ANNALES DE L'INSTITUT FOURIER

WALTER RUDIN

Spaces of type $H^\infty + C$

Annales de l'institut Fourier, tome 25, nº 1 (1975), p. 99-125

<http://www.numdam.org/item?id=AIF_1975__25_1_99_0>

Ann. Inst. Fourier, Grenoble
25, 1 (1975), 99-125

SPACES OF TYPE $H^\infty + C$

by Walter RUDIN (*)

Introduction.

The starting point of the present paper is the theorem of Sarason [18], [8], which states that $H^\infty + C$ is a *closed subalgebra* of L^∞ *on the unit circle* T.

Here C is the space of all continuous functions on T, and H^∞ consists of all $f \in L^\infty$ whose Fourier coefficients $\hat{f}(n)$ are 0 for all $n < 0$. L^∞ is given the essential supremum norm ; multiplication is pointwise. Throughout this paper, the word *function* will mean *complex-valued* function.

Sarason's theorem answered a question raised by Devinatz [4 ; p. 506] who asked for a characterization of those functions on T which are in the L^∞-closure of the space $P + H^\infty$, where P is the set of all trigonometric polynomials on T. It is clear that $P + H^\infty$ is an algebra (consisting of those $f \in L^\infty$ that have $\hat{f}(n) = 0$ for all $n \leqslant n_0 = n_0(f)$). The question thus leads to the Banach algebra $\overline{P + H^\infty}$. Since $C = \overline{P}$ (Fejér's theorem), it is clear that

$$C + H^\infty \subset \overline{P + H^\infty} \subset \overline{C + H^\infty}.$$

What Sarason did was to prove that $C + H^\infty$ is closed. The above inclusions show then immediately that

$$C + H^\infty = \overline{P + H^\infty}.$$

In particular, the perhaps surprising fact that $C + H^\infty$ is an *algebra* on T comes for free !

(*) This research was partially supported by NSF Grant GP-33987X.

In his recent survey article [19 ; p. 290] Sarason sketches a simple version of Zalcman's proof [23] of the above theorem. ([23] deals with a certain class of infinitely connected regions ; a very recent paper on this topic is [3]). An examination of that proof shows that it really uses nothing but the following properties of the Fejér kernels K_n : Convolution with K_n carries L^∞ to C and H^∞ to H^∞ ; $\|K_n * f\|_\infty \leqslant \|f\|_\infty$ for every $f \in L^\infty$; and $\|g - K_n * g\|_\infty \to 0$ as $n \to \infty$, for every $g \in C$.

This observation led to the formulation of a theorem concerning sums of subspaces of a Banach space (Theorem 1.2) whose proof is almost a triviality, and which would not be worth mentioning if it did not have some interesting consequences. It gives a proof of Sarason's theorem (Section 1.3) which is even more direct than the one given in [19] ; no biduals of quotient spaces are needed, and the F. and M. Riesz theorem is avoided.

More significantly, Theorem 1.2 implies almost immediately that various natural analogues of $H^\infty + C$ are closed. This happens, for instance, when the circle is replaced by other compact abelian groups, and when the unit disc is replaced by polydiscs or by balls in $\mathbb{C}^n$, the space of n complex variables. The question then arises whether these spaces are algebras.

For groups (Theorem 3.6) and for polydiscs (Theorem 2.2) the answer turns out to be negative. The algebra feature of Sarason's theorem thus looks more like an exception than like a rule. It was therefore a very pleasant surprise to find that $H^\infty + C$ *does* turn out to be an algebra when the underlying function theory is that of the unit ball in $\mathbb{C}^n$ (Theorem 2.3). This is probably the most interesting result of the present paper. (See also Theorem 2.13).

In Part IV, Theorem 1.2 is used to show that sums of certain closed ideals in Banach algebras are closed, and some examples of closed ideals are given whose sums are not closed.

1. Sums of subspaces of a Banach space.

1.1. DEFINITION. – *If X is a Banach space, $\mathcal{B}$(X) denotes the Banach algebra of all bounded lienar operators Λ on* X, *with the usual norm*

$$\|\Lambda\| = \sup\{\|\Lambda x\| : \|x\| \leqslant 1 \quad , \quad x \in X\}.$$

1.2. THEOREM. – *Suppose* Y *and* Z *are closed subspaces of a Banach space* X, *and suppose that there is a collection* $\Phi \subset \mathcal{B}(X)$ *with the following properties :*

a) *Every* $\Lambda \in \Phi$ *maps* X *into* Y.

b) *Every* $\Lambda \in \Phi$ *maps* Z *into* Z.

c) $\sup\{\|\Lambda\| : \Lambda \in \Phi\} < \infty$.

d) *To every* $y \in Y$ *and to every* $\epsilon > 0$ *corresponds a* $\Lambda \in \Phi$ *such that* $\|y - \Lambda y\| < \epsilon$.

Then Y + Z *is closed.*

Proof. – Let $x \in X$ be a limit point of Y + Z. Choose $\epsilon_n > 0$ so that $\Sigma \epsilon_n < \infty$. There exist vectors $v_n \in Y + Z$ such that

$$\|x - v_n\| \to 0$$

as $n \to \infty$. Choose a subsequence, again denoted by $\{v_n\}$, such that $\|v_n - v_{n-1}\| < \epsilon_n$ for $n \geqslant 2$. Put $x_1 = v_1$, $x_n = v_n - v_{n-1}$ for $n \geqslant 2$. Then $\|x_n\| < \epsilon_n$ for $n \geqslant 2$ and $x = \sum_1^\infty x_n$.

Since $x_n \in Y + Z$, the exist $y_n \in Y$, $z_n \in Z$, such that $x_n = y_n + z_n$. By (d) there are operators $\Lambda_n \in \Phi$ such that

$$\|y_n - \Lambda_n y_n\| < \epsilon_n \qquad (n \geqslant 1). \tag{1}$$

Since Λ_n is linear, $x_n = \tilde{y}_n + \tilde{z}_n$, where

$$\tilde{y}_n = y_n - \Lambda_n y_n + \Lambda_n x_n \ , \ \tilde{z}_n = z_n - \Lambda_n z_n. \tag{2}$$

Assumption (a) shows that $\tilde{y}_n \in Y$; (b) shows that $\tilde{z}_n \in Z$; if M is the supremum in (c), then $\|\Lambda_n x_n\| \leqslant M\,\|x_n\|$. Thus (1) implies

$$\|\tilde{y}_n\| \leqslant (1 + M)\,\epsilon_n \qquad (n \geqslant 2) \tag{3}$$

and hence

$$\|\tilde{z}_n\| \leqslant \|x_n\| + \|\tilde{y}_n\| \leqslant (2 + M)\,\epsilon_n \qquad (n \geqslant 2). \tag{4}$$

Since Y and Z are closed, they are complete. Hence $\tilde{y} = \sum_1^\infty \tilde{y}_n$ is in Y, by (3), and $\tilde{z} = \sum_1^\infty \tilde{z}_n$ is in Z, by (4). Therefore $x = \tilde{y} + \tilde{z}$ is in Y + Z.

1.3. *Proof of Sarason's theorem.* – Let $X = L^\infty(T)$, $Y = C(T)$, $Z = H^\infty(T)$, and let Λ_n be the operator that assigns to each $f \in L^\infty(T)$ the arithmetic mean of the first n partial sums of its Fourier series. With $\Phi = \{\Lambda_n\}$, assumptions (a) to (d) of Theorem 1.2 hold. In fact, the supremum in (c) is 1, and (d) is Fejér's theorem. Thus $C(T) + H^\infty(T)$ is closed in $L^\infty(T)$.

1.4. *Remark.* – Scalar multiplication played no role in the proof of Theorem 1.2. It is therefore to be expected that an analogue of the theorem holds in (additive) abelian groups G that are complete with respect to some translation-invariant metric d. To obtain such an analogue, let A and B be closed subgroups of G, and assume that Φ is an equicontinuous collection of homomorphisms of G into A which carry B into B, such that to each $a \in A$ and each $\epsilon > 0$ corresponds a $\varphi \in \Phi$ with $d(a, \varphi a) < \epsilon$. [To say that Φ is equicontinuous means that to each $\epsilon > 0$ corresponds a $\delta > 0$ such that $d(\varphi x, \varphi y) < \epsilon$ whenever $\varphi \in \Phi$ and $d(x, y) < \delta$].

Conclusion. – A + B is closed in G.

The proof of this is so similar to that of Theorem 1.2 that it does not seem worthwhile to put the details down, especially since I know of no interesting applications.

2. H^∞ + C in several complex variables.

2.1. *Background and notation.* – For each positive integer n, $\mathbb{C}^n$ is the vector space of all ordered n-tuples $z = (z_1, \ldots, z_n)$ of complex numbers z_i, with the usual inner product

$$\langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_n \bar{w}_n \tag{1}$$

and the corresponding norm $\|z\| = \langle z, z \rangle^{1/2}$. We put

$$B = \{z \in \mathbb{C}^n : \|z\| < 1\} \quad , \quad S = \{z \in \mathbb{C}^n : \|z\| = 1\}. \tag{2}$$

Then B is the open unit ball of $\mathbb{C}^n$. Its boundary S is a sphere of (real) dimension $2n - 1$ which carries a (unique) rotation-invariant probability measure σ, defined on the Borel subsets of S. The notation $L^p(S)$, for the usual Lebesgue spaces, refers to this measure σ.

Note that the dimension n is not mentioned in the notations B, S, σ. This should cause no confusion.

As in [17], the polydisc U^n is the set of all $z \in \mathbb{C}^n$ which have $|z_i| < 1$ for $1 \leqslant i \leqslant n$; thus U^n is the cartesian product of n copies of the open unit disc $U \subset \mathbb{C}$. The distinguished boundary of U^n is the torus T^n, consisting of all $z \in \mathbb{C}^n$ with $|z_i| = 1$ for $1 \leqslant i \leqslant n$. On T^n, the expression "almost all" will refer to the Haar measure of the compact group T^n.

Lebesgue measure on the euclidean space R^k will be denoted by m_k.

$H^\infty(B)$ and $H^\infty(U^n)$ are the sup-normed Banach algebras of all bounded holomorphic functions with domains B and U^n, respectively.

If $f \in H^\infty(B)$ then the radial limits

$$f^*(z) = \lim_{r \to 1} f(rz) \tag{3}$$

exist for almost all $z \in S$. [The easiest way to see this is to apply the classical theorem of Fatou to the functions $f_z \in H^\infty(U)$ that are given by $f_z(\lambda) = f(\lambda z)$]. The mapping $f \to f^*$ is, in fact, an isometric isomorphism of $H^\infty(B)$ onto a closed subalgebra of $L^\infty(S)$ which we call $H^\infty(S)$.

Everything remains correct in the preceding paragraph if B and S are replaced by U^n and T^n. In addition, $H^\infty(T^n)$ turns out to be the class of those $h \in L^\infty(T^n)$ whose Fourier coefficients $\hat{h}(\alpha_1, \ldots, \alpha_n)$ are 0 if $\alpha_i < 0$ for at least one i.

As regards the boundary behavior of members of H^∞, all that we shall need here is the existence of the above-mentioned *radial* limits f^* a.e. But much more is known ; good references are [25 ; Chap. 17], [11], [20].

If f is a function with domain B (or U^n) and if $0 \leqslant r < 1$, then f_r is the function with domain S (or T^n) defined by

$$f_r(z) = f(rz). \tag{4}$$

Thus $f^* = \lim_{r \to 1} f_r$.

The following two theorems state the main result of Part II.

2.2. THEOREM. – $H^\infty(T^n) + C(T^n)$ *is a closed subspace of* $L^\infty(T^n)$ *for all* $n \geqslant 1$, *but is an algebra only when* $n = 1$.

2.3. THEOREM. – $H^\infty(S) + C(S)$ *is a closed subalgebra of* $L^\infty(S)$ *for every* $n \geqslant 1$.

2.4. *Proof that* $H^\infty + C$ *is closed.* – Each $f \in L^\infty(T^n)$ has its Poisson integral $P[f]$, an n-harmonic function with domain U^n [17 ; Chap. 2]. Using the notation 2.1 (4), for $0 \leqslant r < 1$, the relevant properties of $P[f]$ are as follows :

a) $P[f]_r \in C(T^n)$ for every $f \in L^\infty(T^n)$.

b) $P[f]_r \in H^\infty(T^n)$ for every $f \in H^\infty(T^n)$.

c) $\|P[f]_r\|_\infty \leqslant \|f\|_\infty$ for every $f \in L^\infty(T^n)$.

d) $\lim_{r \to 1} \|f - P[f]_r\|_\infty = 0$ for every $f \in C(T^n)$.

(See [17 ; p. 18]). With $X = L^\infty(T^n)$, $Y = C(T^n)$, $Z = H^\infty(T^n)$, and $\Lambda_r f = P[f]_r$, the hypotheses of Theorem 1.2 are thus satisfied. This proves one half of Theorem 2.2.

To show that $H^\infty + C$ is closed on S, replace T^n and U^n in the preceding paragraph by S and B. The Poisson integral is now defined by means of the classical Poisson kernel for the ball (the one that is associated with Newtonian potential theory) ; we could also use the Poisson-Szegö kernel ; see [11] or [20 ; p. 24]. In either case, properties (a) to (d) hold, and the proof is completed as above.

Observe that the technique of this proof can be applied to any bounded region $\Omega \subset \mathbb{C}^n$ which is star-shaped with respect to the origin and whose boundary $\partial\Omega$ is sufficiently well-behaved that the Dirichlet problem with continuous data on $\partial\Omega$ can be solved in Ω.

2.5. *Proof that* $H^\infty(T^n) + C(T^n)$ *is not an algebra when* $n > 1$. – Pick an $F \in H^\infty(U)$ which is not in the disc algebra A, i.e., which does not have a continuous extension to $\overline{U}$. Then F^* does not coincide a.e. with any continuous function on T.

Fix $n > 1$, défine $f \in H^\infty(U^n)$ by

$$f(z_1, \ldots, z_n) = F(z_1) \tag{1}$$

and define $\varphi \in L^\infty(T^n)$ by

$$\varphi(z) = \overline{z}_n f^*(z) = \overline{z}_n F^*(z_1). \tag{2}$$

It suffices to show that $\varphi \notin H^\infty(T^n) + C(T^n)$.

So assume, to get a contradiction, that $\varphi = g + h^*$ for some $g \in C(T)$, $h \in H^\infty(U^n)$. Since $z_n \overline{z}_n = 1$ on T^n, it then follows from (2) that

$$F^*(z_1) = z_n g(z) + z_n h^*(z) \tag{3}$$

for almost all $z \in T^n$. Since $h \in H^\infty(U^n)$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta}\, h(rz_1, \ldots, rz_{n-1}, re^{i\theta})\, d\theta = 0 \tag{4}$$

if $0 \leqslant r < 1$ and $(z_1, \ldots, z_{n-1}) \in T^{n-1}$. Letting $r \to 1$ in (4), it follows from (3) that

$$F^*(z_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta}\, g(z_1, \ldots, z_{n-1}, e^{i\theta})\, d\theta \tag{5}$$

a.e. on T^{n-1}. The continuity of g shows that the right side of (5) is a continuous function of $(z_1, \ldots, z_{n-1})$. Thus F^* coincides a.e. on T with a continuous function. This contradicts our choice of F.

The proof of Theorem 2.2 is now complete.

The proof of Theorem 2.3 (i.e., the proof that $H^\infty + C$ is an *algebra* on S) requires further preparation. It will be completed in Section 2.11.

2.6. Lemma. – *Assume $n > 1$. If φ is a nonnegative measurable function on $\mathbb{C} = R^2$, if $z \in S$, and if*

$$F(\zeta) = \varphi(<\zeta, z>) \qquad (\zeta \in \mathbb{C}^n) \tag{1}$$

then

$$\int_S F\, d\sigma = \frac{n-1}{\pi}\int_U \varphi(\lambda)\,(1 - |\lambda|^2)^{n-2}\, dm_2(\lambda). \tag{2}$$

(Proposition 7.2 of [6] contains a more general version of this lemma).

Proof. — It is obviously enough to prove (2) under the assumption that φ (hence also F) is continuous. Choose coordinates in $\mathbb{C}^n$ (by means of a unitary transformation) so that $z = (1, 0, \dots, 0)$. Then $F(\zeta) = \varphi(\zeta_1)$. Define

$$I(r) = \int_{rB} F\, dm_{2n} \qquad (0 < r < \infty). \tag{3}$$

There are two ways to rewrite $I(r)$. By means of polar coordinates,

$$I(r) = 2nV_{2n} \int_0^r t^{2n-1}\, dt \int_S F(t\zeta)\, d\sigma(\zeta), \tag{4}$$

where V_{2n} is the volume of the unit ball in R^{2n}. Hence

$$I'(1) = 2nV_{2n} \int_S F\, d\sigma. \tag{5}$$

On the other hand, Fubini's theorem gives

$$I(r) = V_{2n-2} \int_{rU} \varphi(\lambda)\, (r^2 - |\lambda|^2)^{n-1}\, dm_2(\lambda), \tag{6}$$

since $F(\zeta) = \varphi(\zeta_1)$. Hence

$$I'(1) = 2(n-1)\, V_{2n-2} \int_U \varphi(\lambda)\, (1 - |\lambda|^2)^{n-2}\, dm_2(\lambda). \tag{7}$$

Now (2) follows from (5) and (7), since $nV_{2n} = \pi V_{2n-2}$. [This last relation can be obtained from (6) by putting $r = 1$ and $\varphi = 1$].

2.7. TOEPLITZ OPERATORS ON S. — We recall that the *Cauchy kernel* for B is

$$C(z, \zeta) = \frac{1}{(1 - \langle z, \zeta \rangle)^n} \tag{1}$$

and that the *Cauchy integral* $C[f]$ of a function $f \in L^1(S)$ is defined by

$$C[f](z) = \int_S C(z, \zeta)\, f(\zeta)\, d\sigma(\zeta) \qquad (z \in B). \tag{2}$$

If $f \in H^\infty(B)$, then the *Cauchy formula*

$$f(z) = C[f^*](z) \qquad (z \in B) \tag{3}$$

holds. See [1], [9], [11], [20], [21]. Frequently, $C(z, \zeta)$ is called the *Szegö kernel* of B.

We associate to each $\varphi \in L^\infty(S)$ the *Toeplitz operator* T_φ by the formula

$$T_\varphi f = C[\varphi f]. \tag{4}$$

Thus T_φ carries each $f \in L^1(S)$ to a holomorphic function in B.

[It would be more in keeping with the terminology that is customary when $n = 1$ to use the symbol $T_\varphi f$ for the *boundary values* of $C[\varphi f]$. However this would require a preliminary discussion of their existence. For our present purpose, this is not needed, and therefore the definition (4) seems preferable].

2.8. DEFINITION. – *If $\varphi \in C(S)$, its modulus of continuity ω_φ is defined for* $0 \leqslant t \leqslant 2$ by

$$\omega_\varphi(t) = \sup \{|\varphi(z) - \varphi(\zeta)| : \|z - \zeta\| \leqslant t\}. \tag{1}$$

If

$$\int_0^2 \omega_\varphi(t) \frac{dt}{t} < \infty \tag{2}$$

then φ is said to satisfy *Dini's condition.*

2.9. LEMMA – *For any nonnegative measurable function ω on* $[0, 2]$ *the integrals*

$$\alpha = \int_S \omega(\|\zeta - z\|) |C(z, \zeta)| \, d\sigma(\zeta) \tag{1}$$

and

$$\beta(r) = \int_S \omega(\|\zeta - z\|) \, |C(z, \zeta) - C(rz, \zeta)| \, d\sigma(\zeta) \tag{2}$$

are independent of z (for $z \in S$, $0 \leqslant r < 1$).

If $\omega = \omega_\varphi$ for some $\varphi \in C(S)$ that satisfies Dini's condition, then $\alpha < \infty$ and $\beta(r) \to 0$ as $r \to 1$.

Proof. – We begin with the (more interesting) case $n > 1$.

Note that $\|\zeta - z\|^2 = 2 \operatorname{Re}(1 - \langle z, \zeta \rangle)$ if $\|z\| = \|\zeta\| = 1$. By

2.7 (1) the integrands in (1) and (2) are thus functions of $\langle z, \zeta \rangle$. It follows from Lemma 2.6 that

$$\alpha = \frac{n-1}{\pi} \int_U G(\lambda)\, dm_2(\lambda) \tag{3}$$

and

$$\beta(r) = \frac{n-1}{\pi} \int_U G(\lambda) \left| 1 - \left(\frac{1-\lambda}{1-r\lambda}\right)^n \right| dm_2(\lambda) \tag{4}$$

where

$$G(\lambda) = \frac{\omega\,(\sqrt{2 - 2\,\mathrm{Re}\,\lambda})}{|1-\lambda|^n}(1 - |\lambda|^2)^{n-2}. \tag{5}$$

This shows the asserted independence. If $\omega = \omega_\varphi$ then ω is non-decreasing ; since $1 - |\lambda|^2 \leqslant 2\,|1 - \lambda|$ in U, it follows that

$$G(\lambda) \leqslant 2^{n-2}\,|1-\lambda|^{-2}\,\omega\,(\sqrt{2\,|1-\lambda|}). \tag{6}$$

The integral of G over U is not larger than the integral of the right side of (6) over the half disc D with center at $\lambda = 1$ and radius 2 that covers U. Write $\lambda = 1 - se^{i\theta}$ $(0 < s < 2,\ |\theta| \leqslant \pi/2)$ and then replace $2s$ by t^2, to obtain

$$\int_D \frac{\omega\,(\sqrt{2\,|1-\lambda|}}{|1-\lambda|^2}\, dm_2(\lambda) = 2\pi \int_0^2 \omega\,(t)\,\frac{dt}{t}. \tag{7}$$

Thus $\alpha < \infty$ if $\omega = \omega_\varphi$ and 2.8 (2) holds.

Next, put $\gamma = (1-\lambda)/(1-r\lambda)$, for $|\lambda| \leqslant 1$, $0 \leqslant r < 1$. Then $|1-\gamma| = (1-r)|\lambda|/|1-r\lambda| \leqslant 1$. Hence $|\gamma| \leqslant 2$, and

$$|1-\gamma^n| = |1-\gamma|\,|1+\gamma+\cdots+\gamma^{n-1}| \leqslant 2^n - 1, \tag{8}$$

a bound that is independent of r and λ. Since $\alpha < \infty$ means that $G \in L^1(U)$, Lebesgue's dominated convergence theorem can be applied to the integral in (4) ; it shows that $\beta(r) \to 0$ as $r \to 1$.

When $n = 1$, S is the unit circle, and the substitution $t = 2\sin|\theta/2|$ gives

$$\alpha = \frac{2}{\pi}\int_0^2 \frac{\omega(t)}{\sqrt{4-t^2}} \cdot \frac{dt}{t} < \infty \tag{9}$$

if $\omega = \omega_\varphi$. (Note that ω_φ is bounded). That $\beta(r) \to 0$ as $r \to 1$ follows as above.

We now come to the principal step in the proof that $H^\infty + C$ is an algebra on S.

2.10. – LEMMA. – *If* $\varphi \in C(S)$ *satisfies Dini's condition and* $f \in H^\infty(S)$ *then* $T_\varphi f \in H^\infty(B)$ *and there is a* $g \in C(S)$ *such that*

$$\varphi f = g + (T_\varphi f)^* \qquad \text{a.e. } [\sigma] \tag{1}$$

Proof. – Apply Lemma 2.9, with $\omega = \omega_\varphi$. Since $\alpha < \infty$, the integral

$$g(z) = -\int_S [\varphi(\zeta) - \varphi(z)]\, C(z, \zeta)\, d\sigma(\zeta) \tag{2}$$

exists (as a Lebesgue integral) for every $z \in S$, and defines a bounded function g on S.

Since $f \in H^\infty(S)$, the Cauchy formula extends f to a bounded holomorphic function in B, which we still denote by f. Explicitly,

$$f(rz) = \int_S C(rz, \zeta)\, f(\zeta)\, d\sigma(\zeta) \tag{3}$$

for $z \in S$, $0 \leqslant r < 1$. Also

$$(T_\varphi f)(rz) = \int_S C(rz, \zeta)\, \varphi(\zeta)\, f(\zeta)\, d\sigma(\zeta). \tag{4}$$

Comparison of these three integrals shows that

$$(T_\varphi f)(rz) - f(rz)\,\varphi(z) + g(z)$$
$$= \int_S [\varphi(\zeta) - \varphi(z)]\,[C(rz, \zeta) - C(z, \zeta)]\, f(\zeta)\, d\sigma(\zeta). \tag{5}$$

Define $\beta(r)$ as in Lemma 2.9, with $\omega = \omega_\varphi$, and use the notation 2.1 (4). Then (5) implies

$$|(T_\varphi f)_r - f_r \varphi + g|(z) \leqslant \beta(r)\, \|f\|_\infty \qquad (z \in S,\ 0 \leqslant r < 1). \tag{6}$$

Note that $(T_\varphi f)_r$ and $f_r \varphi$ are continuous functions on S. Since $\beta(r) \to 0$ as $r \to 1$ (Lemma 2.9), their difference converges, uniformly on S, to $-g$. Thus $g \in C(S)$. Since f is bounded in B, (6) implies that $T_\varphi f$ is bounded, hence that $T_\varphi f \in H^\infty(B)$. Thus $(T_\varphi f)_r \to (T_\varphi f)^*$ and $f_r \to f$, a.e. on S, as $r \to 1$, so that (1) follows from (6).

2.11. – *Proof of theorem 2.3.* – Choose $f \in H^\infty(S)$ and $g \in C(S)$. There exist infinitely differentiable functions φ_i on S which converge uniformly to g. By Lemma 2.10, each $\varphi_i f \in H^\infty + C$. Since

$$\|\varphi_i f - gf\|_\infty \to 0$$

as $i \to \infty$ and since $H^\infty + C$ is closed (Section 3.4) it follows that $gf \in H^\infty + C$. Thus $H^\infty(S) + C(S)$ is closed under the formation of products, i.e., it is an algebra.

2.12. – *Remarks.* – The Dini condition cannot be dropped from the hypotheses of Lemma 2.10 since $T_\varphi(1) = C[\varphi]$ is unbounded in B for some $\varphi \in C(S)$.

Some results similar to Lemma 2.10 already exist in the literature. Henkin [9 ; Theorem 1.2] shows that if f is in the ball algebra (continuous on $\overline{B}$, holomorphic in B) and if φ satisfies a Lipschitz condition (of order 1) on S, then $T_\varphi f$ is in the ball algebra. Stout [21 ; Lemma 2] shows that the real part of $T_\varphi(f^*)$ is bounded in B if f is a holomorphic function in B whose real part is bounded and if φ is real and satisfies a Lipschitz condition on S.

Actually, both Henkin and Stout formulate and prove these results not just for the ball, but for strictly pseudo-convex regions with C^3-boundary (Henkin) and for certain convex regions with C^2-boundary (Stout).

Korányi and Vági [12 ; p. 627] prove, for $1 < p < \infty$, that $C[f]$ lies in the Hardy space $H^p(B)$ if $f \in L^p(S)$. It follows that the Toeplitz operators T_φ map $L^p(S)$ into $H^p(B)$, for $1 < p < \infty$ and for every $\varphi \in L^\infty(S)$.

To conclude Part II, here is another version of Theorem 2.3.

2.13. Theorem. – *If* $C_u(B)$ *denotes the class of all uniformly continuous functions with domain* B, *then* $H^\infty(B) + C_u(B)$ *is a Banach algebra, relative to the supremum norm.*

Proof. – Let X be the sup-normed Banach algebra of all bounded continuous functions on B, put $Y = C_u(B)$, $Z = H^\infty(B)$. Then Y consists of exactly those members of X that extend continuously to the closed ball $\overline{B}$. Hence Y and Z are closed subalgebras of X. To apply Theorem 1.2, define $(\Lambda_r f)(z) = f(rz)$, for $f \in X$, $z \in B$, $0 < r < 1$. These operators Λ_r have the properties (a) to (d) needed in Theorem 1.2. Hence $Y + Z$ is closed in X.

As in Section 2.11, it is now sufficient to show that $\varphi f \in H^\infty(B) + C_u(B)$ if $f \in H^\infty(B)$ and if φ is very smooth (say, infinitely differentiable) on $\overline{B}$. In that case, Lemma 2.10 implies that the function

$$F = C[\varphi f^*] = T_\varphi(f^*) \tag{1}$$

is in $H^\infty(B)$. For $z \in S$, $0 \leqslant r < 1$, let us write

$$(\varphi f - F)(rz) = a(r, z) + b(r, z) \tag{2}$$

where

$$a(r, z) = \varphi(z) f(rz) - F(rz), \tag{3}$$

$$b(r, z) = [\varphi(rz) - \varphi(z)] f(rz). \tag{4}$$

As $r \to 1$, the proof of Lemma 2.10 shows that $a(r, z) \to g(z)$ uniformly on S ; since φ is uniformly continuous and f is bounded, $b(r, z) \to 0$ uniformly on S. Thus $(\varphi f - F)_r \to g$, uniformly on S. Hence $\varphi f - F \in C_u(B)$, and the proof is complete.

3. $H^\infty + C$ on groups.

3.1. Beginning with Helson and Lowdenslager [7], "analytic" functions have been studied on compact abelian groups G whose duals Γ are *ordered* groups. By definition, functions on G are "analytic" in this setting if their Fourier coefficients are 0 on the negative half of Γ. (Chapter 8 of [16] is devoted to this topic). In particular, $H^\infty(G)$ makes sense in those situations, and it is natural to ask to what extent the analogue of Sarason's theorem holds. This is answered by Theorem 3.6. But we shall first see that Theorem 1.2 can also be applied on noncommutative groups.

3.2. DEFINITIONS. – Let G be a locally compact group, not necessarily abelian, with identity element e, and with a left Haar measure m : for every Borel set $E \subset G$ and for every $x \in G$, $m(xE) = m(E)$.

The notation $L^p(G)$ refers to the usual Lebesgue spaces, with respect to m. In particular, $L^\infty(G)$ is the Banach space of all bounded Borel functions on G, modulo those that are locally null, and $L^\infty(G)$ is the dual space of $L^1(G)$; see [10 ; p. 141, p. 148].

Each function f with domain G *has right translates* $R_a f$ and *left translates* $L_a f$ defined by

$$(R_a f)(x) = f(xa)\ ,\ (L_a f)(x) = f(ax) \qquad (x \in G, a \in G). \qquad (1)$$

A set Σ of functions on G is *left-invariant* if $L_a f \in \Sigma$ for all $f \in \Sigma$ and $a \in G$. To define *right-invariant,* replace L_a by R_a.

The symbol $C_{ru}(G)$ denotes the class of all bounded functions f on G that are *right uniformly continuous* : to every $\epsilon > 0$ should correspond a neighborhood V of e in G such that $|f(y) - f(x)| < \epsilon$ whenever $y \in Vx$. It is easily seen that $C_{ru}(G)$ is a closed subalgebra of $L^\infty(G)$, and that $C_{ru}(G)$ is right invariant.

The *convolution* $f * g$ of two Borel functions f and g with domain G is defined by

$$(f * g)(x) = \int_G f(y)\, g(y^{-1}x)\, dy = \int_G f(xy)\, g(y^{-1})\, dy \qquad (2)$$

whenever the integrals are finite if the integrands are replaced by their absolute values ; note that we write dy for $dm(y)$.

3.3. THEOREM. – *If* G *is a locally compact group and if* H *is a left-invariant weak*-closed subspace of* $L^\infty(G)$, *then*

$$H + C_{ru}(G) \qquad (1)$$

is a norm-closed subspace of $L^\infty(G)$.

The weak*-topology of $L^\infty(G)$ is of course the one that $L^\infty(G)$ has by virtue of being the dual space of $L^1(G)$. Note the (1) is the sum of a left-invariant space and a right-invariant one. This left-right theme will occur again in Part IV.

Proof. – Put $X = L^\infty(G)$, $Y = C_{ru}(G)$, $Z = H$, and apply Theorem 1.2, with Φ the collection of all operators Λ_φ given by

$$\Lambda_\varphi f = \varphi * f \qquad (2)$$

where $\varphi \in L^1(G)$, $\varphi \geqslant 0$, $\int_G \varphi\, dm = 1$.

We have to verify properties (a) to (d) of Theorem 1.2.

Since $L^1 * L^\infty \subset C_{ru}$ [10 ; p. 295], (a) holds.

To prove (b), let $Z^\perp$ be the space of all $g \in L^1(G)$ such that

$$\int_G fg \, dm = 0 \tag{3}$$

for every $f \in Z$. Since Z is weak*-closed, it follows that $f \in Z$ if and only if (3) holds for every $g \in Z^\perp$. Since Z is left-invariant,

$$\int_G f(x^{-1} y) \, g(y) \, dy = 0 \tag{4}$$

if $f \in Z$, $g \in Z^\perp$, $x \in G$. Multiply (4) by $\varphi(x)$, where $\varphi \in L^1(G)$, integrate with respect to dx, and use Fubini's theorem. The result is

$$\int_G (\varphi * f)(y) \, g(y) \, dy = 0 \tag{5}$$

for every $g \in Z^\perp$. Thus $\varphi * f \in Z$, and (b) holds.

Since $\|\varphi * f\|_\infty \leqslant \|\varphi\|_1 \, \|f\|_\infty$, $\|\Lambda_\varphi\| = 1$ for every $\Lambda_\varphi \in \Phi$. Hence (c) holds.

Finally, take $f \in Y = C_{ru}(G)$, pick $\epsilon > 0$, and let V be a neighborhood of e in G such that $|f(y) - f(x)| < \epsilon$ whenever $yx^{-1} \in V$. Take φ so that $\varphi = 0$ outside V. Then

$$\begin{aligned} |f(x) - (\Lambda_\varphi f)(x)| &= \left| \int_V \varphi(y) \, [f(x) - f(y^{-1} x)] \, dy \right| \\ &\leqslant \sup_{y \in V} |f(x) - f(y^{-1} x)| \leqslant \epsilon \end{aligned}$$

since $x \, (y^{-1} x)^{-1} = y$. Thus (d) holds, and the proof is complete.

Example 3.5 below is an obvious application of Theorem 3.3. But let us first look at one that is not quite so obvious, even when $G = R$.

3.4. THEOREM. – *If* G *is a locally compact group, if* $a \in G$, *and if* $P_a(G)$ *is the class of all* $g \in L^\infty(G)$ *that satisfy the periodicity relation*

$$R_a g = g, \tag{1}$$

then $C_{ru}(G) + P_a(G)$ *is a closed subspace of* $L^\infty(G)$.

Although $C_{ru}(G)$ *and* $P_a(G)$ *are closed subalgebras of* $L^\infty(G)$, *their sum is not an algebra, except in the trivial case* $a = e$.

Proof. – It is clear that $P_a(G)$ is a subalgebra of $L^\infty(G)$, and that $P_a(G)$ is left-invariant ; to see the latter, note that $R_a L_b = L_b R_a$ for every $b \in G$. To apply Theorem 3.3, we have to show that $P_a(G)$ is weak*-closed in $L^\infty(G)$.

The equality (1) means that $g(xa) = g(x)$ except possibly in some set that is locally null. Thus a function $g \in L^\infty(G)$ belongs to $P_a(G)$ if and only if

$$\int_G (g - R_a g) f \, dm = 0 \tag{2}$$

for every $f \in L^1(G)$. Put $b = a^{-1}$. The relation

$$\int_G (R_a g) f \, dm = \Delta(b) \int_G g R_b f \, dm \tag{3}$$

follows almost directly from the definition of the modular function Δ of G [10 ; pp. 195-6]. Thus (2) is equivalent to

$$\int_G [f - \Delta(b) R_b f] \, g \, dm = 0.$$

We conclude that $g \in P_a(G)$ if and only if $\int_G hg \, dm = 0$ for every h of the form $h = f - \Delta(b) R_b f$, as f ranges over $L^1(G)$. Hence $P_a(G)$ is weak*-closed.

Theorem 3.3 implies now that $C_{ru}(G) + P_a(G)$ is (norm-) closed.

Next, suppose $a \neq e$. Choose a function $\varphi \in P_a(G)$ which is discontinuous at e (more precisely, which does not coincide a.e. with any function that is continuous at e). Choose $f \in C_{ru}(G)$ so that $f(e) = 1$, $f(a) = 0$. Assume

$$\varphi f = g + h \tag{5}$$

for some $g \in P_a(G)$, $h \in C_{ru}(G)$. Since $f(a) = 0$ and φ is bounded, φf is continuous at a. Hence g is continuous at a. Since $g \in P_a(G)$, g is also continuous at e. But φf is not continuous at e. Hence (5) is impossible, $\varphi f \notin C_{ru}(G) + P_a(G)$, and the proof is complete.

3.5. *Example.* – Suppose G is a compact *abelian* group with dual group Γ (i.e., Γ is the group of all continuous characters of G), pick $E \subset \Gamma$, and let L_E^∞ be the class of all $f \in L^\infty(G)$ whose Fourier coefficients

$$\hat{f}(\gamma) = \int_G f\bar{\gamma}\, dm \qquad (1)$$

are 0 for all γ not in E.

It is clear that L_E^∞ is a weak*-closed subspace of $L^\infty(G)$. Moreover, the relation

$$(R_a f)\hat{}(\gamma) = \gamma(a)\hat{f}(\gamma) \qquad (a \in G, \gamma \in \Gamma) \qquad (2)$$

shows that L_E^∞ is translation-invariant. (Left and right are now the same, since G is abelian).

The compactness of G implies that $C_{ru}(G) = C(G)$, the space of all continuous functions on G. The following conclusion can therefore be drawn from Theorem 3.3.

$C(G) + L_E^\infty$ *is a closed subspace of* $L^\infty(G)$.

If E is a semigroup then L_E^∞ is a subalgebra of $L^\infty(G)$. Nevertheless, $C(G) + L_E^\infty$ need not be an algebra. Theorem 2.2 showed this (with $G = T^n$, $n > 1$) and Theorem 3.6 will give a more striking illustration.

Background material on ordered groups may be found in Chapter 8 of [16]. As is usual, abelian groups will be written additively, with identity element 0.

3.6. THEOREM. – *Let* G *be a compact abelian group whose dual* Γ *is an ordered group. Define*

$$H^\infty(G) = \{f \in L^\infty(G) : \hat{f}(\gamma) = 0 \quad \text{for all} \quad \gamma < 0\}.$$

Then $C(G) + H^\infty(G)$ *is a closed subspace of* $L^\infty(G)$, *but is an algebra only when* Γ *is isomorphic to the additive group of the integers (and in the trivial case* $\Gamma = \{0\}$*).*

Proof. – Example 3.5 implies directly that $C(G) + H^\infty(G)$ is closed. The proof of the remaining assertion splits into two cases, depending on whether the order of Γ is archimedean or not.

Case 1 : Γ *is archimedean.*

Then Γ is isomorphic to a subgroup of R_d, the real line R with the discrete topology [16 ; p. 196], and we may as well regard Γ as a subgroup of R_d. If Γ is an infinite cyclic group, Sarason's theorem applies. We exclude this case from now on, as well as the case $\Gamma = \{0\}$. Then Γ is dense in R, with respect to the usual topology of R.

We also assume, without loss of generality, that $1 \in \Gamma$. (This "1" is the real number 1 in R_d).

The well-known "triangular" function Δ, defined by

$$\Delta(t) = \max(1 - |t|, 0) \tag{1}$$

is positive-definite on R, hence on R_d, hence on Γ. By Bochner's theorem [16 ; p. 19] the restriction of Δ to Γ is therefore the Fourier transform of some $\nu \in M(G)$ (the space of all complex Borel measures on G). For any $\gamma \in \Gamma$, $(\gamma\nu)\hat{}$ is a translate of $\hat{\nu}$. Hence there exists a measure $\mu \in M(G)$ with $\hat{\mu}(\gamma) = 0$ for all $\gamma \in \Gamma$ which are outside the segment $(0, 2)$ and with $\hat{\mu}(\gamma) \geqslant 1/2$ for infinitely many $\gamma \in \Gamma$. In particular, μ is not absolutely continuous (relative to the Haar measure m of G).

This implies that there exists an $F \in L^\infty(G)$ such that $\hat{F}\hat{\mu}$ is not the Fourier transform of any continuous function on G. In other words, there is no $g \in C(G)$ such that $g(x) = (F * \mu)(x)$ a.e. $[m]$. (For the circle group in place of G, this is one of the earliest nontrivial multiplier theorems in harmonic analysis ; it was proved by Zygmund [24], [25 ; Vol. I, p. 177] ; the second statement is proved, for arbitrary locally compact G, in [10 ; (35.13)].

Put $f = F * \mu$. Then $f \in L^\infty(G)$, $f \notin C(G)$, and $\hat{f}(\gamma) = 0$ outside $(0, 2)$. In particular, $f \in H^\infty(G)$.

We can choose $\gamma_0 \in \Gamma$ so that $(\gamma_0 f)\hat{}(\gamma) = 0$ for all γ outside the segment $(-3, -1)$. We claim that $\gamma_0 f$ has no decomposition of the form

$$\gamma_0 f = g + h, \ g \in C(G), \ h \in H^\infty(G). \tag{2}$$

To see this, assume (2) holds, and choose $\lambda \in M(G)$ so that $\hat{\lambda} = 1$ on $(-3, -1) \cap \Gamma$ and $\hat{\lambda} = 0$ on $[0, \infty) \cap \Gamma$; we may choose λ so that $\hat{\lambda}$ is the restriction of

$$2\Delta\left(\frac{1}{2}(t+2)\right) - \Delta(t+2) \tag{3}$$

to Γ. Then $(\gamma_0 f) * \lambda = \gamma_0 f$ and $h * \lambda = 0$. Hence (2) implies that $\gamma_0 f = g * \lambda \in C(G)$, whereas $f \notin C(G)$. This contradiction shows that (2) is impossible.

Hence $C(G) + H^\infty(G)$ is not an algebra.

Case 2. Γ *is not archimedean.*

In that case, Γ contains an infinite cyclic subgroup Γ_0 and an element γ_1 such that $\gamma < \gamma_1$ for every $\gamma \in \Gamma_0$. Choose $\gamma_0 > 0$ so that γ_0 generates Γ_0. Since Γ is ordered, G is connected [16 ; p. 197], and since γ_0 is not the trivial character, γ_0 maps G *onto* the circle T.

Choose $F \in H^\infty(T)$ so that $F \notin C(T)$ and define

$$f(x) = F(\gamma_0(x)) \qquad (x \in G). \tag{4}$$

Then

$$\hat{f}(\gamma) = \begin{cases} \hat{F}(n) \text{ if } \gamma = n\gamma_0 \\ 0 \text{ for all other } \gamma \in \Gamma. \end{cases} \tag{5}$$

Perhaps the easiest way to see (5) is to start with $F(e^{i\theta}) = e^{in\theta}$. Then $f(x) = [\gamma_0(x)]^n = (x\,,\, n\gamma_0)$, and (5) follows from the fact that characters form an orthonormal set. The general case of (5) follows from this, by linearity.

We now have an $f \in H^\infty(G)$, $f \notin C(G)$, with $\hat{f}(\gamma) = 0$ if $\gamma \notin \Gamma_0$. We claim that $\overline{\gamma}_1 f$ has no decomposition of the form

$$\overline{\gamma}_1 f = g + h,\ g \in C(G),\ h \in H^\infty(G). \tag{6}$$

If (6) holds, then

$$f = \gamma_1 g + \gamma_1 h. \tag{7}$$

Since Γ_0 is a subgroup of Γ, there is a measure μ on G (the Haar measure of the annihilator of Γ_0) whose Fourier transform $\hat{\mu}$ is 1 on Γ_0, 0 off Γ_0.

Since Γ_0 supports $\hat{f}$, we have $f * \mu = f$.

Since $h \in H^\infty(G)$, $(\gamma_1 h)\hat{}(\gamma) = 0$ unless $\gamma \geqslant \gamma_1$; since Γ_0 supports $\hat{\mu}$, our choice of γ_1 shows that $(\gamma_1 h) * \mu = 0$.

Hence (7) implies

$$f = f * \mu = (\gamma_1 g) * \mu. \tag{8}$$

Since $g \in C(G)$, $(\gamma_1 g) * \mu \in C(G)$. But $f \notin C(G)$. This contradiction shows that (6) is impossible, and we have proved that $H^\infty(G) + C(G)$ is not an algebra.

[Note that the proof in Section 2.5 followed the same pattern as the one that we just completed, although the details were simpler there].

4. Sums of ideals.

4.1. Definitions. – *Let $\mathfrak{A}$ be a Banach algebra, not necessarily commutative, with or without unit. A right ideal of $\mathfrak{A}$ is a subspace $\mathfrak{R}$ of $\mathfrak{A}$ such that $xy \in \mathfrak{R}$ if $x \in \mathfrak{R}$ and $y \in \mathfrak{A}$. A left ideal of $\mathfrak{A}$ is a subspace $\mathfrak{L}$ such that $xy \in \mathfrak{L}$ if $x \in \mathfrak{A}$ and $y \in \mathfrak{L}$.*

A subalgebra $\mathfrak{A}_0$ of $\mathfrak{A}$ is said to have a *bounded left approximate identity* if there is a number $M < \infty$ with the following property :

To every finite set $F \subset \mathfrak{A}_0$ and to every $\epsilon > 0$ corresponds an element $u \in \mathfrak{A}_0$ such that $\|u\| \leqslant M$ and $\|x - ux\| < \epsilon$ for all $x \in F$.

[Quite recently it has been proved [22] that if this holds for all singletons F then it holds in fact (with the same M) for all finite sets F. The principal hypothesis of Theorem 4.2, though formally weaker than the assumption that $\mathfrak{R}$ has a bounded left approximate identity, is thus actually equivalent to it].

4.2. Theorem. – *Suppose that $\mathfrak{L}$ and $\mathfrak{R}$ are closed left and right ideals, respectively, in a Banach algebra $\mathfrak{A}$, and that there is a number $M < \infty$ with the following property :*

If $x \in \mathfrak{R}$ and $\epsilon > 0$, there exists $u \in \mathfrak{R}$ such that $\|u\| \leqslant M$ and $\|x - ux\| < \epsilon$.

Then $\mathfrak{R} + \mathfrak{L}$ is a closed subspace of $\mathfrak{A}$.

Proof. – For each $u \in \mathcal{R}$ with $\|u\| \leqslant M$, define

$$\Lambda_u(x) = ux \qquad (x \in \mathfrak{A}). \tag{1}$$

Let $\Phi \subset \mathcal{B}(\mathfrak{A})$ be the set of these multiplication operators Λ_u. We verify (with $X = \mathfrak{A}$, $Y = \mathcal{R}$, $Z = \mathcal{L}$) that Φ has properties (a) to (d) of Theorem 1.2 :

a) holds because $\mathcal{R}$ is a right ideal.

b) holds because $\mathcal{L}$ is a left ideal.

c) holds because $\|\Lambda_u\| \leqslant \|u\| \leqslant M$.

d) is part of our hypothesis.

4.3. *Remark.* – Of course, the theorem holds also if left and right are interchanged, i.e., if we assume that to each $x \in \mathcal{L}$ and to each $\epsilon > 0$ corresponds a $u \in \mathcal{L}$ with $\|u\| \leqslant M$ and $\|x - xu\| < \epsilon$.

4.4. *Example.* – Suppose $e \in \mathfrak{A}$, $e^2 = e$ (i.e., e is an *idempotent*) and $\mathcal{L}$ is a closed left ideal in $\mathfrak{A}$.

It is clear that $e\mathfrak{A}$ is a right ideal in $\mathfrak{A}$. Note that $x \in e\mathfrak{A}$ if and only if $x = ex$. The set of all $x \in \mathfrak{A}$ for which $x = ex$ is obviously closed. So $e\,\mathfrak{A}$ is closed. Alos, $e \in e\,\mathfrak{A}$. Hence $e\,\mathfrak{A}$ satisfies the condition imposed on $\mathcal{R}$ in Theorem 4.2, with $u = e$, $M = \|e\|$.

Conclusion : $e\mathfrak{A} + \mathcal{L}$ *is closed in* $\mathfrak{A}$.

One sees in the same way that $\mathcal{R} + \mathfrak{A}e$ is closed, for every closed right ideal $\mathcal{R}$ in $\mathfrak{A}$

4.5. *Example.* – Let G be any locally compact group, and let M(G) be the Banach algebra of all complex Borel measures on G, with convolution as multiplication. By the Radon-Nikodym theorem, $L^1(G)$ can be identified with the set of all absolutely continuous members of M(G). This turns $L^1(G)$ into a closed two-sided ideal of M(G) which has a bounded approximate identity [10 ; pp. 272, 303]. Theorem 4.2 thus gives the following conclusion :

If $\mathcal{L}$ and $\mathcal{R}$ are closed left and right ideals in M(G), *respectively, then*

$$\mathcal{L} + L^1(G) \quad \textit{and} \quad \mathcal{R} + L^1(G)$$

are closed subspaces of M(G).

The word "closed" refers to the norm topology of M(G) : $\|\mu\|$ is the total variation of $\mu \in M(G)$.

4.6. *Example.* – A B*-*algebra* is, by definition, a Banach algebra with an involution $x \to x^*$ in which the equality $\|xx^*\| = \|x\|^2$ holds.

It is known [5 ; 1.7.3] that every closed right ideal $\mathcal{R}$ in a B*-algebra $\mathcal{A}$ has a bounded approximate left identity. Since the fact needed to apply Theorem 4.2 is quite easily established (see also the proof of Theorem 4.8.14 in [14]) we sketch the required computation :

Pick $x \in \mathcal{R}$, pick a large positive N, and put $y = Nxx^*$. The spectrum of y lies in $[0, \infty)$. Put $u = y(e + y)^{-1}$. Then $u \in \mathcal{R}$, $\|u\| \leqslant 1$, $u = u^*$ and

$$(x - ux)(x - ux)^* = f(y)/N$$

where $f(t) = t/(1 + t)^2 \leqslant \frac{1}{2}$ on $[0, \infty)$. Thus

$$\|x - ux\|^2 \leqslant \frac{1}{2N}.$$

Hence Theorem 4.2 gives the following conclusion :

If $\mathcal{L}$ *and* $\mathcal{R}$ *are closed left and right ideals, respectively, in a* B*-*algebra* $\mathcal{A}$, *then* $\mathcal{L} + \mathcal{R}$ *is a closed subspace of* $\mathcal{A}$.

(I am indebted to C. Akeman for the information that this result is not new. See [2 ; Prop. 6.2]).

The obvious question to ask now is whether the sum of two closed right ideals in a B*-algebra $\mathcal{A}$ is always closed. The answer is negative, even in the best-known case, namely $\mathcal{A} = \mathcal{B}(H)$, the algebra of all bounded linear operators on a Hilbert space H. (The involution is the passage from an operator to its adjoint). This follows from Theorem 4.7.

Our final item, Theorem 4.9, describes a commutative algebra in which Theorem 4.2 fails.

4.7. THEOREM. – *If* X *is an infinite-dimensional Banach space, then* $\mathcal{B}(X)$ *contains closed right ideals* $\mathcal{R}_1, \mathcal{R}_2$, *and closed left ideals* $\mathcal{L}_1, \mathcal{L}_2$, *such that neither* $\mathcal{R}_1 + \mathcal{R}_2$ *nor* $\mathcal{L}_1 + \mathcal{L}_2$ *are closed in* $\mathcal{B}(X)$.

Proof. – Let M_1, M_2 be closed subspaces of X whose sum $M = M_1 + M_2$ is not closed. (The existence of such subspaces is undoubtedly well known. However, I can quote no reference, and therefore include a proof in Proposition 4.8.) Let $\mathcal{R}_i$ $(i = 1, 2)$ be the set of all $T \in \mathcal{B}(X)$ whose range lies in M_i, and put $\mathcal{R} = \mathcal{R}_1 + \mathcal{R}_2$.

Then $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{R}$ are right ideals in $\mathcal{B}(X)$, $\mathcal{R}_1$ and $\mathcal{R}_2$ are closed, and we claim that $\mathcal{R}$ is not closed :

Our choice of M_1 and M_2 shows that there are vectors x_n, y_n, z in X such that $x_n \in M_1, y_n \in M_2, z \notin M$, $\|z - x_n - y_n\| < 1/n$. Fix some $\Lambda \in X^*$ (the dual space of X) with $\|\Lambda\| = 1$, and define

$$S_n x = (\Lambda x)\, x_n\ ,\ T_n x = (\Lambda x)\, y_n\ ,\ Vx = (\Lambda x)\, z \qquad (x \in X). \quad (1)$$

Then $S_n \in \mathcal{R}_1$, $T_n \in \mathcal{R}_2$, and

$$\|V - S_n - T_n\| = \sup_{\|x\|=1} \|(\Lambda x)(z - x_n - y_n)\| \leqslant 1/n\,, \qquad (2)$$

so that V lies in the closure of $\mathcal{R}$. But since the range of V contains z and $z \notin M$, $V \notin \mathcal{R}$. Thus $\mathcal{R}$ is not closed.

For the other half, let N_1 and N_2 be (norm)-closed subspaces of X^* such that $N = N_1 + N_2$ is not closed, and let $\mathcal{L}_i$ $(i = 1, 2)$ be the set of all $T \in \mathcal{B}(X)$ whose adjoints T^* have their ranges in N_i. Since $T \to T^*$ is an isometry of $\mathcal{B}(X)$ into $\mathcal{B}(X^*)$, $\mathcal{L}_1$ and $\mathcal{L}_2$ are closed left ideals in $\mathcal{B}(X)$. We claim that their sum $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$ is not closed :

There are functionals Φ_n, Ψ_n, $\Lambda \in X^*$ such that $\Phi_n \in N_1$, $\Psi_n \in N_2$, $\Lambda \notin N$, $\|\Lambda - \Phi_n - \Psi_n\| < 1/n$. Fix some $x_0 \in X$ with $\|x_0\| = 1$, and define

$$S_n x = (\Phi_n x)\, x_0\ ,\ T_n x = (\Psi_n x)\, x_0\ ,\ Vx = (\Lambda x)\, x_0 \qquad (x \in X). \quad (3)$$

The adjoints of S_n, T_n, V have one-dimensional ranges in X^*, spanned by Φ_n, Ψ_n, Λ, respectively. Thus $S_n \in \mathcal{L}_1$, $T_n \in \mathcal{L}_2$, but $V \notin \mathcal{L}$, since $\Lambda \notin N$. On the other hand,

$$\|V - S_n - T_n\| = \sup_{\|x\|=1} \|((\Lambda - \Phi_n - \Psi_n)x)x_0\| \leqslant 1/n, \qquad (4)$$

so that V lies in the closure of $\mathcal{L}$.

4.8. Proposition. – *Every infinite-dimensional Banach space* X *contains closed subspaces* Y *and* Z, *with* $Y \cap Z = \{0\}$, *such that* $Y + Z$ *is not closed.*

Proof. – By replacing X by one its closed subspaces, we may assume, without loss of generality, that X has a Schauder basis $\{u_n\}$ [13 ; p. 67] with $\|u_n\| = 1$ for $n = 1, 2, 3, \ldots$ Every $x \in X$ has then a unique expansion

$$x = \sum_{n=1}^{\infty} c_n(x)\, u_n. \qquad (1)$$

The coefficient functionals c_n are continuous. The series converges in the norm topology ; hence

$$\lim_{n\to\infty} c_n(x) = 0 \quad \text{for each } x \in X. \qquad (2)$$

Define

$$Y = \{y \in X : c_{2k}(y) = 0 \quad \text{for} \quad k = 1, 2, 3, \ldots\}, \qquad (3)$$

$$Z = \{z \in X : c_{2k-1}(z) = k^2\, c_{2k}(z) \quad \text{for} \quad k = 1, 2, 3, \ldots\}. \qquad (4)$$

The continuity of the functionals c_n shows that Y and Z are closed ; the uniqueness of (1) shows : $Y \cap Z = \{0\}$. For every k, $u_{2k-1} \in Y$ and $k^2 u_{2k-1} + u_{2k} \in Z$. Hence $Y + Z$ contains every u_n, and thus

$$x_0 = \sum_{k=1}^{\infty} k^{-2}\, u_{2k} \qquad (5)$$

lies in the closure of $Y + Z$.

If $x_0 = y + z$, with $y \in Y$, $z \in Z$, then

$$k^{-2} = c_{2k}(x_0) = c_{2k}(y) + c_{2k}(z) = k^{-2} c_{2k-1}(z), \qquad (6)$$

so that $c_{2k-1}(z) = 1$ for $k = 1, 2, 3, \ldots$. This violates (2). Thus $x_0 \notin Y + Z$, and $Y + Z$ is not closed.

4.9. THEOREM. – *The disc algebra contains two closed ideals whose sum is not closed.*

Proof. – Recall that the disc algebra A is the sup-normed algebra of all continuous functions on the closed unit disc $\bar{U} \subset \mathbb{C}$ which are holomorphic in U. Put

$$\alpha_n = 1 - 2^{-n}\ ,\ \beta_n = \alpha_n + 8^{-n} \qquad (n = 1, 2, 3, \ldots) \tag{1}$$

and define

$$J_1 = \{f \in A : f(\alpha_n) = 0 \text{ for } n = 1, 2, 3, \ldots\}$$

$$J_2 = \{g \in A : g(\beta_n) = 0 \text{ for } n = 1, 2, 3, \ldots\}$$

$$M = \{h \in A : h(1) = 0\}.$$

Then J_1 and J_2 are closed ideals in A. Since $\alpha_j \neq \beta_k$ for all j and k, their sum $J = J_1 + J_2$ is an ideal in A whose closure is the maximal ideal M [15]. If $h_0(z) = 1 - z$, it follows that $h_0 \in \bar{J}$. We shall now show that $h_0 \notin J$.

The Schwarz lemma implies that

$$|g(\alpha_n)| \leqslant \left| \frac{\beta_n - \alpha_n}{1 - \alpha_n \beta_n} \right| \|g\|_\infty \tag{2}$$

if $g \in J_2$. Since $\beta_n - \alpha_n = (1 - \alpha_n)^3$, it follows that

$$|g(\alpha_n)| \leqslant (1 - \alpha_n)^2 \|g\|_\infty \qquad (g \in J_2). \tag{3}$$

If $h = f + g$ with $f \in J_1$, $g \in J_2$, (3) implies

$$|h(\alpha_n)| \leqslant (1 - \alpha_n)^2 \|g\|_\infty \qquad (n = 1, 2, 3, \ldots). \tag{4}$$

Since $h_0(\alpha_n) = 1 - \alpha_n$, (4) shows that $h_0 \notin J$.

BIBLIOGRAPHY

[1] L. BUNGART, Boundary kernel functions for domains on complex manifolds, *Pacific J. Math.*, 14 (1960), 1151-1164.

[2] F. COMBES, Sur les faces d'une C*-algèbre, *Bull. Sci. Math.*, 93 (1969), 37-62.

[3] A.M. DAVIE, T.W. GAMELIN, and J. GARNETT, Distance estimates and pointwise bounded density, *Trans. Amer. Math. Soc.*, 175 (1973), 37-68.

[4] A. DEVINATZ, An extension of a limit theorem of G. Szegö, *J. Math. Anal. Appl.*, 14 (1966), 499-510.

[5] J. DIXMIER, Les C*-algèbres et leurs Représentations, Gauthier-Villars, Paris, 1969.

[6] F. FORELLI, Measures whose Poisson integrals are pluriharmonic, *Illinois J. Math.*, 18 (1974), 373-388.

[7] H. HELSON and D. LOWDENSLAGER, Prediction theory and Fourier series in several variables, *Acta Math.*, 99 (1958), 165-202.

[8] H. HELSON and D. SARASON, Past and future, *Math. Scand.*, 21 (1967), 5-16.

[9] G.M. HENKIN, Integral representations of functions holomorphic in strictly pseudoconvex domains and some applications, *Math. USSR Sbornik*, 7 (1969), 597-616. (Mat. Sbornik 78 (1969)).

[10] E. HEWITT and K.A. ROSS, Abstract Harmonic Analysis, Springer Verlag, Berlin ; Vol. 1, 1963 ; Vol. 2, 1970.

[11] A. KORANYI, Harmonic functions on hermitian hyperbolic space, *Trans. Amer. Math. Soc.*, 135 (1969), 507-516.

[12] A. KORANYI and S. VAGI, Singular integrals in homogeneous spaces and some problems of classical analysis, *Ann. Scuola Normale Superiore Pisa*, 25 (1971), 575-648.

[13] J.T. MARTI, Introduction to the Theory of Bases, Springer Verlag, 1969.

[14] C.E. RICKART, General Theory of Banach Algebras, Van Nostrand, 1960.

[15] W. RUDIN, The closed ideals in an algebra of analytic functions, *Can. J. Math.*, 9 (1957), 426-434.

[16] W. RUDIN, Fourier Analysis on Groups, Interscience, 1962.

[17] W. RUDIN, Function Theory in Polydiscs, Benjamin, 1969.

[18] D. SARASON, Generalized interpolation in H^∞, *Trans. Amer. Math. Soc.*, 127 (1967), 179-203.

[19] D. SARASON, Algebras of functions on the unit circle, *Bull. Amer, Math. Soc.*, 79 (1973), 286-299.

[20] E.M. STEIN, Boundary Behavior of Holomorphic Functions of Several Complex Variables, Princeton University Press, 1972.

[21] E.L. STOUT, On the multiplicative Cousin problem with bounded data, *Ann. Scuola Normale Superiore Pisa*, 27 (1973), 1-17.

[22] J. WICHMANN, Bounded approximate units and bounded approximate identities, *Proc. Amer. Math. Soc.*, 41 (1973), 547-550.

[23] L. ZALCMAN, Bounded analytic functions on domains of infinite connectivity, *Trans. Amer. Math. Soc.*, 144 (1969), 241-269.

[24] A. ZYGMUND, Sur un théorème de M. Fekete, *Bull. Acad. Polonaise*, (1927), 343-347.

[25] A. ZYGMUND, Trigonometric Series, 2^{nd} Ed., Cambridge University Press, 1959.

Manuscrit reçu le 15 mars 1974
accepté par J.P. Kahane.

Walter RUDIN,
Department of Mathematics
University of Wisconsin
Madison, Wisc 53706 (USA).