou filters is a Fatou filter, so the class of Fatou filters, ordered by inclusion, is a decreasing directed set. Each Fatou filter corresponds to a Fatou type boundary limit theorem, and the coarser the filter the stronger the theorem. Since $\boldsymbol{\Gamma}_F$ is a Fatou filter [2], $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'_F)$ is also one, coarser than both $\boldsymbol{\Gamma}_F$ and $\boldsymbol{\Gamma}_A$, using Theorem 14. In going from one Fatou theorem to a stronger one there is a gain in that the approach filter is coarser, a possible loss (actual in going from $\boldsymbol{\Gamma}_A$ to $\boldsymbol{\Gamma}(\boldsymbol{\Gamma}'_F)$) in that the exceptional Lebesgue null set on C for some members of H^∞ may be larger.

23. Fatou sets.

A nonempty subset B of D_1 will be called a Fatou set if, for f in H^∞ and almost every specified w on C (depending on f), the restriction of $\hat{f}$ to $w\,(B \cup \Gamma'_A)$ is a constant function. Every nonempty subset of a Fatou set is a Fatou set, and the closure of a Fatou set is a Fatou set.

If B is a Fatou set, $\mathbf{\Gamma}(\overline{B})$ is a Fatou filter. Conversely if $\mathbf{\Gamma}$ is a Fatou filter, Γ' is a Fatou set and we have already noted that $\mathbf{\Gamma}\,(\mathbf{\Gamma}') \subset \Gamma$ and is a Fatou filter. In the sense of Section 22, strengthening a Fatou theorem with filter $\mathbf{\Gamma} = \mathbf{\Gamma}(\mathbf{\Gamma}')$ means enlarging the corresponding Fatou set $\mathbf{\Gamma}'$. For example, if A is a Fatou set, the smallest L maximal superset A^L of A (see Section 19) is also a Fatou set, giving a stronger Fatou theorem unless A is already L maximal.

If a singleton is a Fatou set its point will be called a Fatou point. Every point of a Fatou set is a Fatou point. A countable union of Fatou sets is a Fatou set.

If a set B is a Fatou set then for u harmonic and bounded on D from above or below, the restriction to $w\,(B \cup \mathbf{\Gamma}'_A)$ of $\hat{u}$ is a constant function, for almost every w on C. Conversely the latter condition implies that B is a Fatou set, even if imposed only for bounded harmonic u. Moreover the condition is sufficient even if imposed only for $u = \mu(.\,, A)$ for every measurable subset A of C, since a bounded harmonic function can be uniformly approximated by linear combinations of harmonic measures. Finally it is therefore even sufficient if (for every A) the restriction to wB of

$$\hat{\mu}(.\,, A) = \nu(.\,, A^s)$$

is identically 1 for almost every w in A. This condition is stated for reference in a trivially equivalent form in the following proposition, using the notation of Section 3.

PROPOSITION. — *A subset* B *of* D_1 *is a Fatou set if and only if whenever* A *is a measurable subset of* C *there is a Lebesgue null set* A_0 *of* A *for which* $S(wz) \subset A^s$ *for* z *in* B *and* w *in* $A - A_0$. *If* $B \subset X$ *this condition reduces to* $wB \subset A^s$ *for* w *in* $A - A_0$.

24.

THEOREM. – *If* B *is a Fatou set, so is* $\bigcup_{z \in B} S(z)$.

Let A be a measurable subset of C, corresponding to $A^s \subset X$. According to Proposition 23, $wS(z) = S(wz) \subset A^s$ simultaneously for all z in B, if w is in A less a null set, and, since $S(wz) \subset X$, the theorem follows from the second assertion of the proposition.

25.

THEOREM. – *If* B *is a closed Fatou set of Šilov points, the set* $\{z \in D_1 : \nu(z, B) > 0\}$ *is also a Fatou set.*

If $f \in H^\infty$, if $w \in C$ and if the restriction of $\hat{f}$ to wB is constant then $\hat{f}$ has that same constant value at every point wz of wD_1 for which $\nu(wz, B) > 0$, according to the remarks in Section 3. Hence the set described in the theorem is a Fatou set.

26. Gleason parts.

(See 3,4). We first outline certain material needed below. A sequence $\{z_n, n \geq 1\}$ in D is called an interpolation sequence if $f \to \{f(z_n), n \geq 1\}$ maps H^∞ onto the space of bounded complex sequences. It is sufficient for this if $|z_n| \to 1$ exponentially fast. Thus every sequence with an accumulation point on C has an interpolation subsequence. A point of D' is called an interpolation point if it is a cluster point of an interpolation sequence. The set of all interpolation points is open relative to D' and contains no Šilov boundary points.

Each point z of D' determines a Gleason part, a subset $G(z)$ of D' containing z. If $z_1 \in G(z)$ then $G(z_1) = G(z)$. If z is not an interpolation point, $G(z) = \{z\}$. The Gleason part of an interpolation point is an analytic disk, in D_1 if z is. Specifically, let $L(. , w)$ be the linear transformation depending on w defined by

$$L(\zeta , w) = \frac{\zeta + w}{1 + \overline{w}\zeta} , \quad |w| < 1,$$

mapping D onto itself. If $w \to w' \in D_1$ along an ultrafilter, $L(. , w)$ has a limit map $L(. , w')$ and $z \mapsto L(z , w')$ is a homeomorphism between D and $L(D , w') = G(w')$. Moreover if $f \in H^\infty$, the restriction of $\hat{f}$ to $L(D , w')$ referred back to D by this map, is holomorphic on D, and yields every element of H^∞ as f varies. If u is harmonic, the restriction of $\hat{u}$ to $L(D , w')$ referred back to D is harmonic on D.

If z_1 and z_2 are in the same Gleason part, $\nu(z_1 , .)$ and $\nu(z_2 , .)$ are mutually absolutely continuous measures. Hence $S(z_1) = S(z_2)$.

27.

THEOREM. – *If* B *is a Fatou set, so is* $G(B) = \bigcup_{z \in B} G(z)$.

It is obviously sufficient to prove the theorem for B a Fatou set of interpolation points. With this hypothesis on B, let A be a measurable subset of C. Then the harmonic measure $\mu(. , A)$ is harmonic on D and $\hat{\mu}(wz , A) = 1$, simultaneously for all z in B, for almost every w in A. If w is not exceptional and if $z \in B$, the restriction to $G(wz)$ of $\hat{\mu}(. , A)$ is a bounded (by 1) harmonic function (when referred canonically back to a disk), with value 1 at a point, and is therefore identically 1. According to Proposition 23, G(B) must be a Fatou set.

28.

THEOREM. – *The set of Fatou points is dense in* D_1. *More specifically, if* $A : \{z_n , n \geqslant 1\}$ *is a sequence in* D, *with Euclidean limit* 1, A' *contains an interpolation point* z *for which* $G(z)$ *is a Fatou set.*

The second assertion implies the denseness of the set of Fatou points because if B is any closed $\overline{D}$ neighborhood of a point of D_1, A can be chosen in $B \cap D$.

Going to a subsequence if necessary, we can suppose that A is an interpolation sequence. Let f be a member of H^∞, $f(\zeta) = \sum_0^\infty a_k \zeta^k$, with Fatou nontangential boundary function f^*. Then

$$\int_0^{2\pi} |f^*(e^{i\theta}) - f(z_n e^{i\theta})|^2 \, d\theta = 2\pi \sum_0^\infty |a_k(1 - z_n^k)|^2 \tag{28.1}$$

so the sequence $\{w \mapsto f(wz_n) \ , \ n \geqslant 1\}$ of functions on C converges to f^* in the mean and in measure. Now according to a theorem of Mokobodzki [6] there is a "rapid" ultrafilter of integers with limit ∞ and such an ultrafilter has the property that a sequence of measurable functions converging in measure on a totally finite complete measure space necessarily converges pointwise along the ultrafilter to a limit which is almost everywhere the limit in measure. Along a rapid ultrafilter, $z_.$ converges to some point z of D_1, $f(wz_.)$ converges to $\hat{f}(wz)$, and $\hat{f}(wz) = f^*(w)$ for almost every w on C. Thus z is a Fatou point and hence by Theorem 27 the set $G(z)$ is a Fatou set.

COROLLARY. – *There is no maximal Fatou set, no minimal Fatou filter.*

The two assertions are equivalent, and we prove the first. A maximal Fatou set B would be a closed subset of D_1 and would include every Fatou point, because the union of two Fatou sets is a Fatou set. But then $B = D_1$ by Theorem 28, and this is absurd.

Note that the set of Fatou points is countably closed, that is the closure of a countable set of Fatou points consists of Fatou points, in fact is a Fatou set because the closure of a countable union of Fatou sets is a Fatou set.

29. Example.

Let F_0 be a Fatou set and define Fatou sets Σ_n and F_n for $n > 0$ inductively by

$$\Sigma_{n+1} = \left[\bigcup_{z \in F_n} S(z)\right]^- \ , \quad F_{n+1} = \{z : \nu(z, \Sigma_{n+1}) > 0\} \ , \quad n \geqslant 0.$$

Then

$$F_n \cap X \subset \Sigma_{n+1} \subset F_{n+1} \cap X \quad , \quad F_n \subset F_{n+1} \ ,$$

so $\Sigma_n \subset \Sigma_{n+1}$. The sets $F_\omega = \bigcup_n F_n$ and $\Sigma_\omega = \bigcup_n \Sigma_n$ are Fatou sets satisfying

$$\bigcup_{z \in F_\omega} S(z) \subset \Sigma_\omega = F_\omega \cap X \ , \ F_\omega = \{z : \nu(z \ , \Sigma_\omega) > 0\}$$
$$= \{z \ : \nu(z \ , \Sigma_\omega) = 1\} \ .$$

Since $G(z) \subset F_{n+1}$ whenever $z \in F_n$, F_ω is a union of Gleason parts. Thus F_ω is a Fatou set with nice properties. If one applies the same inductive process to F_ω as to F_0 one can go on by transfinite induction, defining $\Sigma_.$ and $F_.$ for indices running through the countable ordinals. It would be interesting to have conditions on F_0 ensuring the existence of a countable ordinal γ with $F_\gamma = F_{\gamma+1} = \dots$.

30. G_δ sets.

The corollary to the following theorem implies that $\mathbf{\Gamma}'_A$ and $\mathbf{\Gamma}'_F$ are not G_δ sets. Note that a subset of D_1 is a G_δ set if and only if it is a G_δ set relative to D_1, because D_1 is a G_δ set.

THEOREM. – *Let* $\{A_n , n \geqslant 1\}$ *be a sequence of open subsets of* $\overline{D}$ *and let* Γ *be a convergence stable filter of subsets of* D, *with limit the point* 1. *If* $\Gamma' \subset \bigcap_n A_n$, *there is a member* A *of* Γ *with* $A' \subset \bigcap_n A_n$.

Decreasing the sets A_n if necessary we can suppose that $\overline{A}_{n+1} \subset A_n$. The hypotheses imply that each set A_n is a superset of every member of $\mathbf{\Gamma}$ sufficiently near the point 1. Hence $A_n \cap D \in \mathbf{\Gamma}$, and convergence stability of $\mathbf{\Gamma}$ implies that there is a member A of Γ with the property that each set A_n includes the part of A sufficiently near the point 1. Then $A' \subset A'_{n+1} \subset A_n$ for all n, as was to be proved.

COROLLARY. – *No Fatou superset of* Γ'_A *is a* G_δ.

If B is a G_δ Fatou superset of $\mathbf{\Gamma}'_A$, the theorem asserts that there is a member A of $\mathbf{\Gamma}_A$ with $A' \subset B$. Hence A' is a Fatou set and it follows that each member of H^∞ has a limit at w along wA for almost every w on C. This is impossible according to the Littlewood theorem already used, because A contains a continuous curve tangent to C at 1.

31. Baire functions on D'.

The Baire class of complex functions on D' is defined as usual as the smallest class of functions containing the continuous functions and closed under pointwise sequential convergence.

THEOREM. – *Let* A *be a Fatou set and let* u *be a Baire function on* D'. *Then the restriction of* u *to* wA *is a constant function, for almost every* w *on* C.

The class of functions on D' with the stated property is an algebra containing the constant functions and closed under conjugation and sequential pointwise limits. Moreover this class includes the restrictions to D' of the extensions to $\overline{D}$ of the members of H^∞, and these restrictions separate D'. The class therefore includes the Baire class.

BIBLIOGRAPHY

[1] M. BRELOT and J.L. DOOB, Limites angulaires et limites fines, *Ann. Inst. Fourier,* 13 (1963), 395-415.

[2] J.L. DOOB, Conditional Brownian motion and the boundary limits of harmonic functions, *Bull. Soc. Math. France* 85 (1957), 431-458.

[3] Kenneth HOFFMAN, Banach spaces of analytic functions, Prentice Hall 1962.

[4] Kenneth HOFFMAN, Bounded analytic functions and Gleason parts, *Ann. Math.* 86 (1967), 74-111.

[5] L. LUMER-NAÏM, Sur le rôle de la frontière de R.S. Martin dans la théorie du potentiel, *Ann. Inst. Fourier* 7 (1957), 183-281.

[6] Gabriel MOKODOBZKI, Ultrafiltres rapides sur N. Construction d'une densité relative de deux potentiels comparables, Séminaire Théorie Potentiel Brelot-Choquet-Deny 1967/68 Exp. 12.

[7] M. ROSENFELD and MAX L. Weiss, A function algebra approach to a theorem of Lindelöf, *J. London Math. Soc.* (2) 2 (1970), 209-215.

[8] M. TSUJI, Potential theory in modern function theory, Tokyo 1959.

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J.L. DOOB
University of Illinois
Urbana, Illinois 61801