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Ian Reay

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SUBDUALS AND TENSOR PRODUCTS OF SPACES OF HARMONIC FUNCTIONS

by Ian REAY

Introduction.

In this paper we shall be working in the axiomatic potential theory of M. Brelot. For the fundamentals of this theory the reader is referred to Brelot [7] and Hervé [18]. We shall also assume a knowledge of convexity theory that is to be found in, for example, Alfsen [1] and, in summary, in Effros and Kasdan [14]. Let Ω be a topological space to which Brelot's theory applies and let ω be an open, relatively compact subset. The set $X = \{h : h$ is harmonic, $\geqslant 0$ and $h(x_0) = 1\}$, for a fixed arbitrary x_0, is well known to be a compact Choquet simplex in the topology of uniform convergence on compacta. As such it is the state space of the linear function space, A(X), of continuous affine functionals on X. The question arises, and was first proposed by D.A. Edwards, as to an intrinsic description of the space A(X) in the context of potential theory. Such a description is to be found in the statement of Theorem 1. In the case that Ω satisfies the hypothesis of proportionality it is seen to related to the Martin boundary of the space considered. Some ancillary results are also given.

In the second part of the paper Theorem 1 is applied in proving that the space of differences of positive separately harmonic functions (Gowrisankaran [16]) is a tensor product of two spaces of harmonic functions. Also in this part it is demonstrated that by using tensor product techniques whenever possible many proofs of results in the subject of separately harmonic functions can be simplified. For example, Corollary 18, which was originally proved by Gowrisankaran. Some ancillary results are also given in the second part. The readers attention is drawn to Theorems 1, 11 and 14 which are the three central results of the paper.

This work is a summary of the author's doctoral dissertation at the University of Oxford, under the supervision of Dr. G.F. Vincent-Smith, to whom I am indebted for his comprehensive help. I also owe a debt of thanks to K. Gowrisankaran for pointing out that Theorems 1 and 14 are valid without the necessity of assuming the hypothesis of proportionality and for the proof of Proposition 5.

Part 1. – A function space with state space X.

1. We consider the topological space Ω, with the presheaf of harmonic functions $\mathcal{H}(\Omega)$ satisfying Brelot's axioms I, II, III. We also assume that Ω has a countable base of open sets. Let $H^+(\Omega)$ be the cone of positive harmonic functions on Ω. It is well known that $X = \{h \in H^+(\Omega) : h(x_0) = 1\}$ is a compact metrizable Choquet simplex in the topology of uniform convergence on compacta $\subseteq \Omega$. We denote the set of extreme points of X, by Δ_1, such points are called the minimal harmonic functions in X. It can be shown by an application of Choquet's Theorem that to every harmonic function, h, in X there exists a unique probability measure, μ_h, concentrated on Δ_1, such that

$$h(x) = \int_{\Delta_1} u(x)\, d\mu_h(u) \qquad \text{for all} \quad x \in \Omega .$$

This integral is referred to as Martin's Integral Representation.

Martin's Integral Representation defines a map $m : X \to \mathfrak{M}_+^1(\overline{\Delta}_1)$ the simplex of probability measures on $\overline{\Delta}_1$. This map is not a bijection but we will construct a subspace $L \subsetneq C(\overline{\Delta}_1)$ such that the composition of the map m with the dual map : $\mathfrak{M}_+^1(\overline{\Delta}_1) \to S(L)$, where S(L) is the state space of L, is a bijection.

Let $\mathfrak{M}_c(\Omega)$ be the space of all real measures on Ω of compact support. Define the subspace $L \subseteq C(\overline{\Delta}_1)$ as follows ;

$$L = \{f \in C(\overline{\Delta}_1) : f(h) = \nu(h) \text{ for all } h \in \overline{\Delta}_1 \text{ and some } \nu \in \mathfrak{M}_c(\Omega)\}.$$

THEOREM 1. – L *is a linear function space which contains the constants and separates the points of* $\overline{\Delta}_1$, *and its state space* S(L) *is affinely homeomorphic to* X. *Symbolically,* $A(X) \cong \overline{L}$, $S(L) \cong X$.

Proof. – L is a linear subspace of $C(\overline{\Delta}_1)$ trivially. Also (a) it contains the constants, because $\delta_{x_0}(h) = h(x_0) = 1$ for all $h \in X$, and δ_{x_0} is in $\mathfrak{M}_c(\Omega)$, (b) it separates the points of $\overline{\Delta}_1$ since if $h_1 \neq h_2$, both being in $\overline{\Delta}_1$, there exists an $x \in \Omega$ such that $h_1(x) \neq h_2(x)$, so that $\delta_x(h_1) \neq \delta_x(h_2)$ hence there exists an $f \in L$ such that $f(h_1) \neq f(h_2)$.

Define a map $\Gamma : S(L) \to X$ in the following way ; If $p \in S(L)$ then p is a positive linear function on L and $p(1) = 1$. By the Hahn-Banach Theorem p extends, non-uniquely, to a positive linear functional on $C(\overline{\Delta}_1)$ which by the Riesz Representation Theorem can be regarded as a probability measure, μ, on $\overline{\Delta}_1$. This measure defines a harmonic function, h, in X by the Martin Representation. Now, although Γ is not uniquely defined by p, h is, since if μ_1, μ_2 are both extensions of p, then $\mu_1(f) = \mu_2(f)$ for all f in L, in other words,

$$\int\int h(x)\, d\nu(x)\, d\mu_1(h) = \int\int h(x)\, d\nu(x)\, d\mu_2(h).$$

By Fubini's Theorem, since the map $X \times \Omega \to R$ given by $(h, x) \to h(x)$ is continuous,

$$\int\int h(x)\, d\mu_1(h)\, d\nu(x) = \int\int h(x)\, d\mu_2(h)\, d\nu(x),$$

which can be written $\nu(h_1) = \nu(h_2)$ where h_i corresponds to μ_i $(i = 1, 2)$ in the Martin Representation. This holds for all $\nu \in \mathfrak{M}_c(\Omega)$, which clearly implies that $h_1 = h_2$, by taking $\nu = \delta_x$ for instance. So the map Γ is well defined.

Γ is injective, since for the harmonic function, h, the family of measures representing h on $\overline{\Delta}_1$ all have the same value for

$$\int\int_{\overline{\Delta}_1} u(x)\, d\mu(u)\, d\nu(x) \quad \text{for} \quad \nu \in \mathfrak{M}_c(\Omega),$$

and thus for $\int_{\overline{\Delta}_1} \nu(u)\, d\mu(u)$, in other words, when restricted to L the measures all coincide, so that they all collapse to the same element of S(L), which is the only one that can map into h under Γ.

Γ is also a surjection, since if $u \in X$, take one of its representing measures on $\overline{\Delta}_1$ and restrict to L, to obtain $p \in S(L)$, then clearly $\Gamma(p) = u$. So Γ is a bijection between S(L) and X.

Γ is affine, since if $\alpha \in (0, 1)$, $p_i \in S(L)$ and $\Gamma(p_i) = h_i$, where $h_i \in X$, $(i = 1, 2)$,

$$\begin{aligned}\alpha\Gamma(p_1) + (1-\alpha)\,\Gamma(p_2) &= \alpha h_1 + (1-\alpha)\,h_2 \\ &= \alpha \int u(\,.\,)\,d\mu_{h_1}(u) + (1-\alpha)\int u(\,.\,)\,d\mu_{h_2}(u) \\ &= \int u(\,.\,)\,d\{\alpha\mu_{h_1} + (1-\alpha)\,\mu_{h_2}\}\,(u) \\ &= \Gamma(\alpha p_1 + (1-\alpha)\,p_2),\end{aligned}$$

since the map $h \to \mu_h$ is affine.

Finally, Γ is a homeomorphism for X endowed with the topology of uniform convergence on compacta and S(L), the weak* topology. Both X and S(L) are metrizable so it is enough to consider sequences. Now, $p_n \to p$ in S(L) if and only if $p_n(f) \to p(f)$ for all $f \in L$, if and only if $\nu(h_n) \to \nu(h)$ for all $\nu \in \mathfrak{M}_c(\Omega)$ where $h_n = \Gamma(p_n)$, $h = \Gamma(p)$, using the definition of L and Fubini's Theorem, and $\nu(h_n) \to \nu(h)$ for all such ν if and only if $h_n \to h$, because one can take $\nu = \delta_x$, and by the result of Hervé, (Brelot [7], Lemme on page 23), pointwise convergence in H^+ is equivalent to uniform convergence on compacta. This completes the proof.

COROLLARY 2. – *The Choquet boundary of* $\overline{\Delta}_1$ *with respect to* L *is the set of minimal harmonic functions* $\Delta_1 = ex X$. *Symbolically,* $\partial_L(\overline{\Delta}_1) = \Delta_1$.

Proof. – There is a canonical injection $\overline{\Delta}_1 \to X$ and the extreme points of X are just the minimal harmonic functions, the Choquet boundary is just the inverse image of Δ_1 under this map.

COROLLARY 3. – *If* Δ_1 *is closed,* $\overline{L}$ *is a lattice and* $\overline{L} = C(\Delta_1)$.

Proof. – This follows immediately from Theorem 2.6 of Effros and Kazdan [14].

COROLLARY 4. – L *has the weak Riesz separation property and* $\overline{L}$ *has the strong Riesz separation property.*

Proof. – Since L is a simplex space, the result follows from Proposition 9 of Edwards and Vincent-Smith [13].

Remark. – *We emphasise that if* μ_1, μ_2 *are two measures in* $\mathfrak{M}_c(\Omega)$ *they correspond to the same function* $f \in \mathrm{L}$ *if and only if* $\mu_1(h) = \mu_2(h)$ *for all* $h \in \mathrm{X}$.

Proof. – Clearly, if $\mu_1(h) = \mu_2(h)$ for all $h \in \mathrm{X}$ then in particular they do so for all $h \in \overline{\Delta}_1$ and so correspond to the same element of L. Conversely, if $\mu_1(h) = \mu_2(h) = f(h)$ for $h \in \overline{\Delta}_1$, by Martin's Representation every $u \in \mathrm{X}$ can be represented by a probability measure, μ_u, on $\overline{\Delta}_1$ but

$$\int_{\overline{\Delta}_1} \mu_1(h)\, d\mu_u(h) = \int_{\overline{\Delta}_1} \mu_2(h)\, d\mu_u(h),$$

and using Fubini again we obtain $\mu_1(u) = \mu_2(u)$.

In the case that Ω satisfies the hypothesis of proportionality, ie. to every point $x \in \Omega$ there exists a potential p_x of support $\{x\}$, unique up to scalar multiple. Then Theorem 1 and its corollaries can be phrased in terms of the Martin Boundary. Under this new hypothesis the Martin Boundary is a compact, metric space, $\partial\Omega_{\mathrm{M}}$, such that

$$\Delta_1 \subseteq \overline{\Delta}_1 \subseteq \partial\Omega_{\mathrm{M}} \subseteq \mathrm{X},$$

and since $\overline{\Delta}_1$ can clearly be replaced by any closed set, C, such that $\Delta_1 \subseteq \mathrm{C} \subseteq \mathrm{X}$, in the statements of Theorem 1 and its corollaries, they will remain valid, in these circumstances, if at every point at which "$\overline{\Delta}_1$" occurs "$\partial\Omega_{\mathrm{M}}$" is substituted.

2. In this paragraph we study the relationship between the spaces $\mathfrak{M}_c(\Omega)$ and L. It will be seen in the definition of L given in Theorem 1 that we have a positive, linear map $\mathrm{T} : \mathfrak{M}_c(\Omega) \to \mathrm{L} \subseteq \mathrm{C}(\overline{\Delta}_1)$ defined by $\mathrm{T}(\nu)\ (h) = \nu(h) = \int h d\nu$ for $h \in \overline{\Delta}_1$. For a compact $\mathrm{K} \subseteq \Omega$ this map, T, restricts in the obvious way to $\mathrm{T}_{\mathrm{K}} : \mathfrak{M}(\mathrm{K}) \to \mathrm{C}(\overline{\Delta}_1)$. We give $\mathfrak{M}(\mathrm{K})$ the weak* topology and $\mathrm{C}(\overline{\Delta}_1)$ the supnorm topology. The proof of the following proposition is due to K. Gowrisankaran.

Proposition 5. – $\mathrm{T}_{\mathrm{K}} : \mathfrak{M}^+(\mathrm{K}) \to \mathrm{C}^+(\overline{\Delta}_1)$ *is continuous.*

Proof. – Since $\mathfrak{M}^+(\mathrm{K})$ is metrizable in the weak* topology we may consider sequences. To prove the proposition we have to show that whenever $\nu_n \to \nu$ in the weak* topology, in $\mathfrak{M}^+(\mathrm{K})$, $\mathrm{T}(\nu_n)(h) \to \mathrm{T}(\nu)(h)$

uniformly for all $h \in \overline{\Delta}_1$. Without loss of generality we may assume that $\|\nu\| \leqslant 1$, $\|\nu_n\| = 1$ for $n \geqslant 1$. We can find a finite number of points $h_1, h_2, \ldots, h_m$ in $\overline{\Delta}_1$, and open neighbourhoods $V_1, \ldots, V_m$, of the form

$$V_j = \{h \in \overline{\Delta}_1 \ ; \ |h(x) - h_j(x)| < \epsilon/4 \ \text{ for all } \ x \in K\},$$

such that $\cup V_j = \overline{\Delta}_1$. Then, given any $h \in \overline{\Delta}_1$, $\exists$ a V_j containing h and

$$\left|\int h d\nu_n - \int h d\nu\right| \leqslant \left|\int h d\nu_n - \int h_j d\nu_n\right| + \left|\int h_j d\nu_n - \int h_j d\nu\right| + \\ + \left|\int h_j d\nu - \int h d\nu\right| \leqslant \epsilon/2 + \left|\int h_j d\nu_n - \int h_j d\nu\right|.$$

Hence there exists an N such that for all $n \geqslant N$, and for all $h \in \overline{\Delta}_1$,

$$\left|\int h d\nu_n - \int h d\nu\right| < \epsilon .$$

This completes the proof.

Let $L_K \subseteq L^+$ be defined, for each compact $K \subseteq \Omega$, as $L_K = \{f \in L^+ :$ there exists a measure, ν, supported by K with $T(\nu) = f\}$.

COROLLARY 6. – *Any norm bounded subset of* L_K *is equicontinuous.*

Proof. – The set $\mathfrak{M}^+_\alpha(K) = \{\nu \in \mathfrak{M}^+(K) : 0 \leqslant \nu(1) \leqslant \alpha\}$ is weak* compact. Since T_K is continuous on $\mathfrak{M}^+(K)$, $T_K(\mathfrak{M}^+_\alpha(K))$ is a compact subset of $C(\overline{\Delta}_1)$ and so is equicontinuous by Ascoli. But if $N \subseteq L_K$ is bounded in norm by α, $N \subseteq L_K \cap T_K(\mathfrak{M}^+_\alpha(K))$, and so N is equicontinuous.

3. In this paragraph we give a function space whose state space is the simplex B, to be defined later, which is a base of the cone S^+ of positive superharmonic functions on Ω. In this paragraph we assume the hypothesis of proportionality of Hervé ie. to every $x \in \Omega$ there corresponds, up to scalar multiple, a unique potential, p, such that p is harmonic in $\mathcal{C}\{x\}$.

It is well known that S^+ is a lattice cone and has as base the set B defined as follows :

$$B = \{V \in S^+ \ : \ V_f(x_0) + V_{1-f}(x_1) = 1\}$$

where $f \in C(\overline{\Omega})$ ($\overline{\Omega}$ is the one point compactification of Ω) takes the value 1 at the point at infinity, ∞ ; x_0 and x_1 are fixed points in Ω, and the map $f \mapsto V_f$ is the kernel operator. It is left as a puzzle to show that B may be taken to contain X with no loss of generality. In the T-topology (Hervé [18]) B is compact and thus is a Choquet simplex. Since the T-topology induces the topology of uniform convergence on compacta on H^+, B contains X as a closed face.

We can thus write $i : X \to B$ for the canonical injection, from this we can construct the restriction map $i_* : A(B) \to A(X)$. It is a simple corollary of the Edward's Separation Theorem, [12], that i_* is surjective and it is clearly continuous. Theorem 1 tells us that $A(X) \cong \overline{L} \subseteq C(\partial\Omega_M)$. From this it appears natural to look for a subspace $J \subseteq C(\Omega_M^*)$, where $\Omega_M^* = \Omega \cup \partial\Omega_M$ is the Martin Compactification, such that i_* becomes the restriction map $C(\Omega_M^*) \to C(\partial\Omega_M)$, and such that $J \cong A(B)$.

We define $J \subseteq C(\partial\Omega_M^*)$ to be maximal with respect to the property :

$$J|\partial\Omega_M \subseteq \overline{L}.$$

In other words,

$$J = \{f \in C(\Omega_M^*) : f|\partial\Omega_M \in \overline{L}\}.$$

Note that J contains the constants, trivially, since L does, and that J separates the points of Ω_M^*. This is true because L separates the points of $\partial\Omega_M$ and for any f in J and any compact set $K \subseteq \Omega$ one can alter f on K in an arbitrary manner provided the function obtained remains continuous on Ω_M^*. Now it is well known that one can visualise Ω as sitting inside exB, by means of the canonical homeomorphism, ϕ, which sends $x \mapsto p_x$, the potential of support $\{x\}$.

We need the following lemma, the proof of which we do not give as it is almost a standard corollary of the Edward's Separation Theorem, [12].

LEMMA 7. – *If* X *is a closed face of a simplex* B, *such that* $X \supseteq \overline{\mathrm{ex}B} \setminus \mathrm{ex}B$; *in other words* $X \cup \mathrm{ex}B$ *is closed, then any* $f \in C(X \cup \mathrm{ex}B)$ *such that* $f|_X$ *is affine has a norm preserving extension to an element of* A(B). *The extension is unique.*

This lemma can be reformulated in the following way ;

LEMMA 7a. – *With the notation of Lemma 7, if*

$$C_A(X_\cup exB) = \{f \in C(X_\cup exB) : f|_X \in A(X)\}$$

then $A(B) \cong C_A(X_\cup exB)$, *and the correspondance is an order isometry.*

We can now state and prove the proposition central to this paragraph.

PROPOSITION 8. – *If* J *is the function system defined above then* J *is isometrically order isomorphic to* A(B), *hence* B *is affinely homeomorphic to the state space of* J. *Symbolically,* $J \cong A(B)$, $S(J) \cong B$.

Proof. – It is well known that $X \supseteq \overline{exB} \backslash exB$, in other words $\phi(\Omega)_\cup X$ is a closed subset of B containing exB. X is also well known to be a closed face of B so by Lemma 7a.

$$A(B) \cong C_A(X_\cup exB) \cong J.$$

COROLLARY 9. – *If* Δ_1 *is the set of minimal harmonic functions in* X *then the Choquet boundary of* J *is* $\Omega_\cup \Delta_1$.

Proof. – The extreme boundary of B is just the set $\phi(\Omega)_\cup \Delta_1$ and the Choquet boundary is the inverse image of this under the map $\Omega^*_M \to B$.

PROPOSITION 10. – *If* $J_0 = \{f \in C(\Omega) : f$ *is the restriction of an element in* J$\}$, *then* Ω^*_M *is the* J_0*-compactification of* Ω.

Proof. – Let Ω^*_J be the J_0-compactification of Ω. Then

(a) Ω is homeomorphic to a dense subset of Ω^*_J.

(b) Every element of J_0 extends to an element of $C(\Omega^*_J)$.

(c) The extensions separate the points of $\partial\Omega_J = \Omega^*_J \backslash \Omega$, and Ω^*_J is uniquely defined by the properties (a), (b) and (c), up to homeomorphism. But Ω^*_M satisfies all three properties and so $\Omega^*_M \cong \Omega^*_J$.

Part 2. – Separately harmonic functions and tensor products of simplexes.

1. In this paragraph, unless otherwise indicated, we shall assume that each harmonic space satisfies the axioms I, II, III of Brelot. Separately harmonic functions, as their name suggests, are functions of two variables in harmonic spaces Ω_0 , Ω_1, which are harmonic in each variable. More formally,

DEFINITION. – *Let* Ω_0 , Ω_1, *be two harmonic spaces, then a function,* h, *on* $\Omega_0 \times \Omega_1$ *is separately harmonic if it is harmonic in each variable for each fixed value of the other variable. I.e. for each* $x \in \Omega_0$, $y \mapsto h(x , y)$ *lies in* $\mathrm{H}(\Omega_1)$ *and for each* $y \in \Omega_1$, $x \mapsto h(x , y)$ *lies in* $\mathrm{H}(\Omega_0)$.

In a similar way a function h can be defined to be separately harmonic on each open subset $\omega \subseteq \Omega_0 \times \Omega_1$. Let $\mathrm{MH}^+(\Omega_0 , \Omega_1)$ be the convex cone of positive separately harmonic functions on $\Omega_0 \times \Omega_1$, let $\mathrm{MH}(\Omega_0 , \Omega_1)$ be the vector space of separately harmonic functions on $\Omega_0 \times \Omega_1$ and let $\mathrm{MH}_0(\Omega_0 , \Omega_1)$ be the vector space of differences of positive separately harmonic functions. Also the symbols $\mathrm{MH}^+(\omega)$, $\mathrm{MH}(\omega)$ and $\mathrm{MH}_0(\omega)$ will denote the same objects corresponding to the open set $\omega \subseteq \Omega_0 \times \Omega_1$.

The above definition has been taken from Gowrisankaran [16]. In fact, Gowrisankaran talked about multiply harmonic functions which are separately harmonic functions that are also continuous, but because any positive separately harmonic function is necessarily continuous and, in this paper we shall only be interested in positive functions or differences of such functions we do not need to impose this extra condition. In [16], Gowrisankaran showed that the separately harmonic functions satisfy three axioms, the first, IM, and the third, IIIM, corresponding exactly to axioms I and III of Brelot's system ; the second, IIM, to a somewhat weaker form of the axiom II of Brelot. Then, among other things, Gowrisankaran develops the theory along similar lines to the development of axiomatic potential theory and proves an integral representation theorem analogous to Martin's Integral Representation. We propose, in the next paragraph, to deduce this theorem by different methods and in this paragraph we will cite the axioms IM, IIM, IIIM, and give a new proof of IIM, based on tensor product ideas.

AXIOM IM. – *Let* $\omega \subseteq \Omega_0 \times \Omega_1$ *be open. If* $u \in \mathrm{MH}(\omega)$ *then* $u \in \mathrm{MH}(\delta)$ *for all* $\delta \subseteq \omega$. *If* u *is separately harmonic in a neighbourhood of each point in* ω *then* $u \in \mathrm{MH}(\omega)$.

Proof. – See [16].

Call the presheaf of separately harmonic functions on the directed system of open subsets of $\Omega_0 \times \Omega_1$, $\mathrm{M}\mathcal{H}(\Omega_0, \Omega_1)$.

AXIOM IIIM. – *Let* $\delta \subseteq \Omega_0 \times \Omega_1$ *be a domain and let* $\{u_i\}_{i \in \mathrm{I}}$ *be an increasing, filtering family of functions in* $\mathrm{MH}(\delta)$. *Then the upper envelope* u *of this family is either* $+\infty$ *on* δ *or lies in* $\mathrm{MH}(\delta)$.

Proof. – See [16].

One can immediately deduce the

LEMMA. – *If* $u \in \mathrm{MH}^+(\delta)$, *for a domain* δ, *then either* $u > 0$ *everywhere in* δ *or* $= 0$ *everywhere in* δ.

Proof. – See [16].

An immediate corollary of this is that the set

$$\mathrm{X_M} = \{h \in \mathrm{MH}^+(\Omega_0, \Omega_1) : h(x_0, y_0) = \\ = 1 \text{ for fixed } (x_0, y_0) \in \Omega_0 \times \Omega_1\}$$

is a base for $\mathrm{MH}^+(\Omega_0, \Omega_1)$. We shall discuss the topology on $\mathrm{MH}^+(\Omega_0, \Omega_1)$ in the next paragraph.

In order to prove the validity of Axiom IIM, we need to set up some terminology. Let $\omega_0 \subseteq \Omega_0$, $\omega_1 \subseteq \Omega_1$ be open and relatively compact. Define $\mathrm{MA}(\omega_0, \omega_1) = \{h : h$ is continuous and separately harmonic in $\omega_0 \times \omega_1$ and has a continuous extension to $\overline{\omega}_0 \times \overline{\omega}_1$ such that $h(x, .)$ is harmonic in ω_1 for $x \in \overline{\omega}_0$ and similarly for x and y reversed$\}$. MA is clearly a uniformly closed subspace of $\mathrm{C}(\overline{\omega}_0 \times \overline{\omega}_1)$. Since we are assuming throughout that the constants are harmonic then MA is a closed function space that contains the constants and separates the points of $\overline{\omega}_0 \times \overline{\omega}_1$, let S(MA) be its state space.

AXIOM IIM. – *Let* $\omega_0 \subseteq \Omega_0$, $\omega_1 \subseteq \Omega_1$ *be regular domains. Let* $\delta(\omega_0 \times \omega_1) = \partial\omega_0 \times \partial\omega_1$. *For any* $f \in \mathrm{C}(\delta(\omega_0 \times \omega_1))$ *there exists a function* J_f *on* $\overline{\omega}_0 \times \overline{\omega}_1$ *with the following properties :*

(a) $J_f \geqslant 0$ *if* $f \geqslant 0$.

(b) $J_f = f$ *on* $\delta(\omega_0 \times \omega_1)$ *and lies in* MA.

(c) J_f *is uniquely determined by* (a) *and* (b).

Proof. – Let the symbol "v" refer to completion with respect to the weak tensor product norm. Then there is a positive isometry

$$I : A_0 \overset{\vee}{\otimes} A_1 \to MA,$$

where $A_0 = A(\omega_0) = \{h \in H(\omega_0) : h$ has a continuous extension to $\overline{\omega}_0\}$, $A_1 = A(\omega_1)$. This map is defined on the dense subspace, $A_0 \otimes A_1$, by the rule

$$I : \sum_{i=1}^{n} u_i \otimes v_i \mapsto \sum_{i=1}^{n} u_i v_i ,$$

for $u_i \in A_0$, $v_i \in A_1$, and it is enough to show that it is an isometry on this set since one can then extend by continuity. Let $\| . \|_{\otimes}$ be the weak tensor product norm, then

$$\left\| \sum_{i=1}^{n} u_i \otimes v_i \right\|_{\otimes} = \sup \left\{ \left\| \sum_{i=1}^{n} < u_i , p > v_i \right\| : p \text{ lies in the unit ball of } A_0^* \right\}.$$

Now, by a direct calculation based on the triangle inequality for the norm, the map $p \mapsto \left\| \sum_{i=1}^{n} < u_i , p > v_i \right\|$ is seen to be convex. It is also continuous since it is the composition of the maps

$$p \mapsto \sum_{i=1}^{n} < u_i , p > v_i \mapsto \left\| \sum_{i=1}^{n} < u_i , p > v_i \right\|$$

each of which is continuous. Hence, by the Bauer Maximum Principle (see e.g. [1]), it is sufficient to take the supremum over the extreme boundary of the unit ball of A_0^*. But this is precisely the set

$$\mathrm{exS}(A_0) \cup - \mathrm{exS}(A_0),$$

where $S(A_0)$ is the state space of A_0, and by a well known result (see e.g. Effros and Kazdan [14]) it is contained in

$$\{\delta_x : x \in \overline{\omega}_0\} \cup \{-\delta_x : x \in \overline{\omega}_0\},$$

where δ_x is the evaluation at x, so

$$\left\| \sum_{i=1}^{n} u_i \otimes v_i \right\|_{\otimes} = \sup \left\{ \left\| \sum_{i=1}^{n} u_i(x)\, v_i \right\| \ : \ x \in \overline{\omega}_0 \right\},$$

$$= \sup \left\{ \left| \sum_{i=1}^{n} u_i(x)\, v_i(y) \right| \ : \ x \in \overline{\omega}_0 \ , \ y \in \overline{\omega}_1 \right\},$$

$$= \left\| \sum_{i=1}^{n} u_i v_i \right\| .$$

So I is an isometry, it is clearly positive. Because MA is closed in $C(\overline{\omega}_0 \times \overline{\omega}_1)$ and therefore complete in the supnorm topology I extends to the completion, $A_0 \check{\otimes} A_1$, and remains an isometry there. Now, $\delta(\omega_0 \times \omega_1) = \partial\omega_0 \times \partial\omega_1$ is a closed set and is the extreme boundary of the simplex

$$\mathfrak{M}^1_+(\partial\omega_0 \times \partial\omega_1) \cong \mathfrak{M}^1_+(\partial\omega_0) \otimes \mathfrak{M}^1_+(\partial\omega_1).$$

So $\mathfrak{M}^1_+(\partial\omega_0 \times \partial\omega_1)$ is the state space of

$$C(\partial\omega_0) \check{\otimes} C(\partial\omega_1) \cong A_0 \check{\otimes} A_1 .$$

So by the extension theorem of Bauer (see e.g. [1]) every continuous function $f \in C(\delta(\omega_0 \times \omega_1))$ extends continuously to an element of MA. (a) follows immediately from the positivity of I and Bauer's Maximum Principle. This completes the proof.

This proof essentially uses the fact that there exists a positive isometry $A_0 \check{\otimes} A_1 \to MA$. This leads one to ask the questions : (a) Can we say that $A_0 \check{\otimes} A_1 \cong MA$ in a sense which preserves the order structures ? In other words, is $I : A_0 \check{\otimes} A_1 \to MA(\omega_0 , \omega_1)$ bipositive and surjective ? (b) If so, does (a) still remain true for ω_0 and ω_1 no longer regular ?

The answer to question (b), and hence also to (a), is affirmative provided that we assume that Ω_0 and Ω_1 both have a countable base of open sets, that both spaces satisfy the axiom of proportionality of Hervé and Axiom D (see e.g. [7]). These assumptions ensure, firstly, that Ω_0 and Ω_1 are metrizable and, secondly, that A_0 and A_1 are simplex spaces (the latter being ensured by Axiom D). One can however avoid the use of Axiom D by assuming instead that the two sets ω_0 and ω_1 are weakly determining domains as this also ensures that A_0 and A_1 are simplex spaces (see Boboc and Cornea [2]).

The reader who is not acquainted with the theory of tensor products of simplexes and simplex spaces is referred to the papers [11], [19] and [21].

THEOREM 11. – *Under the hypotheses mentioned above on the two spaces* Ω_0 *and* Ω_1, $A_0 \check{\otimes} A_1$ *is isometrically order isomorphic to* $MA(\omega_0, \omega_1)$, *where* $A_0 = A(\omega_0)$, $A_1 = A(\omega_1)$ *and* $\omega_0 \subseteq \Omega_0$, $\omega_1 \subseteq \Omega_1$ *are open relatively compact sets. Symbolically,*

$$A_0 \check{\otimes} A_1 \cong MA.$$

Proof. – Let S_0, S_1 be the state spaces of A_0, A_1 respectively. We recall that by the results of Davies and Vincent-Smith [11], $S_0 \otimes S_1$ is the state space of $A_0 \check{\otimes} A_1$ and is a simplex. Moreover, $A_0 \check{\otimes} A_1$ is isometrically order isomorphic to $BA(S_0, S_1)$, the Banach space of jointly continuous biaffine functionals on $S_0 \times S_1$. Since each function in $BA(S_0, S_1)$ achieves its maximum at a point in the product of the extreme boundaries of S_0 and S_1 and this is just the product of the two sets of regular boundary points, $\partial_r\omega_0$, $\partial_r\omega_1$, the restriction map combined with the inverse of the natural injection $\overline{\omega}_0 \times \overline{\omega}_1 \to S_0 \times S_1$ provides an isometry

$$G : BA(S_0, S_1) \to MA(\omega_0, \omega_1)$$

which is bipositive. Hence, we also have an isometric order isomorphism

$$A_0 \check{\otimes} A_1 \to MA(\omega_0, \omega_1).$$

We will identify $A_0 \check{\otimes} A_1$ with its image under this map.

Now, as we have just remarked, the state space of $A_0 \check{\otimes} A_1$ is a simplex and $A_0 \check{\otimes} A_1$ is a function space containing the constants and separating the points of $\overline{\omega}_0 \times \overline{\omega}_1$, such that

$$A_0 \check{\otimes} A_1 \subseteq MA(\omega_0, \omega_1) \subseteq C(\overline{\omega}_0 \times \overline{\omega}_1)$$

and if we can show that the Choquet boundaries of $A_0 \check{\otimes} A_1$ and $MA(\omega_0, \omega_1)$ are equal then the density theorem of Edwards and Vincent-Smith [13] will give us that the former subspace is dense in the latter, and, since they are both closed, they will be equal.

To show that the Choquet boundaries are equal we remark firstly that the Choquet boundary of $A_0 \check{\otimes} A_1$ is just the set $\partial_r\omega_0 \times \partial_r\omega_1$ as this is the extreme boundary of the simplex $S_0 \otimes S_1$. To prove that this set is also the Choquet boundary of $MA(\omega_0, \omega_1)$ we use the characterisation of the Choquet boundary as the set of points in $\overline{\omega}_0 \times \overline{\omega}_1$ which do not possess a representing measure over MA larger than $\delta_{x,y}$ in the ordering defined by the min-stable wedge W(MA) generated by MA (see e.g. Effros and Kazdan [14]).

Let (x, y) lie in the Choquet boundary of MA, and let $\mu \succ \delta_x$ in the ordering defined by W_0 – the min-stable wedge generated by A_0 – then form the measure $\mu \otimes \delta_y$. It is not hard to see that this measure is greater than $\delta_{x,y}$ in the ordering defined by W(MA), e.g. if $f = h_1 \wedge h_2 \wedge \ldots \wedge h_n$ lies in W(MA), where

$$h_i \in \mathrm{MA} \ (i = 1, \ldots, n)$$

then

$$\int f d\mu \otimes \delta_y = \int h_1 \wedge h_2 \wedge \ldots \wedge h_n(\,.\,, y)\, d\mu\,,$$

and the function $x \mapsto h_1 \wedge h_2 \wedge \ldots \wedge h_n(x, y)$ lies in W_0, by definition of $MA(\omega_0, \omega_1)$, so

$$\int f d\mu \otimes \delta_y \leqslant f(x, y).$$

This shows that $\mu \otimes \delta_y \succ \delta_{x,y}$ and it is clear that this implies $\mu \otimes \delta_y = \delta_{x,y}$ since (x, y) lies in the Choquet boundary and it is immediate from this that $\mu = \delta_x$, hence x lies in the Choquet boundary of $\overline{\omega}_0$ with respect to A_0, i.e. in $\partial_r\omega_0$. By exactly the same reasoning $y \in \partial_r\omega_1$ and so $(x, y) \in \partial_r\omega_0 \times \partial_r\omega_1$.

The converse inequality, that the Choquet boundary of MA contains $\partial_r\omega_0 \times \partial_r\omega_1$, follows immediately from the fact that $\mathrm{MA} \supseteq A_0 \check{\otimes} A_1$. This completes the proof.

COROLLARY 12. – $S(\mathrm{MA}) = S_0 \otimes S_1$. *Hence*

(a) S(MA) *is a simplex.*

(b) $\mathrm{exS(MA)} = \partial_r\omega_0 \times \partial_r\omega_1$.

(c) *The Shilov boundary of* MA $= \overline{\partial_r\omega_0 \times \partial_r\omega_1}$.

Proof. – This is the dual statement of Theorem 11. (a) and (b) follow from the results of Davies and Vincent-Smith in [11]. (c) is just the statement that the Shilov boundary is the closure of the Choquet boundary. This completes the proof.

Part (c) of this corollary was obtained by J. Walsh in [26] for Euclidean space, using probabilistic techniques.

It is clear from the statement and proof of Axiom IIM that for ω_0 , ω_1 regular, the map $f \mapsto J_f$ from $C(\partial\omega_0 \times \partial\omega_1)$ to $MA(\omega_0, \omega_1)$ is a positive linear map and so the map $f \mapsto J_f(x, y)$ is a positive measure on $\partial\omega_0 \times \partial\omega_1$, called $\rho_{x,y}^{\omega_0,\omega_1}$.

We complete this paragraph by quoting a proposition that is a straight generalisation of the convergence theorem of Hervé. We do not give the proof since it follows very closely the proof of the original result, to be found, for instance, in Brelot [7]. We merely remark that the proof requires the fact that positive separately harmonic functions are continuous which is Lemma 1 of [16].

PROPOSITION 13. – *Let $\{h_n\}$ be a sequence of separately harmonic functions in* $MH^+(\omega_0, \omega_1)$, *pointwise convergent to* $h \in MH^+$, *then*

(a) $h = \sup\limits_n \widehat{(\inf\limits_{p>n} h_p)}$, *where "$\wedge$" denotes the lower semi-continuous regularisation.*

(b) $\{h_n\}$ *is uniformly convergent on compacta to* h.

2. In this paragraph we show that the space of functions obtained by forming the tensor product of two spaces of differences of positive harmonic functions is the space of differences of positive separately harmonic functions. Many of the results in [16] are deduced from this fact. The hypothesis on each space is, in addition to the basic three axioms of Brelot, the existence of a countable base of open sets.

We consider the two spaces Ω_0 , Ω_1. Recall that

$$MH^+ = MH^+(\Omega_0, \Omega_1)$$

is the cone of positively separately harmonic functions. By Lemma 1 of [16] every function in this cone and in the space $MH_0 = MH^+ - MH^+$ generated by it are continuous. Moreover, as we remarked above MH^+ has the convex base,

$$X_M = \{h \in MH^+ : h(x_0, y_0) = 1\},$$

for a fixed, arbitrary pair $(x_0, y_0) \in \Omega_0 \times \Omega_1$. We give MH^+ the topology of uniform convergence on compacta.

THEOREM 14. – *Let* $H^+(\Omega_0), H^+(\Omega_1)$ *be the positive cones of harmonic functions on the spaces* Ω_0, Ω_1, *respectively and let*

$$X_0 = \{h \in H^+(\Omega_0) : h(x_0) = 1\} \quad , \quad X_1 = \{h \in H^+(\Omega_1) : h(y_0) = 1\}$$

be their respective bases. Then X_M *is affinely homeomorphic to* $X_0 \otimes X_1$. *Symbolically,* $X_M \cong X_0 \otimes X_1$.

Proof. – Let L_0, L_1 be the subduals of $H^+(\Omega_0) - H^+(\Omega_0)$, $H^+(\Omega_1) - H^+(\Omega_1)$ respectively as in Theorem 1. If $f \in L_0 \otimes L_1$ then f has an expression of the form $f = \sum_{r=1}^{n} f_r \otimes f'_r$ for $f_r \in L_0$, $f'_r \in L_1$, and corresponds to the measure (non-uniquely) of compact support $\subseteq \Omega_0 \times \Omega_1$, $\mu = \sum_{r=1}^{n} \mu_r \otimes \mu'_r$, where μ_r corresponds to f_r and is a measure of compact support on Ω_0, and μ'_r corresponds to f'_r and is a measure of compact supportoon Ω_1 $(r = 1, \ldots, n)$. Now define the map $S : X_M \to X_0 \otimes X_1$ in the following way : for $h \in X_M$,

$$S(h)\,(f) = \int h(x, y)\, d\mu(x, y) = \sum_{r=1}^{n} \iint h(x, y)\, d\mu_r(x)\, d\mu'_r(y).$$

S is a well-defined map by the remark after Corollary 4, because h is separately harmonic and hence by a simple application of Fubini's Theorem, the map $y \mapsto \int h(x, y)\, d\mu_r(x)$ is also harmonic. It is clear that $S(h)$ is a positive linear map from $L_0 \otimes L_1 \to R$, and $L_0 \otimes L_1$ has an order unit so it is also continuous and hence extends to a positive linear form on $\overline{L_0} \check{\otimes} \overline{L_1}$ (as in the proof of the validity of Axiom IIM the sup norm on $L_0 \otimes L_1$ coincides with the weak tensor product of the sup norms on L_0 and L_1). Hence, $S(h)$ lies in the positive cone of the dual of $\overline{L_0} \check{\otimes} \overline{L_1}$, that is to say, by definition of the tensor product of simplexes, in a scalar multiple of $X_0 \otimes X_1$. But if $f = 1 \otimes 1$, f corresponds to the measure $\delta_{x_0} \times \delta_{y_0}$, so that

$$S(h)\,(f) = \iint h(x\,,y)\,d\delta_{x_0}(x)\,d\delta_{y_0}(y) = h(x_0\,,y_0) = 1,$$

and f is the unit element in $\overline{L}_0 \check{\otimes} \overline{L}_1$, in other words $S(h) \in X_0 \otimes X_1$.

S is clearly affine by the linearity of integration, and it is an injection because if $S(h_1)\,(f) = S(h_2)\,(f)$ for all $f \in \overline{L}_0 \check{\otimes} \overline{L}_1$, select f_0 such that it corresponds to $\delta_x \otimes \delta_y$, then

$$h_1(x\,,y) = S(h_1)\,(f_0) = S(h_2)\,(f_0) = h_2(x\,,y)\,,$$

and $x\,,y$ are arbitrary. To show that S maps onto $X_0 \otimes X_1$ is rather more difficult.

Let $F \in X_0 \otimes X_1$, then, as $X_0 \otimes X_1$ is a compact, metrizable simplex with extreme boundary $\mathrm{ex}X_0 \times \mathrm{ex}X_1$, there exists a unique probability measure, ν, concentrated on $\mathrm{ex}X_0 \times \mathrm{ex}X_1$ which represents F over $A(X_0 \otimes X_1) \cong \overline{L}_0 \check{\otimes} \overline{L}_1$. Then

$$\langle f, F\rangle = \int_{\mathrm{ex}X_0 \times \mathrm{ex}X_1} \langle f\,, u \otimes v\rangle\, d\nu\,,$$

where $f \in \overline{L}_0 \check{\otimes} \overline{L}_1$. We remark that every element of $\mathrm{ex}X_0 \times \mathrm{ex}X_1$ lies in the image of S ; if $(u\,,v) \in \mathrm{ex}X_0 \times \mathrm{ex}X_1$ and $f \in L_0 \otimes L_1$,

$$\begin{aligned} S(uv)\,(f) &= \sum_{r=1}^{n} \iint u(x)\,v(y)\,d\mu_r(x)\,d\mu_r'(y)\,,\\ &= \sum_{r=1}^{n} \int u(x)\,d\mu_r(x) \int v(y)\,d\mu_r'(y)\,,\\ &= \sum_{r=1}^{n} \langle u\,, f_r\rangle \langle v\,, f_r'\rangle\,, \end{aligned}$$

where $f_r \in L_0$, corresponds to μ_r, and f_r', to μ_r',

$$= \langle f\,, u \otimes v\rangle\,.$$

In other words $S : uv \mapsto u \otimes v$. In general, we have the measure ν corresponding to $F \in X_0 \otimes X_1$; define h on $\Omega_0 \times \Omega_1$ by

$$h(x\,,y) = \int_{\mathrm{ex}X \times \mathrm{ex}X_1} u(x)\,v(y)\,d\nu(u\,,v)\,,$$

then h is clearly separately harmonic by Martin's Representation, and lies in X_M since

$$h(x_0\,,\,y_0) = \int u(x_0)\,v(y_0)\,d\nu(u\,,\,v)\,,$$
$$= \int 1\;d\nu(u\,,\,v)\,,$$
$$= 1.$$

Now, for this h and for $f = \sum_{r=1}^{n} f_r \otimes f_r'$ as usual,

$$S(h)\,(f) = \sum_{r=1}^{n} \iiint u(x)\,v(y)\,d\nu(u\,,\,v)\,d\mu_r(x)\,d\mu_r'(y)\,,$$

and the map $(u\,,v\,,x\,,y) \mapsto u(x)\,v(y)$ is continuous from

$$X_0 \times X_1 \times \Omega_0 \times \Omega_1$$

to R so that

$$S(h)\,(f) = \int \left\{ \sum_{r=1}^{n} \iint u(x)\,v(y)\,d\mu_r(x)\,d\mu_r'(y) \right\} d\nu(u\,,\,v),$$
$$= \int S(uv)\,(f)\;d\nu(u\,,\,v)\,,$$
$$= \int \langle f\,,\,u \otimes v \rangle\;d\nu(u\,,\,v)\,,$$
$$= \langle f\,,\,F \rangle.$$

This holds for all $f \in L_0 \otimes L_1$, so, by continuity, for all f in $\overline{L_0} \check{\otimes} \overline{L_1}$. So S is a surjection. Finally, S is a homeomorphism by the next lemma.

LEMMA 15. – *The topology of uniform convergence on compacta* $\subseteq \Omega_0 \times \Omega_1$, *on* X_M *is the same as the topology on* X_M *induced by the map* S *from* $X_0 \otimes X_1$.

Proof. – The topology on $X_0 \otimes X_1$ is the weak* topology in the duality $\langle \overline{L_0} \check{\otimes} \overline{L_1}, H_0(\Omega_0) \otimes H_0(\Omega_1) \rangle$; we may consider sequences in $X_0 \otimes X_1$ since X_0 and X_1 and, hence, $X_0 \otimes X_1$ are metrizable. Now, $p_n \to p$ in $X_0 \otimes X_1$ if and only if $\langle p_n\,,f \rangle \to \langle p\,,f \rangle$ for all $f \in \overline{L_0} \check{\otimes} \overline{L_1}$. Suppose for the moment that f is in $L_0 \otimes L_1$. We have that $\langle S(h)\,,(f) \rangle = \int h(x\,,y)\;d\mu(x\,,y)$, where $h \in X_M$ and μ on

$\Omega_0 \times \Omega_1$ corresponds to f. The support of μ is a finite union of sets of the form $K \times L$, where $K \subseteq \Omega_0$, $L \subseteq \Omega_1$ are compact, and so is compact. If $h_n \to h$ uniformly on compacta, for h_n, $h \in X_M$, then $\int h_n \, d\mu \to \int h d\mu$ for all μ corresponding to an $f \in L_0 \otimes L_1$, so $\langle S(h_n), f \rangle \to \langle S(h), f \rangle$ for all $f \in L_0 \otimes L_1$. Now the map

$$f \mapsto \langle S(h), f \rangle$$

is continuous linear because $S(h)$ lies in the continuous dual of $\overline{L_0} \check{\otimes} \overline{L_1}$; so if $f_m \in L_0 \otimes L_1$ for all $m \geqslant 1$ and $f_m \to f$ as $m \to \infty$, where $f \in \overline{L_0} \check{\otimes} \overline{L_1}$, then

$$\begin{aligned} \langle S(h_n), f \rangle - \langle S(h), f \rangle \leqslant & |\langle S(h_n), f \rangle - \langle S(h_n), f_m \rangle| + \\ & |\langle S(h_n), f_m \rangle - \langle S(h), f_m \rangle| + \\ & |\langle S(h), f_m \rangle - \langle S(h), f \rangle|, \\ \leqslant & \, \epsilon, \end{aligned}$$

for m and n sufficiently large. So $\langle S(h_n), f \rangle \to \langle S(h), f \rangle$ for all $f \in \overline{L_0} \check{\otimes} \overline{L_1}$.

Conversely, if $\langle S(h_n), f \rangle \to \langle S(h), f \rangle$ for all $f \in \overline{L_0} \check{\otimes} \overline{L_1}$, choose f to correspond to the measure $\delta_x \otimes \delta_y$, this shows that $h_n \to h$ pointwise, and by Proposition 13, we have uniform convergence on compacta $\subseteq \Omega_0 \times \Omega_1$. This completes the proof.

COROLLARY 16. – X_M *is a compact Choquet simplex in the topology of uniform convergence on compacta.*

Proof. – Since the tensor product of two simplexes in a simplex, [11], Theorem 3.

One can define *minimal separately harmonic functions* in exactly the same way as minimal harmonic functions by saying that a separately harmonic function is minimal if the only positive separately harmonic functions that it dominates are scalar multiples of it. Then one has,

COROLLARY 17. – *The minimal separately harmonic functions in* X_M *are the functions of the form* uv *for* u *a minimal harmonic function in* X_0, *and* v *a minimal harmonic function in* X_1.

Proof. – Immediate from Theorem 14 and Theorem 4 of [11].

COROLLARY 18. – *(Gowrisankaran) To every* $h \in MH^+(\Omega_0, \Omega_1)$ *corresponds a unique positive measure* ν_h *on* $exX_0 \times exX_1$ *such that*

$$h(x, y) = \int u(x) v(y) \, d\nu_h(u, v)$$

for all $(x, y) \in \Omega_0 \times \Omega_1$.

Proof. – Immediate from Theorem 14 and Choquet's Theorem bearing in mind Corollary 17.

COROLLARY 19. – *(Gowrisankaran)* $MH^+(\Omega_0, \Omega_1)$ *is a lattice in its own order with compact base* X_M.

Proof. – By definition of a simplex.

Gowrisankaran proved, in fact, that MH^+ is a complete lattice.

COROLLARY 20. – *(Gowrisankaran) The set* X_M *is equicontinuous.*

Proof. – By Ascoli's Theorem X_M is equicontinuous if and only if it is relatively compact, and X_M is compact by Lemma 15.

Following Walsh and Loeb [25] we make the following definition. Let F be a topological vector space (locally convex), let $\omega \subseteq \Omega$ be an open subset of a harmonic space, and let $f : \omega \to F$ be a function, then f is called F-*harmonic* if for all $p \in F^*$, the map $\omega \to R$ defined by $x \to \langle f(x), p \rangle$ is harmonic.

COROLLARY 21. – *(c.f. Gowrisankaran* [16], *Lemma 6) The maps* $y \to \nu_y^h$, *where* $h \in MH^+$ *and* ν_y^h *is the canonical measure on* exX_0 *representing the harmonic function* $h(\,.\,, y)$, *are* F-*harmonic, where* F *is the space* $\mathfrak{M}(\overline{exX_0})$ *in the weak* topology.*

Proof. – By Corollary 18 we have the representation

$$h(x, y) = \int u(x) v(y) \, d\nu(u, v)$$

so that $d\nu_y^h(u) = v(y) \, d\nu(u, v)$, by Martin's uniqueness property, since both are measures concentrated on exX_0. The dual of $\mathfrak{M}(\overline{exX_0})$ with the weak* topology is $C(\overline{exX_0})$ and if $f \in C(\overline{exX_0})$,

$$\langle f , \nu_y^h \rangle = \int f(u)\, d\nu_y^h(u) = \int f(u) v(y)\, d\nu(u\ ,\ v) = \int v(y)\, d\nu_f'(v)\,,$$

where ν_f' is a measure on exX_1 so $\langle f , \nu_y^h \rangle$ is a harmonic function of y, i.e. $y \to \nu_y^h$ is F-harmonic.

3. In this paragraph we show that a result on uniform approximation of harmonic functions, originating from Deny, can be extended in a natural way, to separately harmonic functions. We consider in this paragraph harmonic spaces of Brelot satisfying, in addition to the basic three axioms, the countability axiom, the axiom of proportionality and Axiom D. Let $\omega \subseteq \Omega$ be an open, relatively compact subset of such a space. Recall that, under our assumptions, the subspace

$$\mathrm{A}(\omega) = \{f \in \mathrm{C}(\overline{\omega}) : f \text{ is harmonic in } \omega\}\,,$$

is a simplex space and is uniformly closed. We now also define

$\mathrm{D}(\omega) = \{f \in \mathrm{A}(\omega)\ :\ f$ extends to a function harmonic in a neighbourhood of $\overline{\omega}\}$.

It can be shown that the state space of D is also a simplex, call it S(D). It is shown that the Choquet boundary of D is contained in $\partial\omega$ and those points in the Choquet boundary are called stable points of ω. Stable points play the same role with respect to D as regular points play with respect A. We quote the following :

THEOREM 22. — *D is uniformly dense in A if and only if every regular point is stable.*

It is not hard to see that since $\mathrm{D} \subseteq \mathrm{A}$, every stable point is necessarily regular. We intend to show that by suitably redefining D and A for separately harmonic functions we can prove a result, in this context, analogous to Theorem 22. So consider harmonic spaces $\Omega_0\ , \Omega_1$ and their associated objects — their allegiance being defined by the suffix 0 or 1. In paragraph 1, we defined the set $\mathrm{MA}(\omega_0\ , \omega_1)$, $\omega_0\ , \omega_1$ open and relatively compact, as follows ;

$\mathrm{MA}(\omega_0\ , \omega_1) = \{h : h$ is continuous and separately harmonic in $\omega_0 \times \omega_1$ and has a continuous extension to $\overline{\omega}_0 \times \overline{\omega}_1$ such that $h(x\ , .)$ is harmonic in ω_1 for all $x \in \overline{\omega}_0$ and similarly for x and y reversed$\}$.

We now define the set $MD(\omega_0, \omega_1)$ as follows,

$MD(\omega_0, \omega_1) = \{h \in MA(\omega_0, \omega_1) : h$ extends to a function continuous and separately harmonic in a neighbourhood of $\overline{\omega}_0 \times \overline{\omega}_1\}$.

We give MA and MD the sup norm topologies.

LEMMA 23. – *The space* $\overline{D}_0 \check{\otimes} \overline{D}_1$, *which is the completion of* $\overline{D}_0 \otimes \overline{D}_1$ *with respect to the weak tensor product norm, is isometrically order isomorphic to* $\overline{MD}$; *symbolically* $\overline{D}_0 \check{\otimes} \overline{D}_1 \cong \overline{MD}$.

Proof. – We do not give the proof as it does not differ in any essential way from the proof of Theorem 11. It uses the Edwards-Vincent-Smith density theorem in exactly the same way.

PROPOSITION 24. – $\overline{MD} = MA$ *if and only if every point pair* $(x, y) \in \partial\omega_0 \times \partial\omega_1$ *which is a pair of stable points is also a pair of regular points.*

Proof. – If every pair of stable points is a pair of regular points then by Theorem 22, $\overline{D}_0 = A_0$ and $\overline{D}_1 = A_1$ so by Theorem 11 and Lemma 23,

$$\overline{MD} = \overline{D}_0 \check{\otimes} \overline{D}_1 = A_0 \check{\otimes} A_1 = MA .$$

Conversely, if $\overline{MD} = MA$ then by the easy half of the density theorem

$$S(MD) = S(\overline{MD}) = S(MA) \text{ and so } exS(MD) = exS(MA)$$

but $exS(MD) = exS(D_0) \times exS(D_1)$ because $S(MD) = S(D_0) \otimes S(D_1)$, by dualising Lemma 23, and $exS(MA) = exS(A_0) \times exS(A_1)$. Hence,

$$exS(D_0) \times exS(D_1) = exS(A_0) \times exS(A_1),$$

but the left hand side defines the set of pairs of stable points and the right hand side the set of pairs of regular points. This completes the proof.

The subject of uniform approximations of harmonic functions is dealt with in a number of papers, notably, Brelot [5], de la Pradelle [23], and Vincent-Smith [24].

4. In this, the final, paragraph of this paper we show that Keldych's Theorem also extends to separately harmonic functions. We assume in this paragraph the same conditions on our space Ω as in the previous paragraph.

Keldych's Lemma asserts that if $\omega \subseteq \Omega$ is open and relatively compact and if for all $x \in \partial\omega$, $\partial\omega \backslash \{x\}$ is connected, then for each regular boundary point x, there exists a positive function $f \in \mathrm{C}(\partial\omega)$ such that $f(x) = 0$, $f > 0$ at every other point and f extends continuously to a function H_f^ω harmonic in ω. Keldych's Lemma is the key fact which enables one to prove,

THEOREM 25. – *(Keldych). Suppose* ω *satisfies the conditions stated above. Let* $\mathrm{L}_f(x)$ *be, for each* $f \in \mathrm{C}(\partial\omega)$ *a function on* ω *such that*

(a) $x \mapsto \mathrm{L}_f(x)$ *is harmonic.*

(b) $\mathrm{L}_f = \mathrm{H}_f^\omega$ *whenever* H_f^ω *extends continuously to* f.

(c) *The function* $f \mapsto \mathrm{L}_f : \mathrm{C}(\partial\omega) \to \mathrm{H}(\omega)$ *is positive and linear. Then* $\mathrm{L}_f =$ *the solution of the generalised Dirichlet problem,* H_f^ω, *for all* $f \in \mathrm{C}(\partial\omega)$.

We remark that Keldych's Theorem extends almost word for word to separately harmonic functions. Firstly, one can see that Keldych's Lemma extends : Let $\omega_0 \times \omega_1 \subseteq \Omega_0 \times \Omega_1$ with ω_0 , ω_1 as for ω in Theorem 25. Then for $x_i \in \partial\omega_i$, regular boundary points, we can find peaking functions $f_i \in \mathrm{C}(\partial\omega_i)$ such that $f_i > 0$ everywhere except at x_i where $f_i(x_i) = 0$, and f_i extends to an element of A_i $(i = 0, 1)$. Then $f_0 \otimes f_1 \in \mathrm{C}(\partial\omega_0 \times \partial\omega_1)$ extends to an element of $\mathrm{A}_0 \otimes \mathrm{A}_1 \subseteq \mathrm{A}_0 \check{\otimes} \mathrm{A}_1 \subseteq \mathrm{C}(\overline{\omega}_0 \times \overline{\omega}_1)$. It is easy to see that $f_0 \otimes f_1$ is a peaking function for $\mathrm{MA}(\omega_0 , \omega_1) = \mathrm{A}_0 \check{\otimes} \mathrm{A}_1$. This encourages one to believe that we can extend Keldych's Theorem, although we do not use it in the proof, in fact.

PROPOSITION 26. – *Let* ω_0 , ω_1 *be as in Theorem 25. Let* $\Lambda_f(x , y)$ *be, for each* $f \in \mathrm{C}(\partial\omega_0 \times \partial\omega_1)$, *a function on* $\omega_0 \times \omega_1$ *such that ;*

(a) $(x , y) \mapsto \Lambda_f(x , y)$ *is separately harmonic in* $\omega_0 \times \omega_1$.

(b) $\Lambda_f = \mathrm{J}_f$ *whenever* f *extends continuously to an element of* MA. *(See Paragraph 1 for the definition of* J_f*).*

(c) *The function* $f \mapsto \Lambda_f : \mathrm{C}(\partial\omega_0 \times \partial\omega_1) \to \mathrm{MH}(\omega_0 , \omega_1)$ *is positive and linear.*

Then Λ_f *is unique.*

Proof. – Fix $g \in C(\partial\omega_1)$, $g > 0$ such that g extends continuously to $H_g^{\omega_1}$, its Dirichlet solution in ω_1, and fix $y \in \omega_1$. Then consider the map

$$f \mapsto \Lambda_{f \otimes g}(x\,,\,y)/H_f(y)\,,$$

It can be seen that this map satisfies conditions (a), (b) and (c) of Keldych's Theorem and so

$$\Lambda_{f \otimes g}(x\,,\,y) = H_f^{\omega_0}(x)\,H_g^{\omega_1}(y)\,,$$

for all $x \in \omega_0\,, y \in \omega_1$. Now, fix $f \in C(\partial\omega_0)$, $f > 0$, and $x \in \omega_0$, and consider the map

$$g \mapsto \Lambda_{f \otimes g}(x\,,\,y)/H_f^{\omega_0}(x)\,.$$

By what has just been said this map satisfies the conditions of Keldych's Theorem as well so that

$$\Lambda_{f \otimes g}(x\,,\,y) = H_f^{\omega_0}(x)\;H_g^{\omega_1}(y)$$

for all strictly positive $f \in C^+(\partial\omega_0)$ and $g \in C^+(\partial\omega_1)$, and all

$$(x\,,\,y) \in \omega_0 \times \omega_1\,.$$

By the linearity of Λ and the decomposition $f = (f^+ + \epsilon) - (f^- + \epsilon)$, $g = (g^+ + \epsilon) - (g^- + \epsilon)$, one can see that this equation remains valid for f and g arbitrary.

Finally, if $f = \sum_{i=1}^{n} f_i \otimes g_i$ is in $C(\partial\omega_0 \times \partial\omega_1)$ then by linearity we must have

$$\Lambda_f(x\,,\,y) = \sum_{i=1}^{n} H_{f_i}^{\omega_0}(x)\,H_{g_i}^{\omega_1}(y)\,.$$

Thus Λ_f is uniquely defined on a dense subspace of $C(\partial\omega_0 \times \partial\omega_1)$ and since for all $x\,,y$, $\Lambda_f\,(x\,,y)$ is a positive linear functional on an order unit norm space it is continuous and so extends uniquely to the whole space. This completes the proof.

Because of this result we call Λ_f the solution of the generalised Dirichlet problem for separately harmonic functions. For Keldych's Theorem see Brelot [6], [7].

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I. REAY,
Mathematisches Institut
Universität Erlangen D-852
Bismarck Str. 1 1/2
Allemagne Fédérale.