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SPACES OF BAIRE FUNCTIONS I

by J. E. JAYNE

1. Introduction.

In this paper it is proved that if a compact Hausdorff space X contains a non-empty perfect subset, then for each countable ordinal α the Banach space of bounded real-valued Baire functions of class α on X is a proper subspace of the Banach space of bounded real-valued Baire functions of class $\alpha + 1$ on X. This was announced in [14] and [16] and contained in [15]. This result for uncountable compact subsets of Euclidean space, which implies the result for all uncountable compact metric spaces, is due to Lebesgue [19]. A proof is given by Hausdorff [12, p. 207]. If a compact space X does not contain a non-empty perfect subset, then it is known that the space of all bounded real-valued Baire functions on X coincides with the space of bounded Baire functions of class 1 [22].

In the study of the space of bounded real-valued Baire functions on a space X the compactness of X is not essential. For example, an uncountable subset of the real line **R** is a Baire (equivalently, Borel) subset if and only if it is Baire isomorphic to the unit interval [0, 1] ([18, pp. 447 and 489]). Thus the Banach space of bounded real-valued Baire functions on any uncountable Baire subset of **R** is isometrically isomorphic to the space of bounded real-valued Baire functions on [0, 1].

With this in mind the results will be phrased for a class of completely regular Hausdorff spaces which will contain all compact spaces, all complete separable metric spaces, all Baire subsets of these spaces, and more generally, all subsets of these spaces which are obtainable from the Baire subsets by Souslin's operation (A). These spaces, which will be called *disjoint analytic spaces*, are defined to be those completely regular Hausdorff spaces which are the images of analytic subsets of the Baire 0-dimensional product space N^N ($N = \{1, 2, 3, \dots\}$) under disjoint upper semi-continuous compact-valued maps. This class of spaces, as will be seen, is a proper subclass of Choquet's completely regular K-analytic spaces ([7] and [8]). The precise definitions are given below.

Characterizations of disjoint analytic spaces are given in theorem 3 and significant properties in theorems 4 and 5.

The central result on the existence of Baire classes is then

THEOREM 6. — *If* X *is a disjoint analytic space, then the following are equivalent:*

1) *For each countable ordinal* α *the Banach space of bounded real-valued Baire functions on* X *of class* α *is a proper subspace of the space of bounded real-valued Baire functions of class* $\alpha + 1$.

2) *There exists a bounded real-valued Baire function on* X *of class* 2 *which is not of class* 1.

3) *The family of Baire subsets of* X *is not invariant under Souslin's operation* (A).

4) *The space of all bounded real-valued Baire functions on* X *is a proper subspace of the space of all bounded real-valued functions on* X *which are continuous in the topology having the family of Baire subsets of* X *as a base for the open sets.*

5) *The weakest topology on* X *for which all of the real-valued Baire functions on* X *are continuous does not have the Lindelöf property.*

6) X *contains a non-empty compact perfect subset.*

An analogous set of equivalences is given for pseudocompact spaces in theorem 9. This result is used to give an example of a completely regular Hausdorff space X which contains no non-empty compact perfect subsets, but yet has the property that for each countable ordinal α the Banach space of bounded real-valued Baire functions on X of class α is a proper subspace of the space of such functions of class $\alpha + 1$.

In the seventh section the sequential stability index of a space of real-valued functions defined on a set X is considered; that is, the smallest ordinal α such that the iteration of the process of adjoining the pointwise limits of bounded sequences of functions stops producing new functions on precisely the α-th iteration. In these terms theorem 6 implies that the only possible indices for the space of bounded continuous real-valued functions on a disjoint analytic space are 0, 1 and Ω (the first uncountable ordinal).

In theorem 10 it is shown that for any infinite completely regular Hausdorff space the Banach space of bounded real-valued Baire functions of class 1 contains closed linear subspaces with index α for each countable ordinal α.

Finally, the sequential stability index of a closed linear subspace of the space of continuous real-valued functions on a compact space is characterized in terms of weak* sequential convergence in the second dual of the subspace, thus giving invariance of the index under isomorphic embeddings in the space of continuous real-valued functions on any compact space.

The second part of this paper will consider the problem of evaluating the sequential stability index of the space of continuous real-valued affine functions on a compact convex subset of a Hausdorff locally convex real topological vector space.

2. Preliminary definitions and notation.

All topological spaces considered will be completely regular Hausdorff spaces. The word space will be used to refer to such a topological space. A perfect subset of a space X is a closed subset which in its relative topology has no isolated points.

The space of bounded continuous real-valued functions on a space X will be denoted by $C(X)$. Let

$$B_0(X) = C(X)$$

and inductively define $B_\alpha(X)$ for each ordinal $\alpha \leqslant \Omega$ (Ω denotes the first uncountable ordinal) to be the space of pointwise limits of bounded sequences of functions in

$$\bigcup_{\beta<\alpha} B_\beta(X)$$

With the pointwise operations and the supremum norm each $B_\alpha(X)$ is a lattice-ordered Banach algebra [12, § 41].

The family of Baire sets of a space X is the smallest family of sets containing the zero sets of continuous real-valued functions (i.e. of the form $Z(f) = \{x \in X : f(x) = 0\}$), and closed under countable unions and countable intersections.

The Baire sets of X of multiplicative class 0, denoted $Z_0(X)$, are the zero sets of continuous real-valued functions. The sets of additive class 0, denoted $CZ_0(X)$, are the complements of the sets in $Z_0(X)$. Define inductively for each countable ordinal α the sets of multiplicative class $\alpha + 1$, denoted $Z_{\alpha+1}(X)$, to be the countable intersections of the sets of additive class α and the sets of additive class $\alpha + 1$, denoted $CZ_{\alpha+1}(X)$, to be their complements. The sets of multiplicative class λ (λ a limit ordinal), denoted $Z_\lambda(X)$, are defined to be the countable intersections of countable unions of sets in $\bigcup_{\alpha<\lambda} Z_\alpha(X)$, and the sets of additive class λ, denoted $CZ_\lambda(X)$ are defined to be their complements.

For every $\alpha < \Omega$ the family of sets of ambiguous class α is defined to be

$$A_\alpha(X) = \{S \subseteq X : S \in Z_\alpha(X) \quad \text{and} \quad S \in CZ_\alpha(X)\}.$$

The family of subsets $EA_\alpha(X)$ of exactly ambiguous class α of X is defined by

$$EA_\alpha(X) = A_\alpha(X) \setminus \bigcup_{\xi<\alpha} A_\xi(X)$$

We have

$$Z_\Omega(X) = \bigcup_{0 \leqslant \alpha < \Omega} EA_\alpha(X),$$

and

$$EA_\alpha(X) \cap EA_\xi(X) = \varnothing \quad \text{for} \quad \alpha \neq \xi.$$

Note that for any space X we have

$$Z_\Omega(X) = CZ_\Omega(X).$$

Thus we need only consider ordinals $\alpha \leqslant \Omega$.

Baire sets and functions are related by the following classical theorem :

THEOREM (Lebesgue-Hausdorff [18, p. 393]). — *Let* f *be a bounded real-valued function on a space* X. *For each ordinal* $\alpha < \Omega$ $f \in B_\alpha(X)$ *if and only if* $f^{-1}[F] \in Z_\alpha(X)$ *for every closed subset* F *of* **R**.

We also have that a set $B \subseteq X$ is in $Z_\alpha(X)$ if and only if there is an $f \in B_\alpha(X)$ such that $B = Z(f)$. The sufficiency is clear from the preceeding theorem. For the necessity note that if α is not a limit ordinal then

$$B = \bigcap_{n=1}^{\infty} B_n, \qquad B_n \in \bigcup_{\beta<\alpha} CZ_\beta(X)$$

and thus each $B_n \in A_\alpha(X)$. This implies that χ_{B_n}, the characteristic function of B_n, is in $B_\alpha(X)$. Define

$$f = \sum_{n=1}^{\infty} 2^{-n}\chi_{B_n}$$

Then $B = Z(f)$ and, since this series converges uniformly, $f \in B_\alpha(X)$. The same argument applies for a limit ordinal λ since the countable union of sets

$$B_n \in \bigcup_{\alpha<\lambda} Z_\alpha(X)$$

is in $A_\lambda(X)$.

If H is a family of subsets of a space X, then the Souslin — H subsets are those of the form

$$\bigcup_{\sigma \in N^N} \bigcap_{s<\sigma} H_s \in H,$$

where $N = \{1, 2, 3, \ldots\}$ and $s < \sigma$ means that s is a finite restriction of σ.

For any space X the Souslin — $Z_\alpha(X)$ sets coincide with the Souslin — $Z_0(X)$ sets for each $\alpha \leqslant \Omega$. This family of subsets of X will be denoted by ZS(X).

A map f of a space X into a space Y is called proper if it is continuous, closed, and $f^{-1}(y)$ is compact for every $y \in Y$.

For a space X we denote by X_β the set X with the weak topology generated by $B_\Omega(X)$. Since every Baire subset of X is the union of zero sets of X, the topology of X_β coincides with the weak topology generated by $B_\alpha(X)$ for each $\alpha > 0$. This topology also coincides with the weakest topology on X such that every function in the cone of non-negative lower semi-continuous real-valued Baire functions is continuous; that is, this topology is the *fine* topology (see Brelot [4]) associated with this cone. E. R. Lorch has considered this topology and named it the *iota* topology (see [20] and [21]).

3. Preliminary existence results.

THEOREM 1. — *If a space X contains a non-empty compact perfect subset, then*

1) $Z_\alpha(X) \neq Z_\Omega(X)$ *for all* $\alpha < \Omega$,

2) $Z_\Omega(X) \neq ZS(X)$, *and*

3) X_β *is not a Linderlöf space.*

Proof. — 1) Suppose X contains a compact perfect subset K. Then there exists a continuous map

$$f: K \to [0, 1]$$

of K onto the unit interval [24, p. 214]. For each $\alpha < \Omega$ there exists an

$$H_\alpha \in EA_\alpha([0, 1])$$

[12, p. 207]. Since f^{-1} preserves unions and intersections,

$$f^{-1}[H_\alpha] \in EA_\xi(K) \quad \text{for some} \quad \xi \leqslant \alpha.$$

We claim that

$$f^{-1}[H_\alpha] \in EA_\alpha(K) \quad \text{if} \quad \alpha = 0, 1, 2, \quad \text{or} \quad \alpha \geqslant \omega_0,$$

and that

$$f^{-1}[H_\alpha] \in EA_{\alpha-1}(K) \cup EA_\alpha(K) \quad \text{if} \quad 2 < \alpha < \omega_0.$$

Suppose $\alpha \geqslant \omega_0$ and $f^{-1}[H_\alpha] \in EA_\xi(K)$ for $\xi < \alpha$. Then by a transfinite induction argument we have that there exists a sequence

$$S_0 = \{Z(g_n) : g_n \in C(K), n = 1, 3, 5, \ldots\}$$

from which $f^{-1}[H_\alpha]$ is obtained on the ξ-th iterations of the process of which the first step is to form the family S_0^1 of all countable unions of subfamilies of S_0 and the second step is to form the family of all countable intersections of subfamilies of S_0^1. Similarly, there exists a countable family

$$S_e = \{K \setminus Z(g_n) : g_n \in C(K), n = 2, 4, 6, \dots\}$$

from which $f^{-1}[H_\alpha]$ is obtained on the ξ-th iteration of the analogous process.

Define

$$g : K \to \mathbf{R}^N$$

by

$$g(x) = [g_1(x), g_2(x), g_3(x), \dots].$$

Then $g[K]$ is a compact metric space and

$$g^{-1} \circ g[f^{-1}[H_\alpha]] = f^{-1}[H_\alpha].$$

Consider the map

$$f \times g : K \to [0, 1] \times g[K]$$

defined by

$$f \times g(x) = [f(x), g(x)].$$

Then, since $f \times g$ preserves the two iteration processes described above,

$$(f \times g)^{-1} \circ f \times g[f^{-1}[H_\alpha]] = f^{-1}[H_\alpha]$$

and

$$f \times g[f^{-1}[H_\alpha]] \in EA_\xi(f \times g[K]).$$

Let π be the restriction to $f \times g[K]$ of the projection of $[0, 1] \times g[K]$ onto $[0, 1]$. Then

$$\pi[f \times g[f^{-1}[H_\alpha]]] = H_\alpha$$

and

$$\pi^{-1} \circ \pi[f \times g[f^{-1}[H_\alpha]]] = f \times g[f^{-1}[H_\alpha]].$$

This implies that π restricted to

$$f \times g[f^{-1}[H_\alpha]]$$

is a proper map. This contradicts the fact that, for Baire sets in complete separable metric spaces, proper maps preserve both additive and multiplicative Baire class for $\alpha \geqslant \omega_0$ [31, p. 585]. Therefore

$$f^{-1}[H_\alpha] \in EA_\alpha(K) \quad \text{for} \quad \alpha \geqslant \omega_0.$$

The cases $\alpha = 0$ and $\alpha = 1$ are clear. The case $\alpha = 2$ follows from the observation that

$$f^{-1}[H_2] \quad \text{and} \quad f^{-1}[[0, 1] \setminus H_2]$$

are σ-compact, which implies that

$$f \circ f^{-1}[H_2] = H_2$$

and

$$f \circ f^{-1}[[0, 1] \setminus H_2] = [0, 1] \setminus H_2$$

are σ-compact. But this implies that

$$H_2 \in EA_1([0, 1]),$$

which is not the case.

If $2 < \alpha < \omega_0$ the same argument as given for $\alpha \geqslant \omega_0$ leads to a contradiction of the fact that, for Baire sets in complete separable metric spaces, proper maps do not raise additive or multiplicative Baire class by more than one for $2 < \alpha < \omega_0$ [31, p. 585]. Therefore

$$f^{-1}[H_\alpha] \in EA_{\alpha-1}(K) \cup EA_\alpha(K) \quad \text{for} \quad 2 < \alpha < \omega_0.$$

Now since K is compact every continuous real-valued function on K extends to a continuous real-valued function on X [11, p. 43]. Thus for each $Z \in Z_0(K)$ there is a $Z' \in Z_0(X)$ such that

$$Z = Z' \cap K$$

and inductively we have that for each $\alpha < \Omega$ that there is a $Z'_\alpha \in Z_\alpha(X)$ such that

$$f^{-1}[H_\alpha] = Z'_\alpha \cap K.$$

If

$$Z_\alpha \in \bigcup_{\xi < \alpha} EA_\xi(X),$$

then a transfinite induction argument implies that

$$f^{-1}[H_\alpha] \in \bigcup_{\xi < \alpha} EA_\xi(K),$$

which for $\alpha \geqslant \omega_0$ is not the case.

Therefore

$$Z_\alpha(X) \neq Z_\Omega(X) \quad \text{for all} \quad \alpha < \Omega.$$

2) As in part 1) let K be a non-empty compact perfect subset of X and f a continuous map of K onto [0, 1].

There exists an

$$A \in ZS([0, 1]) \setminus Z_\Omega([0, 1])$$

[12, p. 207]. We have

$$A = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z_s, \quad Z_s \in Z_0([0, 1]),$$

and

$$f^{-1}[A] = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} f^{-1}[Z_s].$$

Therefore $f^{-1}[A] \in ZS(K)$.

If

$$f^{-1}[A] \in Z_\Omega(K), \quad \text{then} \quad K \setminus f^{-1}[A] \in Z_\Omega(K).$$

But then

$$f[f^{-1}[A]]$$

and

$$f[K \setminus f^{-1}[A]] = [0, 1] \setminus A$$

would be in $ZS([0, 1])$ [9, pp. 160-161], which implies that $A \in Z_\Omega([0, 1])$ [12, p. 218]. Therefore

$$f^{-1}[A] \in ZS(K) \setminus Z_\Omega(K).$$

Now, since (as noted above) for each $Z \in Z_0(K)$ there is a $Z' \in Z_0(X)$ such that $Z = Z' \cap K$, there is an $A' \in ZS(X)$ such that

$$A = A' \cap K.$$

If A' were in $Z_\Omega(X)$, then A would be in $Z_\Omega(K)$. Therefore

$$A' \in ZS(X) \setminus Z_\Omega(X).$$

3) If K is a non-empty compact perfect subset of X, then, being closed in X, K is closed in X_β. Since for each $Z \in Z_0(K)$ there is a $Z' \in Z_0(X)$ such that

$$Z = Z' \cap K,$$

the relative topology induced on K by X_β coincides with that of K_β. Therefore, if X_β is Lindelöf, then K_β is Lindelöf. But this is the case only if K contains no non-empty perfect subsets [23, p. 27]. Therefore X contains no compact perfect subsets.

Remark. — The first two parts of theorem 1 for compact X were announced in [14]. It is demonstrated in [6] that if X is dense-in-itself and a G_δ subset of its Stone-Čech compactification, then $Z_\alpha(X) \cup CZ_\alpha(X) \neq Z_\Omega(X)$ for all $\alpha < \Omega$, and that $ZS(X) \setminus Z_\Omega(X) \neq \varnothing$. The same conclusion is drawn in [6] if X is a dense-in-itself pseudocompact space. For a dense-in-itself perfectly normal compact space it is proved in [25] that $Z_\alpha(X) \neq Z_\Omega(X)$ for all $\alpha < \Omega$ and in [5] that $ZS(X) \setminus Z_\Omega(X) \neq \varnothing$.

Since the Baire sets of a space X are a base for the topology of X_β and the countable intersection of Baire sets is a Baire set, we have that the countable intersection of open sets in X_β is open. Therefore every zero set in X is clopen (closed and open) in X_β and the family of clopen subsets of X_β is closed under countable unions and countable intersections. More generally, we have

THEOREM 2. — *For any space X the family of clopen subsets of X_β is invariant under Souslin's operation* (A).

Proof. — Let

$$A = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z_s,$$

where each Z_s is clopen in X_β. Then A is open in X_β, being the union of open sets.

If $x \in X_\beta \setminus A$, then

$$x \notin \bigcap_{s<\sigma} Z_s \quad \text{for any} \quad \sigma \in N^N.$$

Therefore for each $\sigma \in N^N$ there is a $s_\sigma < \sigma$ such that $x \in Z_{s_\sigma}$. Then

$$x \in \bigcap_{\sigma \in N^N} (X \setminus Z_{s_\sigma})$$

and, since there are only countably many distinct $s'_\sigma s$,

$$\bigcap_{\sigma \in N^N} (X \setminus Z_{s_\sigma})$$

is a clopen subset of X_β contained in $X \setminus A$. Therefore A is closed in X_β.

4. Analytic spaces.

In this section we single out a class of domain spaces, the so-called disjoint analytic spaces, for the study of Banach spaces of real-valued Baire functions.

Let K(X) denote the family of compact subsets of a space X. A map from a space Y

$$F : Y \to K(X)$$

is called upper semi-continuous if

$$\{y \in Y : F(y) \subseteq U\}$$

is open in Y for each open set U in X. The map F is called disjoint if

$$F(y) \cap F(y') = \varnothing \quad \text{for} \quad y \neq y'.$$

A space X is called analytic if there is an upper semi-continuous map

$$F : N^N \to K(X)$$

with

$$X = \bigcup_{\sigma \in N^N} F(\sigma).$$

If this map F is disjoint, then X is called a Borelian [9] or descriptive Borel space [26].

Note that a subset A of a complete separable metric space X is a Souslin $- Z_0(X)$ set (i.e. $A \in ZS(X)$) if and only if it is analytic in the above sense, and a subset B of (a complete separable metric space) X is a Borel subset (i.e. $B \in Z_\Omega(X)$) if and only if it is a descriptive Borel space ([8] and [9]).

Recall that a map $f: X \to Y$ is called proper if it is continuous, closed, and $f^{-1}(y)$ is compact for each $y \in Y$.

A space X is called proper analytic if it admits a proper map onto an analytic subset of a complete separable metric space. It is known that a space X is proper analytic if and only if $X \in ZS(\beta X)$, where βX denotes the Stone-Čech compactification of X, and if and only if X is homeomorphic to a closed subset of a product of a compact space and a metrizable analytic space [17].

Finally, we call a space X disjoint analytic if there is an analytic subset $A \subseteq N^N$ and a disjoint upper semi-continuous map

$$F : A \to K(X)$$

with

$$X = \bigcup_{\sigma \in A} F(\sigma).$$

THEOREM 3. — *For any space* X *the following are equivalent:*

1) X *is a disjoint analytic space.*

2) X *is the one-to-one continuous image of a proper analytic space.*

3) X *is the one-to-one continuous image of a space which admits a proper map onto an analytic subset of* N^N.

Proof. — 1) $\Longrightarrow$ 3) Let A be an analytic subset of N^N and F an upper semi-continuous map;

$$F : A \to \mathcal{K}(X), \quad X = \bigcup_{\sigma \in A} F(\sigma).$$

The set of points σ such that $F(\sigma) \neq \varnothing$ is a closed subset of A and so an analytic subset of N^N. Thus we may suppose that

$$F(\sigma) \neq \varnothing \quad \text{for all } \sigma \in A.$$

Let T denote the smallest topology on the set X containing the open sets of X and the sets $\{F[U] : U \text{ is open in } A\}$. Let $(X, \mathcal{T})$ denote X with this topology. Then $(X, \mathcal{T})$ is a completely regular Hausdorff space and the function

$$F^{-1} : (X, \mathcal{T}) \to A$$

is a proper map (see [9, p. 164]).

Clearly the identity map

$$\mathrm{id} : (X, \mathscr{T}) \to X$$

is continuous.

3) $\Longrightarrow$ 2) Trivial.

2) $\Longrightarrow$ 1) Let Y be a proper analytic space and

$$f : Y \to X$$

a one-to-one continuous map onto X. Let A be an analytic subset of a complete separable metric space M and

$$g : Y \to A$$

a proper map onto A. Let C be a closed subset of N^N and

$$h : C \to M$$

a one-to-one continuous map onto M (see e.g. [18, p. 447]). Then $h^{-1}[A]$ is an analytic subset of C and thus of N^N.

The map

$$\Phi : h^{-1}[A] \to K(X)$$

with

$$X = \bigcup_{x \in h^{-1}[A]} \Phi(x)$$

defined by

$$\Phi(x) = g^{-1}[h(x)]$$

is an upper semi-continuous map, since the composition of two upper semi-continuous maps is upper semi-continuous. This also uses the fact that a continuous map is proper if and only if its inverse is an upper semi-continuous compact valued map.

Remarks. — 1) G. Choquet has defined a Hausdorff space to be K-analytic if it is the continuous image of a $K_{\sigma\delta}$ (countable intersection of countable unions of compact sets) in some compact Hausdorff space (see [7] and [8]). The analytic spaces defined here coincide with completely regular K-analytic spaces (see [9] or [10]).

2) The equivalence of 1) and 3) in theorem 3 is analogous to Frolík's theorem [9, theorem 13] that a space is descriptive Borel if and only if it is the one-to-one continuous image of

a space which admits a proper map onto a closed subset of N^N. The proof given here is by the same method as used by Frolík.

3) There are disjoint analytic spaces which are not proper analytic. The theorem of Frolík just quoted implies that every descriptive Borel space is disjoint analytic. Let x be a point of $\beta N \setminus N$. Then $N \cup \{x\}$ with the relative topology induced by βN is a disjoint analytic space, since it is the one-to-one continuous image of a countable discrete space. But $N \cup \{x\}$ is not a proper analytic space, since if it were then there would be a zero set

$$Z \in Z_0(\beta(N \cup \{x\}))$$

with

$$x \in Z \subseteq N \cup \{x\}.$$

Therefore, since

$$\{x\} = Z \cap \bigcap_{n=1}^{\infty} (\beta(N \cup \{x\}) \setminus \{n\})$$

we would have that x is a G_δ subset of $\beta(N \cup \{x\})$, which (see e.g. [11, p. 89]) is equal to βN. But this is not the case [11, p. 132].

4) There are analytic spaces which are not disjoint analytic. Frolík's example ([9, p. 166] and [10, p. 427]) of a σ-compact space which is not descriptive Borel is such a space. Namely, let X be the one point compactification of an uncountable discrete space and let $x_0 \in X$ be the compactifying point. In the product space $X \times N$ ($N = \{1, 2, 3, \ldots\}$ as usual) identify the points

$$\{(x_0, n) : n \in N\}.$$

The resulting quotient space Y is σ-compact. If there were an analytic subset A of N^N and a disjoint upper semi-continuous map

$$F : A \to K(Y)$$

with

$$Y = \bigcup_{\sigma \in A} F(\sigma),$$

then $F[A \setminus \{\sigma_0\}] = Y \setminus F(\sigma_0)$, where σ_0 is the unique element of A with

$$\{(x_0, n) : n \in N\} \subseteq F(\sigma_0),$$

is a Lindelöf space (since the upper semi-continuous compact valued image of a Lindelöf space is Lindelöf). But this is not the case since $Y \setminus F(\sigma_0)$ is discrete and uncountable.

5) It is clear from part 2) of theorem 3 that every proper analytic space is disjoint analytic. Since every analytic subset of N^N is the continuous image of N^N and since the composition of two upper semi-continuous compact valued maps is again such a map, we have that every disjoint analytic space is analytic. Therefore, we have

$$\text{proper analytic} \Longrightarrow \text{disjoint analytic} \Longrightarrow \text{analytic} \quad (= \text{K-analytic})$$

and none of the implications are reversible.

Recall that a metrizable analytic space is either countable or it contains a non-empty compact perfect subset [18, p. 479]. The following theorem extends this result to disjoint analytic spaces and is the main property of disjoint analytic spaces used in the next section. The word countable here includes the possibility of being finite.

A space is called dispersed if it contains no non-empty perfect subsets.

THEOREM 4. — *Let* X *be a disjoint analytic space. Then either*

1) X *contains a non-empty compact perfect subset, or*

2) X *is the countable union of dispersed compact subspaces. In addition, these conditions are mutually exclusive.*

Proof. — First, suppose that X contains a non-empty compact perfect subset K. Since K is perfect there exists a continuous map

$$f: K \to [0, 1]$$

of K onto the unit interval [24, p. 214]. Since X is disjoint analytic it is Lindelöf and thus normal. Therefore, by Tietze's extension theorem f has an extension to a continuous function

$$f: X \to [0, 1]$$

of X onto the unit interval. This implies that X is not the countable union of dispersed compact spaces, since every continuous real-valued function on a dispersed compact space (and so on the countable union of such spaces) has a countable range [28].

Now suppose that X does not contain a compact perfect subset. Let A be an analytic subset of N^N and

$$F : A \to K(X),$$
$$X = \bigcup_{\sigma \in A} F(\sigma)$$

be a disjoint upper semi-continuous map.

We claim that A must be countable. On the contrary, if A were uncountable, then there would be a non-empty compact perfect subset $C \subseteq A$.

Now, as in the proof of 1) $\Longrightarrow$ 3) of theorem 3, consider the smallest topology on the set X containing the open sets of X and the sets $\{F[U] : U \text{ is open in } A\}$. As before, this is a completely regular Hausdorff topology. Let $(X, \mathscr{T})$ denote the set X with this topology. Then, as before, the map

$$F^{-1} : (X, \mathscr{T}) \to A$$

is a proper map onto A. Thus

$$F : A \to \mathscr{K}((X, \mathscr{T}))$$

is upper semi-continuous. Therefore, since the upper semi-continuous image of a compact sets is compact,

$$F[C] = \bigcup_{\sigma \in C} F(\sigma)$$

is a compact subset of $(X, \mathscr{T})$.

Since the identity map

$$\mathrm{id} : (X, \mathscr{T}) \to X$$

is continuous, the restriction to $F[C]$ is a homeomorphism.

The compact subset

$$\mathrm{id} \circ F[C]$$

of X must contain a non-empty compact perfect subset, since otherwise the continuous map

$$F^{-1} : F[C] \to A$$

of F[C] onto A would have a countable range [28] contradicting our assumption that A is uncountable.

Thus we have that A is countable (or finite) and therefore

$$X = \bigcup_{\sigma \in A} F(\sigma)$$

is the countable union of compact spaces each of which must be dispersed since we are supposing X to contain no non-empty compact perfect subsets.

Remark. — I do not know if theorem 4 holds for all analytic spaces.

The next theorem implies that the continuum hypothesis holds for the class of disjoint analytic spaces in which every point is a G_δ.

THEOREM 5. — *Let* X *be a disjoint analytic space with an uncountable number of points. If every point of* X *is a* G_δ, *then the cardinality of* X *is that of the continuum.*

Proof. — If X does not contain a non-empty compact perfect subset, then X is the countable union of dispersed compact spaces (by theorem 4) each of which has all G_δ points. Thus, since a dispersed compact space all of whose points are $G_\delta's$ is necessarily countable [1, p. 34], X must be countable. Therefore X contains a non-empty compact perfect subset and consequently has cardinality at least that of the continuum [1, p. 29].

From theorem 3 there is a proper analytic space P, a one-to-one continuous map f of P onto X, and a proper map g of P onto a metrizable analytic space A; that is,

$$f : P \to X, \quad g : P \to A.$$

Let $x_0 \in X$. Then since x_0 is a G_δ, $f^{-1}(x_0)$ is a G_δ in P. Thus there is an $h \in C(P)$ such that

$$Z(h) = \{f^{-1}(x_0)\}$$

Let $\hat{f}$ be the extension of f to βP and $\hat{h}$ the extension of h to βP.

Define

$$\hat{g} \times \hat{h} : \beta P \to A \times \mathbf{R}$$

by

$$\hat{g} \times \hat{h}(p) = (\hat{g}(p), \hat{h}(p))$$

Then $(\hat{g} \times \hat{h})^{-1} \circ (\hat{g} \times \hat{h})(f^{-1}(x_0)) = f^{-1}(x_0)$, which implies that $f^{-1}(x_0)$ is a G_δ set in βP. Therefore $f^{-1}(x_0)$ has a countable neighbourhood base in βP, and so likewise in P. Thus P satisfies the first axiom of countability.

The theorem now follows from the general result that a first countable Lindelöf space has cardinality at most that of the continuum [3].

Remark. — There are uncountable disjoint analytic spaces in which every point is a G_δ but yet which are not 1-st countable. Let X be the disjoint union of the unit interval and the set $N \cup \{x\}$, where $x \in \beta N \setminus N$, with the relative topology induced by βN. Then X is a disjoint analytic space since it is the one-to-one continuous image of the complete separable metric space $[0, 1] \cup N$ (disjoint union). Every point of X is a G_δ subset, but the point x does not have a countable neighbourhood base (see e.g. [11, p. 131]).

5. Baire classes on disjoint analytic spaces.

We are now in a position to prove theorem 6 in the introduction. Reformulating in terms of the notation and terminology that we have developed we have

THEOREM 6. — *If* X *is a disjoint analytic space, then the following are equivalent:*

1) $B_\Omega(X) = B_\alpha(X)$ *for some* $\alpha < \Omega$,

2) $B_\Omega(X) = B_1(X)$,

3) $B_\Omega(X) = C(X_\beta)$,

4) $Z_\Omega(X) = ZS(X)$,

5) X_β *is a Lindelöf space,*

6) X *is the countable union of dispersed compact spaces,*

7) X *contains no non-empty compact perfect subsets.*

Proof. — From theorem 1 we have 1) $\Longrightarrow$ 7), 4) $\Longrightarrow$ 7), and 5) $\Longrightarrow$ 7). From theorem 2 we have 3) $\Longrightarrow$ 4). It therefore suffices to demonstrate 7) $\Longrightarrow$ 6), 6) $\Longrightarrow$ 2), 6) $\Longrightarrow$ 3), and 6) $\Longrightarrow$ 5).

7) $\Longrightarrow$ 6) Since X is a disjoint analytic space, there is a proper analytic space Y, a one-to-one continuous map

$$f: Y \to X$$

of Y onto X, and a proper map

$$g: Y \to M$$

of Y onto a metrizable analytic space M. If M is uncountable then it contains a non-empty compact perfect subset K [12, p. 205]. Then

$$f \circ g^{-1}[K]$$

is a compact subset of X, since the inverse of a proper map is a compact valued upper semi-continuous map and the image of a compact set under such a map is compact. Since $f \circ g^{-1}[K]$ admits a continuous map onto a compact perfect space, it contains a perfect subset [28, p. 39]. Therefore M must be countable, which implies that X is the countable union of compact dispersed spaces.

For future reference let

$$M = \{x_n : n = 1, 2, 3, \ldots\}.$$

Then

$$X = \bigcup_{n=1}^{\infty} f \circ g^{-1}(x_n)$$

and each $f \circ g^{-1}(x_n)$ is a dispersed compact zero set of X. We will denote $f \circ g^{-1}$ by F.

6) $\Longrightarrow$ 2) From above we have

$$X = \bigcup_{n=1}^{\infty} F(x_n)$$

since for each $n = 1, 2, \ldots, F(x_n)$ is compact and dispersed, we have by [22, p. 36] that

$$Z_1(F(x_n)) = Z_\Omega(F(x_n)).$$

Let $B \in Z_\Omega(X)$. Then

$$B = \bigcup_{n=1}^{\infty} (F(x_n) \cap B)$$

and

$$X \setminus B = \bigcup_{n=1}^{\infty} (F(x_n) \cap (X \setminus B)).$$

Since each

$$F(x_n) \cap B \in Z_1(X)$$

and

$$F(x_n) \cap (X \setminus B) \in Z_1(X)$$

we have $B \in Z_1(X)$. Therefore $B_\Omega(X) = B_1(X)$ according to the Lebesgue-Hausdorff theorem quoted in § 2.

6) $\Longrightarrow$ 3) From the proof of 7) $\Longrightarrow$ 6) we have

$$X = \bigcup_{n=1}^{\infty} F(x_n)$$

Since each $Z \in Z_0(F(x_n))$ is the intersection of $F(x_n)$ with a $Z' \in Z_0(X)$, the relative topology on $F(x_n)$ induced by X_β coincides with that of $F(x_n)_\beta$.

Let $f \in C(X_\beta)$. For each n

$$f|_{F(x_n)} \in C(F(x_n))$$

Thus

$$f|_{F(x_n)} \in B_1(F(x_n)),$$

since for a dispersed compact space, say E, we have from [22, p. 36] that $B_1(E) = C(E_\beta)$. Then, since the $F(x_n)$ are zero sets in X, $f \in B_1(X)$.

6) $\Longrightarrow$ 5) From above, we have that

$$X = \bigcup_{n=1}^{\infty} F(x_n),$$

and the relative topology on each $F(x_n)$ induced by X_β is that of $F(x_n)_\beta$. Since each $F(x_n)$ is dispersed, each $F(x_n)_\beta$ is a Lindelöf space [23, p. 27]. Therefore X_β is a Lindelöf space.

6. Baire classes on pseudocompact spaces.

Recall that a space X is called pseudocompact if every continuous real-valued function on X is bounded, and is called realcompact if it is homeomorphic to a closed subset of a product of real lines. As is well known, a space is compact if and only if it is both pseudocompact and realcompact.

The Hewitt realcompactification of a space X will be denoted by νX. The reader is referred to [11] for a treatment of this topic.

The following theorem due to P. R. Meyer (1961, unpublished) is the key result in determining the existence of Baire classes on a pseudocompact space.

THEOREM 7. — *(P. R. Meyer) Let* X *be a (completely regular Hausdorff) space. Every* $f \in B_\alpha(X)$ *has a unique extension to an* $f \in B_\alpha(\nu X)$.

Proof. — Every non-empty zero set in νX intersects X [11, p. 118]. Every Baire set B in νX, being a Souslin-$Z_0(\nu X)$ set, has a representation

$$B = \bigcup_{\sigma \in N^N} \bigcap_{s < \sigma} Z_s, \quad Z_s \in Z_0(\nu X),$$

and so is the union of zero sets in νX. Thus every non-empty Baire set in νX intersects X.

As is well known [11, p. 118], every $f \in C(X)$ has a unique extension to an $f \in C(\nu X)$. Proceeding by induction let $f \in B_{\alpha+1}(X)$ and $f_n \in B_\alpha(X)$ such that

$$f_n(x) \to f(x) \quad \text{for all} \quad x \in X.$$

Let $x_0 \in \nu X \setminus X$ and

$$B_n = \{x \in \nu X : \hat{f}_n(x) - \hat{f}_n(x_0) = 0\}.$$

Since each B_n is a Baire set in νX and $x_0 \in \bigcap_{n=1}^{\infty} B_n$ it follows that

$$\left(\bigcap_{n=1}^{\infty} B_n\right) \cap X = \varnothing.$$

Therefore $\hat{f}_n(x_0)$ converges to a finite limit as $n \to \infty$ and there is an $\hat{f} \in B_{\alpha+1}(\nu X)$ such that

$$\hat{f}_n(x) \to \hat{f}(x) \quad \text{for all} \quad x \in \nu X$$

and

$$\hat{f}(x) = f(x) \quad \text{for all} \quad x \in X.$$

Suppose $\hat{f}$ and $\hat{f}'$ are two extensions of f. Then

$$\nu X \setminus \{x \in \nu X : \hat{f}(x) = \hat{f}'(x)\}$$

is a Baire set in νX which does not intersect X and is thus empty; that is,

$$\hat{f}(x) = \hat{f}'(x) \quad \text{for all} \quad x \in \nu X.$$

THEOREM 8. — *For any space* X *and ordinal* $\alpha < \Omega$.

1) $B_\Omega(X) = B_\alpha(X)$ *if and only if* $B_\Omega(\nu X) = B_\alpha(\nu X)$, *and*

2) $Z_\Omega(X) = ZS(X)$ *if and only if* $Z_\Omega(\nu X) = ZS(\nu X)$.

Proof. — The first statement follows immediately from theorem 7. The second follows from the fact that for $Z(f) \subseteq X$, $f \in C(X)$, we have

$$cl_{\nu X} Z(f) = Z(\hat{f}),$$

where $\hat{f} \in C(\nu X)$ is the extension of f and $cl_{\nu X} Z(f)$ is the closure of $Z(f)$ in νX. This implies that for each $A \in ZS(X)$ there is an $A' \in ZS(\nu X)$ such that

$$A = X \cap A'$$

and $A \in Z_\Omega(X)$ if and only if $A' \in Z_\Omega(\nu X)$.

THEOREM 9. — *If* X *is a pseudocompact space, then the following are equivalent:*

1) $B_\Omega(X) = B_\alpha(X)$ *for some* $\alpha < \Omega$.

2) $B_\Omega(X) = B_1(X)$.

3) $B_\Omega(\nu X) = C((\nu X)_\beta)$.

4) $Z_\Omega(X) = ZS(X)$.

5) *Every (supremum norm) continuous linear functional on* $C(X)$ *is of the form*

$$F(f) = \sum_{n=1}^{\infty} a_n f(x_n),$$

where (x_n) *is a fixed sequence in* $\nu X (= \beta X)$ *and* $\sum_{n=1}^{\infty} |a_n| < \infty$. *If* X *has these properties or if* X_β *is a Lindelöf space, then* X *contains no non-empty perfect subsets.*

Proof. — Since X is pseudocompact, the Stone-Čech compactification βX coincides with νX. Therefore the equivalence of 1), 2), 3), and 4) follows from theorems 7 and 8, which also imply that these are equivalent to βX being dispersed.

But βX is dispersed if and only if every continuous linear functional on $C(\beta X)$ (which is isometrically isomorphic to $C(X)$) is of the form

$$F(f) = \sum_{n=1}^{\infty} a_n f(x_n),$$

where (x_n) is a fixed sequence in βX and $\sum_{n=1}^{\infty} |a_n| < \infty$ [24, p. 214]. Therefore 5) is equivalent to 1) through 4).

If X contained a non-empty perfect subset K, then $cl_{\beta X} K$ would be a perfect subset of βX, which, as noted above, is incompatible with 1) through 5).

If X_β in Lindelöf, then X must be Lindelöf. Therefore X must be compact and theorem 1 applies.

Remarks. — 1) There are spaces X which contain no non-empty compact perfect subsets and which have

$$B_\alpha(X) \neq B_{\alpha+1}(X) \quad \text{for all} \quad \alpha < \Omega,$$

and

$$Z_\Omega(X) \neq ZS(X).$$

Let, as usual, $N = \{1, 2, 3, \ldots\}$ and for each infinite subset $C \subseteq N$ let x_C be a point in $(cl_{\beta N} C) \setminus N$.

Consider the space

$$X = N \cup \{x_C : C \text{ is an infinite subset of } N\}$$

with the relative topology induced from βN.

The space X is pseudocompact. If it were not there would be an unbounded continuous function

$$f: X \to \mathbf{R}.$$

Since N is dense in X there would then be a sequence $S = (n_i) \subseteq N$ such that $|f(n_i)| \geqslant i$. Then we would have

$$x_S \in cl_{\beta N}\{n_i : i \geqslant k\}$$

for all $k = 1, 2, 3, \ldots,$ and so $|f(x_S)| \geqslant k$ for every k, which is impossible.

Now, since $N \subseteq X \subseteq \beta N$, we have that $\beta X = \beta N$ [11, p. 89], and since βN contains a non-empty compact perfect subset theorem 9 implies that

$$B_\alpha(X) \neq B_{\alpha+1}(X) \quad \text{for all} \quad \alpha < \Omega$$

and

$$Z_\Omega(X) \neq ZS(X).$$

It remains to show that X contains no non-empty compact perfect subsets. In fact, every compact subset of X is finite. This follows from the fact that every infinite compact subset of βN has cardinality 2^c, where x denotes the cardinality of the continuum [11, p. 130-133], since the cardinality of X is at most c.

This example was used in [13] to illustrate other phenomena.

2) There are pseudocompact non-compact spaces whose Stone-Čech compactifications are dispersed. The space of ordinals $\{\alpha : \alpha < \Omega\}$ with the interval topology is such a space [11, p. 74]. Its Stone-Čech compactification is $\{\alpha : \alpha \leqslant \Omega\}$.

3) All finite spaces X have the property that

$$C(X) = B_\Omega(X),$$

all dispersed infinite compact spaces have

$$C(X) \neq B_1(X) = B_\Omega(X),$$

and all other compact spaces have

$$B_\Omega(X) \neq B_{\alpha+1}(X) \quad \text{for all} \quad \alpha < \Omega$$

Do there exist spaces, X_α say, with the property that for $2 \leqslant \alpha < \Omega$

$$\bigcup_{\xi<\alpha} B_\xi(X_\alpha) \neq B_\alpha(X_\alpha) = B_{\alpha+1}(X_\alpha)?$$

Under the assumption of the continuum hypothesis it is known [30, p. 43] that there exist subsets X of the line **R** such that

$$B_1(X) \neq B_2(X) = B_3(X).$$

I am not aware of any other examples of this nature.

7. The sequential stability index.

Let S be a set of bounded real-valued functions on a set X. Let S_1 be the set of pointwise limits of bounded sequences in S. For each ordinal α inductively define

$$S_\alpha = \left(\bigcup_{\xi<\alpha} S_\xi\right)_1.$$

We call the smallest ordinal α such that

$$S_\alpha = S_{\alpha+1}$$

the sequential stability index of S. We denote this index by $i[S]$.

As noted at the end of § 6, without axiomatic assumptions only the existence of (completely regular Hausdorff) spaces X with

$$i[C(X)] = 0,\ 1,\ \text{and}\ \Omega$$

is known, and assuming the continuum hypothesis there is a subset X of **R** with

$$i[C(X)] = 2.$$

This index was defined in [2] where it was shown that if X is a non-dispersed compact space, then for each ordinal $\alpha < \Omega$ there exists a uniformly closed linear subspace

$$M^\alpha \subseteq (C(X))_1 = B_1(X)$$

with

$$i[M^\alpha] = \alpha.$$

Here we observe that this result holds for every space X with an infinite number of points.

THEOREM 10. — *If a (completely regular Hausdorff) space* X *has an infinite number of points, then for each ordinal* $\alpha < \Omega$ *there exists a uniformly closed linear subspace* M^{α} *of* $B_1(X)$ *with* $i[M^{\alpha}] = \alpha$.

Proof. — The key to the proof is the following theorem of D. Sarason [29]: For each ordinal $\alpha < \Omega$ there exists a uniformly closed linear subspace M^{α} of the space of bounded complex-valued functions on $N = \{1, 2, \ldots\}$ with $i[M^{\alpha}] = \alpha$.

Given such a space M^{α} of complex-valued functions on N we obtain a corresponding linear space $\tilde{M}^{\alpha}$ of bounded real-valued functions on N with

$$i[\tilde{M}^{\alpha}] = \alpha$$

as follows: Let $\tilde{N} = N_1 \cup N_2$, the disjoint union of two copies N_1 and N_2 of N.

For each $f = u + iv \in M^{\alpha}$ define

$$\hat{f} : \tilde{N} \to \mathbf{R}$$

by

$$\hat{f}(x) = \begin{cases} u(x) & \text{if } x \in N_1 \\ v(x) & \text{if } x \in N_2. \end{cases}$$

Then it is not difficult to check that

$$\tilde{M}^{\alpha} = \{\hat{f} : f \in M^{\alpha}\}$$

is a uniformly closed linear subspace of C(N) with $i[\tilde{M}^{\alpha}] = \alpha$.

Now let X be a space with an infinite number of points. Let $h \in C(X)$ be such that $h[X]$ is infinite and let F be a closed countably infinite subset of $h[X]$ (considering $h[X]$ with the relative topology).

Considering the above space $\tilde{M}^{\alpha}$ to be defined on the countable set F and extending each $g \in \tilde{M}^{\alpha}$ to all of $h[X]$ by defining

$$g(x) = 0 \quad \text{for} \quad x \in h[X] \setminus F$$

we have

$$\tilde{M}^\alpha \subseteq B_1(h[X]).$$

Then the linear subspace

$$P^\alpha = \{g \circ h : g \in \tilde{M}^\alpha\} \subseteq B_1(X)$$

is uniformly closed and $i[P^\alpha] = \alpha$.

Remarks. — 1) The question remains open as to whether or not for every infinite space X and ordinal $\alpha < \Omega$ there exists a linear subspace M^α of $C(X)$ with $i[M^\alpha] = \alpha$. A positive answer is given in [2] for the spaces $[0, 1)$ and N^N.

2) If in the definition of the sequential stability index we define S_1 to be the space of bounded functions on X which are the pointwise limits of sequences in S (not necessarily bounded sequences), then we obtain a quite different index, which we denote by i'. For example, if X is a dispersed compact space, then every subset S of $B_1(X)$ has $i'[S] \leqslant 1$ ([22]). On the other hand by theorem 10 every infinite dispersed compact space has for each $\alpha < \Omega$ subspaces $M^\alpha \subseteq B_1(X)$ with $i[M^\alpha] = \alpha$.

Let B be a Banach space. Define B^1_{w*} to be the subset of B^{**}, the second dual of B, consisting of weak* limits of sequences in the canonical injection of B in B^{**}. Inductively for each ordinal α define B^α_{w*} to be the subset of B^{**} consisting of weak* limits of sequences in $\bigcup_{\xi<\alpha} B^\xi_{w*}$. The smallest ordinal α such that

$$\bigcup_{\xi<\alpha} B^\xi_{w*} \neq B^\alpha_{w*} = B^{\alpha+1}_{w*}$$

will be called the weak* sequential stability index of B and will be denoted by $i_{w*}[B]$.

THEOREM 11. — *If X is a compact space and B is a closed linear subspace of $C(X)$, then*

$$i[B] = i_{w*}[B].$$

Consequently the sequential stability index of B is the same for every isomorphic (i.e. linear homeomorphic) embedding of B

into a space of continuous real-valued functions on a compact space.

Proof. — Let as usual B_α be the space of functions obtained from B on the α-th iteration of the point-wise sequential limiting operation on X. Let $f \in B_1$ and $(f_n) \subseteq B$, $\|f_n\| \leqslant M$, $n = 1, 2, \ldots,$ such that

$$f_n(x) \to f(x) \quad \text{for all} \quad x \in X.$$

Then

$$\int f_n d\mu \to \int f \, d\mu$$

for each finite signed Baire measure μ on X by the bounded convergence theorem. Thus from the Riesz representation theorem and the weak* sequential completeness of B^{**} there is a unique element, say $\hat{f}$, in B^{**} which is the weak* limit of the canonical image of the sequence (f_n) in B^{**}.

Conversely if $g \in B^1_{w^*}$ and $(f_n) \subseteq B \subseteq B^{**}$ such that

$$f_n \xrightarrow{w^*} g,$$

then

$$F_X(f_n) \to F_X(g)$$

where F_X, $x \in X$ is the functional which evaluates at x. Thus the function f defined by $f(x) = F_X(g)$ is in B_1 and $\hat{f} = g$.

Thus the map

$$\varphi : B_1 \to B^1_{w^*}$$

defined by

$$\varphi(f) = \hat{f}$$

is one-to-one and onto.

It is not difficult to check that the map φ is a linear isometry.

Arguing inductively we obtain that the map φ extends for each $\alpha \leqslant \Omega$ to a linear isometry of B_α onto $B^\alpha_{w^*}$. This concludes the proof.

The second part of this paper will consider the problem of evaluating the sequential stability index of the space of continuous (real-valued) affine functions on a compact convex subset of a Hausdorff locally convex real topological vector space.

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