ANNALES DE L'INSTITUT FOURIER

EVGENY B. DYNKIN

Additive functionals of Markov processes and stochastic systems

Annales de l'institut Fourier, tome 25, nº 3-4 (1975), p. 177-200

<http://www.numdam.org/item?id=AIF_1975__25_3-4_177_0>

Ann. Inst. Fourier, Grenoble
25, 3 et 4 (1975), 177-200.

ADDITIVE FUNCTIONALS OF MARKOV PROCESSES AND STOCHASTIC SYSTEMS

by E. B. DYNKIN

Dédié à Monsieur M. Brelot à l'occasion de son 70^e anniversaire.

A voluminous literature is devoted to the problem of describing additive functionals of Markov processes (see, for example, [1], [4], [9], [13], [14], [16]). Only continuous and natural functionals have been investigated before a general solution was outlined in [6]. In the present paper we propose a new method based on a general duality theory developed in [7], [8], and on the technique of central projection.

We consider inhomogeneous functionals. The homogeneous case will be treated in another place.

1. Introduction.

1.1. Let $(x_t, \mathbf{P})$ be a Markov process with a sample space Ω, defined on a time interval $T = (\alpha, \beta)$. For an open set $G \subseteq T$, denote by $\mathcal{F}(G)$ a σ-algebra on Ω generated by $x_t(t \in G)$. For any $B \in \mathcal{B}_T$ (1) define $\mathcal{F}(B)$ as an intersection of $\mathcal{F}(G)$ over all $G \supseteq B$. We shall use abbreviations

$$\mathcal{F}_{<t}, \ \mathcal{F}_{\leqslant t}, \ \mathcal{F}_{\neq t}, \ \mathcal{F}_{>t}, \ \mathcal{F}_{\geqslant t}$$

for $\mathcal{F}(\alpha, t)$, $\mathcal{F}(\alpha, t]$, $\mathcal{F}\{(\alpha, t) \cup (t, \beta)\}$, $\mathcal{F}(t, \beta)$, $\mathcal{F}[t, \beta)$.

(1) We denote by $\mathcal{B}_T$ the σ-algebra of Borel subsets of T.

A function $A(\omega, B)(\omega \in \Omega, B \in \mathscr{B}_T)$ is called an *additive functional* of $(x_t, \mathbf{P})$ if:

1.1.α. For almost all ω, $A(\omega, .)$ is a σ-finite measure on $\mathscr{B}_T$.

1.1.β. For any open interval $I \subseteq T$, the function $A(., I)$ is measurable with respect to $\overline{\mathscr{F}}(I)$ (2).

Functionals coinciding for almost all ω will be considered as indistinguishable.

We say that a function ξ on $T \times \Omega$ is *well-measurable* relative to $\overline{\mathscr{F}}_{\leqslant t}$ (to $\overline{\mathscr{F}}_{\geqslant t}$) if it is measurable with respect to the σ-algebra generated by functions $f(t, \omega)$ with the following properties: for any t, $f(t, .)$ is measurable relative to $\overline{\mathscr{F}}_{\leqslant t}$ (respectively, to $\overline{\mathscr{F}}_{\geqslant t}$) and, for almost all ω, $f(., \omega)$ is right (left) continuous in t (cf. [11], Chapter 8 and [3], Chapter 4).

A functional A will be called *normal* if the function

$$A\{t\} = A(\omega, \{t\})$$

satisfies the following conditions:

1.1.γ. For any t, $A\{t\}$ is measurable with respect to $\overline{\mathscr{F}}_{\neq t}$.

1.1.δ. $A\{t\}$ is well-measurable relative to $\overline{\mathscr{F}}_{\leqslant t}$ and relative to $\overline{\mathscr{F}}_{\geqslant t}$.

A functional A is *continuous* if, for almost all ω, $A\{t\} = 0$ for all $t \in T$. All continuous functionals are normal.

Condition 1.1.γ is fulfilled for all functionals if $\overline{\mathscr{F}}(T) = \overline{\mathscr{F}}_{\neq t}$ for any $t \in T$ (such are the cases of right and left continuous processes and regular processes in the sence of [5]).

Condition 1.1.δ is satisfied if $A(B) < \infty$ a.s. for any closed interval $B \subset T$: in this case the well-measurability of $A\{t\}$ follows immediately from the relations

$$A\{t\} = A(s, t] - A(s, t) = A[t, u) - A(t, u)$$

for all $s < t < u \in T$ (a.s.). (Some finiteness conditions

(2) For any σ-algebra $\mathscr{F}$ on Ω, $\overline{\mathscr{F}}$ means the minimal σ-algebra which contains $\mathscr{F}$ and all the sets of measure 0.

were used in all the papers devoted to the additive functionals. The above-mentioned one is perhaps the mildest of them.)

We describe all the normal additive functionals of a Markov process $(x_t, \mathbf{P})$ under the following assumptions:

1.1.A. The state space E_t at each time t is a Borel measurable space (i.e. is isomorphic to a Borel subset of a Polish space).

1.1.B. The sample space Ω coincides with the set of all the paths $\omega(t) \in E_t$.

1.1.C. The two-dimensional probability distributions

$$m_{st}(\Gamma) = \mathbf{P}\{(x_s, x_t) \in \Gamma\}$$

of $(x_t, \mathbf{P})$ are absolutely continuous with respect to the product of the corresponding one-dimensional distributions $m_t(B) = P(x_t \in B)$ (3).

1.2. The process x_t is involved in the definition of an additive functional only through the σ-algebras $\mathscr{F}(I)$.

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and let a σ-algebra $\mathscr{F}(I) \subseteq \mathscr{F}$ be associated with any open interval $I \subseteq T$ in such a way that:

1.2.A. $\mathscr{F}(I_1) \subseteq \mathscr{F}(I_2)$ if $I_1 \subseteq I_2$.

1.2.B. If $I_n \uparrow I$, then $\mathscr{F}(I)$ is generated by σ-algebras $\mathscr{F}(I_n)$.

1.2.C. Let $I_n \uparrow I$. Let $\mathbf{P}_n$ be a probability measure on $\mathscr{F}(I_n)$ and $\mathbf{P}_n(C) = \mathbf{P}_{n-1}(C)$ for $C \in \mathscr{F}(I_{n-1})$. Then there exists a measure $\mathbf{P}_\infty$ on $\mathscr{F}(I)$ which coincides with $\mathbf{P}_n$ on $\mathscr{F}(I_n)$, $n = 1, 2, \ldots$

According to [7], [8] we say that $(\mathscr{F}(I), \mathbf{P})$ is a *stochastic system* in the space $(\Omega, \mathscr{F})$.

Define $\mathscr{F}(G)$ for any open set $G \subseteq T$ as the minimal σ-algebra which contains $\mathscr{F}(I)$ for all the intervals $I \subseteq G$. According to n° 1.1, for an arbitrary $B \in B_T$, denote by $\mathscr{F}(B)$ the intersection of $\mathscr{F}(G)$ for all open $G \supseteq B$. In

(3) This result was announced in [6] however condition 1.1 δ was omitted. (Counterexamples show that, without 1.1, δ, Theorem 1.1 is not true.)

this paper we consider only stochastic systems satisfying the following additional condition:

1.2.D. $\mathscr{F}(\mathrm{I})$ is equal to the minimal σ-algebra which contains $\mathscr{F}\{t\}$ for all $t \in \mathrm{I}$.

We suppose that the σ-algebra $\mathscr{F}$ is complete for the measure $\mathbf{P}$.

Under conditions 1.1.A and 1.1.B, σ-algebras $\mathscr{F}(\mathrm{I})$ satisfy the requirements 1.2.A-1.2.D. Hence the problem of n° 1.1 is a special case of the general problem: to describe all the normal additive functionals of a stochastic system $(\mathscr{F}(\mathrm{I}), \mathbf{P})$.

To extend condition 1.1.C, we need the following definitions. A random process x_t taking values in Borel spaces E_t is called a *Markov representation* of the stochastic system $(\mathscr{F}(\mathrm{I}), \mathbf{P})$ if x_t is measurable relative to $\mathscr{F}(\mathrm{I})$ for $t \in \mathrm{I}$ and, for any $t \in \mathrm{T}$, $\xi \in \mathscr{F}_{<t}$, $\eta \in \mathscr{F}_{>t}$ (4),

$$(1.1) \qquad \mathbf{P}(\xi\eta|x_t) = \mathbf{P}(\xi|x_t)\mathbf{P}(\eta|x_t) \quad \text{(a.s.)}.$$

The representation x_t is called *absolutely continuous* if its two-dimensional distributions are absolutely continuous with respect to the products of the corresponding one-dimensional distributions.

All the normal additive functionals of the stochastic system $(x_t, \mathbf{P})$ will be described under the only condition that it has an absolutely continuous Markov representation.

1.3. The advantages of the new statement of the problem are the following: beeng not bound with a specific Markov representation we have the freedom to choose it in the best possible way.

It is proved in [7], [8] that, if a stochastic system has an absolutely continuous Markov representation, then there exist a Markov representation x_t and probability measures $\mathbf{P}_{t,x}$ on $\mathscr{F}_{>t}$ and $\mathbf{P}^{s,x}$ on $\mathscr{F}_{<t}$ with the following properties:

1.3.A. For $\xi \in \mathscr{F}_{<t}$, $\eta \in \mathscr{F}_{>t}$,

$$\mathbf{P}(\eta|\mathscr{F}_{\leqslant t}) = \mathbf{P}_{t,x_t}\eta, \ \mathbf{P}(\xi|\mathscr{F}_{\geqslant t}) = \mathbf{P}^{t,x_t}\xi \quad \text{(a.s.)}.$$

(4) For a function ξ, the symbol $\xi \in \mathscr{F}$ means that ξ is non-negative and $\mathscr{F}$-measurable. By $\mathbf{P}\xi$ we mean the mathematical expectation of ξ i.e. the integral of ξ with respect to $\mathbf{P}$.

1.3.B. For $s < t$, $\xi \in \mathscr{F}_{<s}$, $\eta \in \mathscr{F}_{>t}$,

$$\mathrm{P}_{s,x}(\eta|\mathscr{F}(s, t]) = \mathbf{P}_{t,x_t}\eta \quad (\text{a.s. } \mathbf{P}_{s,x}),$$
$$\mathrm{P}^{t,x}(\xi|\mathscr{F}[s, t)) = \mathbf{P}^{s,x_s}\xi \quad (\text{a.s. } \mathbf{P}^{t,x}).$$

1.3.C. If $\mathbf{P}_{t,x} = \mathbf{P}_{t,y}$ and $\mathbf{P}^{t,x} = \mathbf{P}^{t,y}$, then $x = y$.

1.3.D. Let $\xi \in \mathscr{F}_{\leqslant u}$, $\eta \in \mathscr{F}_{\geqslant u}$,

$$h_1(t, x) = \begin{cases} \mathbf{P}_{t,x}\eta & \text{for } t < u, \\ 0 & \text{for } t \geqslant u; \end{cases}$$
$$h_2(t, x) = \begin{cases} \mathrm{P}^{t,x}\xi & \text{for } t > u, \\ 0 & \text{for } t \leqslant u. \end{cases}$$

Then for almost all ω the function $h_1(t, x_t)$ is right continuous in t and has left-hand limits; the function $h_2(t, x_t)$ is left continuous and has right-hand limits.

We call $\mathbf{P}_{t,x}$ *transition* and $\mathbf{P}^{t,x}$ *cotransition probabilities.*

Additive functionals of $(\mathscr{F}(\mathrm{I}), \mathbf{P})$ will be described by means of x_t (5).

1.4. Functions $\xi_t(\omega)$ and $\eta_t(\omega)$ are called *indistinguishable* if $\mathbf{P}(\xi_t = \eta_t \text{ for all } t) = 1$. We call *evanescent* the functions which are indistinguishable from 0 and the sets which have the evanescent indicators. Set

$$\mathscr{F}_{\mathrm{T}\times\Omega} = \mathscr{B}_{\mathrm{T}} \times \mathscr{F}(\mathrm{T})$$

and denote by $\overline{\mathscr{F}}_{\mathrm{T}\times\Omega}$ the minimal σ-algebra which contains $\mathscr{F}_{\mathrm{T}\times\Omega}$ and all the evanescent sets.

Let $\mathscr{E}$ be the union of E_t for all $t \in \mathrm{T}$ and let $\mathscr{B}_{\mathscr{E}}$ be σ-algebra on $\mathrm{T} \times \Omega$ generated by all the functions (1.2). If $\Gamma \in \mathscr{B}_{\mathscr{E}}$, then $\{(t, \omega) : x_t(\omega) \in \Gamma\} \in \overline{\mathscr{F}}_{\mathrm{T}\times\Omega}$ and the expression

$$\mu(\Gamma) = \mathbf{P} \int_{\mathrm{T}} 1_\Gamma(x_t)\mathrm{A}\,(dt) \tag{1.3}$$

has meaning (6). It evaluates the mathematical expectation of time when x_t belongs to Γ if the time is measured by A.

(5) We rely only on properties 1.3.A-1.3.D of the representation x_t. Of course Markov representations with these properties can exist for stochastic systems which have no absolutely continuous Markov representation. All results of the paper remain valid in such cases. Condition 1.3.B is used rather weakly and, probably, a similar theory can be developed without 1.3.B.

(6) We denote by 1_{C} the indicator function of C that is the function with the value 1 on C and 0 outside C.

The measure μ defined by (1.3) will be called the *spectral measure* of the additive functional A.

A set Γ of $\mathscr{B}_{\mathcal{E}}$ is *inaccessible* if $\mathbf{P}(x_t \bar{\in} \Gamma$ for all $t \in T) = 1$. It is *scanty* if a.s. the set $\{t : x_t \in \Gamma\}$ is at most countable.

The main result of the paper is summarized in the following theorem:

THEOREM 1.1. — *The spectral measure of any normal additive functional is σ-finite and vanishes on all the inaccessible sets. Each measure on $\mathscr{B}_{\mathcal{E}}$ with these properties is a spectral measure of one and only one normal additive functional. The functional is continuous if and only if its spectral measure vanishes on all the scanty sets.*

1.5. The following fundamental identity plays the central role for the demonstration of Theorem 1.1:

$$\text{(1.4)} \qquad \mathbf{P} \int_T \xi_t A\,(dt) = \int_E \pi\xi \, d\mu$$

Here $\xi_t(\omega)$ is a $\overline{\mathscr{F}}_{T\times\Omega}$-measurable function such that, for any t, ξ_t is measurable relative to $\overline{\mathscr{F}}_{\neq t}$ (the functions with these properties will be called *solid*); and π is an operator which associates a class of equivalent measurable functions on $\mathscr{E}$ with every solid function ξ. (Two functions f_1, f_2 on $\mathscr{E}$ are called *equivalent* if $f_1 = f_2$ outside an inaccessible set.)

Solid functions are investigated in section 2. In particular, all the values A(B) of a normal additive functional A are proved to be *solid functions*. The projection π is defined in section 3. In section 4 we prove that all the spectral measures are σ-finite [7]. Relying on this fact, we prove the fundamental identity (1.4) in section 5.

Let φ be a solid function independent of t and let $B \in \mathscr{B}_T$. Formula (1.4) implies that

$$\text{(1.5)} \qquad \mathbf{P}\varphi A(B) = \int_{\mathcal{E}(B)} \pi\varphi \, d\mu$$

[7] For the first time, the σ-finiteness of μ was proved (in a different way) by Šur [15] under the additional condition: $A(s, t) < \infty$ a.s. for any

$$s < t \in T.$$

Here $\mathscr{E}(\mathrm{B})$ is the union of $\mathscr{E}_t$ over all $t \in \mathrm{B}$. It is easily seen from (1.5) that every normal additive functional is uniquely determined by its spectral measure. Further we see that $\mathrm{A}(\mathrm{B})$ is the Radon-Nikodym derivative $d\Phi_{\mathrm{B}}/d\mathbf{P}$ where $\Phi_{\mathrm{B}}(\varphi)$ is the expression on the right side of (1.5). This observation suggests a way to construct A starting from μ: first, calculate the measure Φ_{B} and then consider its derivative with respect to $\mathbf{P}$. This is done in section 6 ([8]).

With the help of (1.4), it is easy to prove that the functional A is continuous if μ vanishes on all the scanty sets. It is sufficient to remark that, for $\xi_t = \mathrm{A}\{t\}$, the left side of (1.4) is equal to $\mathbf{P} \sum_t \mathrm{A}\{t\}^2$ and the integrand on the right side vanishes outside of a scanty set (see Theorem 5.3).

To prove that the spectral measures are σ-finite, we use in section 4 the following assertion: every scanty set is a countable union of subsets such that, for each of them, the number of hits by x_t has the finite mathematical expectation. This assertion is rather close to a theorem of Dellacherie ([2], Theorem 5) but direct reference is not valid because our situation is different. We prove the above-mentioned fact in the Appendix at the end of the paper.

1.6. To simplify notations, we shall assume that

$$\mathrm{T} = (\alpha, \beta) = (-\infty, +\infty).$$

Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space. A family $\mathscr{M}_t (t \in \mathrm{T})$ of sub-σ-algebras of $\mathscr{F}$ will be called a right filtration of $(\Omega, \mathscr{F}, \mathbf{P})$ if $\mathscr{M}_s \subseteq \mathscr{M}_t$ for $s < t$. A function $\tau(\omega)$ taking values in $(-\infty, +\infty]$ is called a Markov (or stopping) time relative to $\mathscr{M}_t$ if $\{\tau \leqslant t\} \in \mathscr{M}_t$ for every finite t. Set $\mathrm{C} \in \mathscr{M}_\tau$ if $\mathrm{C} \cap \{\tau \leqslant t\} \in \mathscr{M}_t$ for every finite t. A σ-algebra of well-measurable sets is associated with any right filtration $\mathscr{M}_t$. This is the σ-algebra on $\mathrm{T} \times \Omega$ generated by all the evanescent functions and by all the right continuous functions $\xi_t(\omega)$ with the property: for every t, ξ_t is measurable with respect to $\mathscr{M}_t$.

The family $\mathscr{M}_t = \mathscr{F}_{\leqslant t}$ is an example of right filtration.

([8]) A similar idea was used by Metivier [12] to construct a natural increasing process from a stochastic measure with the bounded variation. A discussion with Professor Metivier was very fruitful and stimulating for the author.

The definition of *well-measurability* given in n° 1.1 for this case is equivalent to present one.

The definitions of a *left filtration* $\mathcal{M}^t$ and associated Markov times and well-measurable functions are similar.

Right and left filtration, transition and cotransition probabilities are examples of dual notions which can be obtained from each other by time reversing. To each statement there corresponds an obvious dual version. We formulate one of them and use asterisks for references to the dual statement.

2. Solid functions.

2.1. According to n° 1.5, a function $\xi_t(\omega)$ is called *solid* if it is $\overline{\mathcal{F}}_{T\times\Omega}$-measurable and if, for any t, ξ_t is $\overline{\mathcal{F}}_{\neq t}$-measurable. Obviously the finite sum and product and the limit of a sequence of solid functions are also solid. Every Borel function on T is solid. Now we want to establish the solidity of some other important functions.

2.1.A. The function $x_t(\omega)$ is solid. All the values $x_r(\omega)$ are also solid functions (independent of t).

Indeed, functions h_i defined by (1.2) generate the measurable structure on $\mathcal{E}$ and their restrictions to E_r generate the measurable structure on E_r (the last assertion follows from 1.1.A and 1.3.C; cf. [5], n° 2.1). Therefore all we need is to prove the solidity of $h_i(t, x_t)$ and $h_i(r, x_r)$. It can be easily deduced from 1.3.D.

2.1.B. Let A be a normal additive functional. Then the function $A\{t\}$ and all the functions $A(B)(\beta \in \mathcal{B}_T)$ are solid.

By 1.1.β and 1.1.δ all the functions are $\overline{\mathcal{F}}_{T\times\Omega}$ measurable. By 1.1.γ $A\{t\} \in \overline{\mathcal{F}}_{\neq t}$ and by 1.1.β $A(B) \in \mathcal{F}_{\neq t}$ for $t \in B$.

2.2. Consider the family of functions

$$\xi_t(\omega) = \varphi(\omega) 1_{s<t<u} \psi(\omega) \tag{2.1}$$

where $s < u$ and $\varphi \in \mathcal{F}_{<s}$, $\psi \in \mathcal{F}_{>u}$ are bounded. Evidently functions (2.1) are solid. We prove that together with the evanescent functions they generate the class of all the solid functions.

LEMMA 2.1. — *Denote by* $\mathscr{S}$ *the* σ*-algebra on* $T \times \Omega$ *generated by the functions* (2.1). *For any* $\mathscr{F}_{T\times\Omega}$*-measurable function* ξ, *there exist an* $\mathscr{S}$*-measurable function* η *and a countable set* Λ *such that*

$$\xi_t(\omega) = \eta_t(\omega) \quad \textit{for all} \quad t \bar{\in} \Lambda, \quad \omega \in \Omega.$$

Proof. — The set of the functions ξ for which the statement is true is closed under addition, multiplication and passage to the limit. It contains the indicator functions of all the open intervals. Taking into account 1.2.D, it is sufficient to prove the statement of Lemma for the functions

$$\xi_t(\omega) = \varphi(\omega)$$

where $\varphi \in \mathscr{F}\{r\}$. But

$$\lim_{\varepsilon \downarrow 0} (\varphi 1_{t>r+\varepsilon} + 1_{t<r-\varepsilon}) = \varphi 1_{t\neq r}.$$

The left side is $\mathscr{S}$-measurable and the right side is equal to ξ_t for $t \neq r$.

2.3. THEOREM 2.1. — *Let* $\mathscr{H}$ *be a class of non-negative functions on* T *with the properties*: *a*) *If* $f_1, f_2 \in \mathscr{H}$, *then* $f_1 + f_2 \in \mathscr{H}$. *b*) *If* $f \in \mathscr{H}$ *and* c *is a positive number, then* $cf \in \mathscr{H}$. *c*) *If* $f_1, f_2 \in \mathscr{H}$ *and* $f_1 - f_2 \geqslant 0$, *then* $f_1 - f_2 \in \mathscr{H}$. *d*) *If* $f_n \in \mathscr{H}$ *and* $f_n \uparrow f$, *then* $f \in \mathscr{H}$.

If all the functions (2.1) *and all the evanescent functions belong to* $\mathscr{H}$, *then all the non-negative solid functions belong to* $\mathscr{H}$.

Proof. — The family (2.1) is closed under multiplication. Therefore $\mathscr{H}$ contains all the $\mathscr{S}$-measurable functions. (It is easily seen, for example, from [11], Chapter 1, T 20 or from [4], Appendix, Lemma 0.2).

First, we prove that $\varphi 1_{t=r}$ belongs to $\mathscr{H}$ for every $\varphi \in \overline{\mathscr{F}}_{\neq r}$. According to 1.2.D, the σ-algebra $\mathscr{F}_{\neq r}$ is generated by functions $\varphi = \varphi_1\varphi_2$ where $r_1 < r < r_2$ and $\varphi_1 \in \mathscr{F}_{<r_1}$, $\varphi_2 \in \mathscr{F}_{<r_2}$ are bounded. But

$$\varphi_1\varphi_2 1_{t=r} = \lim_{\substack{r_1 \uparrow r \\ r_2 \downarrow r}} \varphi_1 1_{r_1<t<r_2} \varphi_2$$

and the right side is, evidently, $\mathscr{S}$-measurable.

Now let ζ be an arbitrary non-negative solid function. There exists an $\overline{\mathscr{F}}_{T\times\Omega}$-measurable function ξ which is indistinguishable from ζ. Consider an $\mathscr{S}$-measurable function η and a countable set Λ described in Lemma 2.1. We have

$$\xi_t = \eta_t 1_T + \sum_{r\in\Lambda} \xi_r 1_{t=r}.$$

The first term is $\mathscr{S}$-measurable and hence belongs to $\mathscr{H}$. The second term belongs to $\mathscr{H}$ because $\xi_r \in \overline{\mathscr{F}}_{\neq r}$. Thus $\xi \in \mathscr{H}$ and $\zeta \in \mathscr{H}$.

2.4. Let us prove now that, for any solid function $\xi \geqslant 0$, the function $\xi'_t = \mathbf{P}(\xi_t | \mathscr{F}_{\leqslant r})$ is solid too.

By virtue of Theorem 2.1, it is sufficient to verify the assertion only for the functions (2.1). In the case $r \geqslant u$, $\mathscr{F}_{>u}$ is generated by functions $\psi = \psi_1\psi_2$ where $\psi_1 \in \mathscr{F}(u, r]$, $\psi_2 \in \mathscr{F}_{>r}$, are bounded. The corresponding

$$\xi'_t = \varphi 1_{s<t<u} \psi_1 \mathbf{P}_{r, x_r} \psi_2$$

belong to the family (2.1) and hence are solid. In the case $s \leqslant r < u$, we have $\xi'_t = (\varphi 1_{s<t<u}) \mathbf{P}_{r, x_r}\psi$. The function in parenthesis belongs to the family (2.1). The second factor is solid by 2.1.A. The case $r < s$ can be treated in a similar way.

3. Central projection.

3.1. We introduce two dual classes of functions on $T \times \Omega$: *right functions* are well-measurable functions associated with the right filtration $\mathscr{M}_t = \overline{\mathscr{F}}_{\leqslant t}$; *left functions* are well-mesurable functions associated with the left filtration $\mathscr{M}^t = \overline{\mathscr{F}}_{\geqslant t}$.

Let $B \in \mathscr{B}_T$. We say that η and η' are *indistinguishable* on B and write $\eta = \eta'$ on B if $\mathbf{P}(\eta_t = \eta'_t$ for all $t) = 1$. The following proposition is an immediate consequence of [3] (Chapter 4, T14):

3.1.A. *Let* η *and* η' *be right functions. Then* $\eta \equiv \eta'$ *on* B *if and only if, for any Markov time* τ (9),

$$\mathbf{P}\eta_\tau 1_B(\tau) = \mathbf{P}\eta'_\tau 1_B(\tau).$$

(9) By Markov times without any reference to filtration, we mean Markov times relative to $\mathscr{M}_t = \overline{\mathscr{F}}_{\leqslant t}$.

3.2. Let $\xi \in \overline{\mathscr{F}}_{T\times\Omega}$. We say that η is the *right projection* of ξ and write $\eta = \Pi^+\xi$ if η is a right function and

$$\mathbf{P}(\xi_\tau|\mathscr{M}_\tau) = \eta_\tau \quad (\text{a.s. } \tau < \infty) \tag{3.1}$$

for every Markov time τ [10].

For bounded ξ existence of the right projection is proved in [3] (Chapter 5, T14). The projection of unbounded ξ can be expressed by formula [11]

$$\Pi^+\xi = \lim \Pi^+(\xi \wedge n) \tag{3.2}$$

(It is clear from 3.1. A that, outside an evanescent set,

$$\Pi^+(\xi \wedge n) \geqslant \Pi^+(\xi \wedge m)$$

for $n > m$, hence the limit on the right side of (3.2) does exist.) The concept of right projection can be extended to functions taking positive and negative values by formula

$$\Pi^+\xi = \Pi^+\xi^+ - \Pi^+\xi^-, \tag{3.3}$$

where $\xi^+ = \xi \vee 0$, $\xi^- = (-\xi) \vee 0$ (formula (3.1) remains valid if ξ_τ is integrable).

The left projection can be defined analogously.

Note some properties of the right projection.

3.2.A. *If* $\xi = \xi'$ *on* B, *then* $\Pi^+\xi = \Pi^+\xi'$ *on* B.

This is an obvious consequence of 3.1.A.

It follows from 3.2.A that any two right projections of ξ are indistinguishable.

3.2.B. *The operator* Π^+ *preserves inequalities and linear operations. Let* $0 \leqslant \xi_n \uparrow \xi$ *and let* ζ *be a right function. Then, outside an evanescent set,* $\Pi^+\xi_n \uparrow \Lambda^+\xi$ *and*

$$\Pi^+(\zeta\xi) = \zeta\Pi^+\xi.$$

This follows easily from the definition of Π^+.

3.2.C. *If* ξ *—is independent of* t *and* $\xi' = \mathbf{P}(\xi|\mathscr{F}_{\leqslant r})$, *then* $\Pi^+\xi = \Pi^+\xi'$ *on* $(-\infty, r]$.

[10] Writting (a.s. $\tau < \infty$) means « for almost all ω satisfying the condition $\tau(\omega) < \infty$ ».

[11] We denote by $a \wedge b$ the smaller and by $a \vee b$ the greater of the two numbers a, b.

Proof. — By virtue of 1.2.D, it is enough to consider $\xi = \varphi\psi$ where $\varphi \in \mathscr{F}_{\leqslant r}$, $\psi \in \mathscr{F}_{>r}$. According to 1.3.A, $\xi' = \varphi \mathbf{P}_{r, x_r}\psi$. Let $\eta = \Pi^+\xi$, $\eta' = \Pi^+\xi'$. For any Markov time τ, we have

$$\mathbf{P}\,\eta_\tau 1_{\tau \leqslant r} = \mathbf{P}\varphi\psi 1_{\tau \leqslant r}.$$

Since $\varphi 1_{\tau \leqslant r} \in \mathscr{M}_r$, the right side is equal to

$$\mathrm{P}\varphi 1_{\tau \leqslant r}\mathbf{P}_{r, x_r}\psi = \mathbf{P}\xi' 1_{\tau \leqslant r} = \mathbf{P}\,\eta' 1_{\tau \leqslant r},$$

and 3.1.A implies 3.2.C.

3.3. Central functions are the functions which are simultaneously solid, right and left. By 1.1.δ and 2.1.B, the function $\mathrm{A}\{t\}$ corresponding to any normal additive functional A is central.

Lemma 3.1. — *For any measurable f on $\mathscr{E}$, the function $f(t, x_t)$ is central.*

Proof. — It is sufficient to prove this for the functions h_i defined by (1.2). The solidity of $h_i(t, x_t)$ follows from 2.1.A. By symmetry, it is enough to check that $h_i(t, x_t)$ are right. For h_1 it is clear from 1.3.D. Put $\xi_t = \lim_{r \downarrow t} h_2(r, x_r)$. Obviously, the function $\xi_{t-\varepsilon}$ is right for every $\varepsilon > 0$. It remains to note that

$$h_2(t, x_t) \equiv \lim_{\varepsilon \downarrow 0} \xi_{t-\varepsilon}.$$

Theorem 3.1. — *If ξ is solid, then*

$$\Pi^+\Pi^-\xi \equiv \Pi^-\Pi^+\xi \equiv f(t, x_t) \tag{3.4}$$

where f is a measurable function on $\mathscr{E}$. In particular, if

$$\xi_t(\omega) = \varphi(\omega) 1_{s<t<u}\psi(\omega)(s < u;\ \varphi \in \mathscr{F}_{<s},\ \psi \in \mathscr{F}_{>u} \text{ are bounded}), \tag{3.5}$$

then f can be expressed by formula

$$f(t, x) = \mathbf{P}^{t,x}\varphi 1_{s<t<u}\mathbf{P}_{t,x}\psi \tag{3.6}$$

Proof. — By 3.2.B and Theorem 2.1, we have only to prove the assertion concerning the functions (3.5). According to the strong Markov property (see [5], formula (3.2)), for any Markov time τ,

$$\mathbf{P}(\xi_\tau|\mathscr{M}_\tau) = \varphi 1_{s<\tau<u}\mathbf{P}(\psi|\mathscr{M}_\tau) = \varphi 1_{s<\tau<u}\mathbf{P}_{\tau, x_\tau}\psi = \eta_\tau^+ \text{ (a.s.)}$$

where

$$\eta_t^+ = \varphi 1_{s<t<u}\mathbf{P}_{t, x_t}\psi.$$

Clearly, η_t^+ is a right function. Therefore $\Pi^+\xi = \eta^+$. Similarly $\Pi^-\xi = \eta^-$, where

$$\eta_t^- = (\mathbf{P}^{t, x_t}\varphi)1_{s<t<u}\psi.$$

Using the strong Markov property once more, we note that for any Markov time τ

$$\mathbf{P}(\eta_\tau^-|\mathscr{M}_\tau) = (\mathbf{P}^{\tau, x_\tau}\varphi)1_{s<\tau<u}\mathbf{P}_{\tau, x_\tau}\psi.$$

Therefore $(\mathbf{P}^{t, x_t}\varphi)1_{s<t<u}\mathbf{P}_{t, x_t}\psi$ is the right projection of η^-. It is also the left projection of η^+.

3.4. Theorem 3.1 implies that, for any solid ξ,

$$\Pi^+\Pi^-\xi = \Pi^-\Pi^+\xi$$

is a central function. Denote it by $\Pi\xi$ and call the central projection of ξ. The corresponding function f will be called the *projection* of ξ on $\mathscr{E}$ and denoted by $\pi\xi$. (It is unique up to equivalence.)

If ξ is central, then $\xi \equiv \pi\xi$. Hence

$$\xi \equiv f(t, x_t) \tag{3.7}$$

where $f = \pi\xi$.

3.5. Lemma 3.2. — *Let ξ be solid and independant of t and let $\xi' = \mathbf{P}(\xi|\mathscr{F}(\mathrm{B}))$ where $\mathrm{B} = [s, u]$ is a closed interval. Then $\Pi\xi \equiv \Pi\xi'$ on* B. *Outside an inaccessible set, $\pi\xi$ and $\pi\xi'$ coincide on $\mathscr{E}(\mathrm{B})$.*

Proof. — Using 1.2.D and 1.3.A, it is easy to prove that

$$\xi' = \mathbf{P}(\hat{\xi}|\mathscr{F}_{\leqslant u}) \qquad \text{where} \qquad \hat{\xi} = \mathbf{P}(\xi|\mathscr{F}_{\geqslant s}).$$

By 3.2.C $\Pi^+\xi' \equiv \Pi^+\hat{\xi}$ (on $(-\infty, u]$. By (3.4) and 3.2.A*, it follows from here that $\Pi\xi' \equiv \Pi^-\Pi^+\xi \equiv \Pi^-\Pi^+\hat{\xi} \equiv \Pi\hat{\xi}$

on $(-\infty, u]$. On the other hand, by 3.2.C* $\Pi^-\hat{\xi} \equiv \Pi^-\xi$ on $[s, +\infty)$, and by 3.2.A $\Pi\hat{\xi} \equiv \Pi^+\Pi^-\hat{\xi} \equiv \Pi^+\Pi^-\xi \equiv \Pi\xi$ on $[s, +\infty)$.

4. σ-finiteness of spectral measures.

4.1. The aim of this section is to prove the following.

THEOREM 4.1. — *The spectral measure of any normal additive functional is σ-finite.*

To that end, we use the decomposition $A = A_0 + A_1$ of the measure $A(B)$ into a continuous part $A_0(B)$ and a discret part

$$A_1(B) = \sum_{t \in B} A\{t\}.$$

According to nº 3.3 and formula (3.7),

$$A\{t\} \equiv f(t, x_t) \tag{4.1}$$

where f is the projection of $A\{t\}$ on $\mathscr{E}$. By 1.1.α, the set $\{t: A\{t\} \neq 0\}$ is a.s. at most countable. Therefore the set $\{f \neq 0\}$ is scanty.

Put

$$\gamma(C) = \mathbf{P} \sum_t 1_C(x_t)$$

It is clear that γ is a measure and the spectral measure μ_1 of the functional A_1 can be expressed by formula

$$\mu_1(C) = \mathbf{P} \sum_t (1_C f)(x_t) = \int_C f \, d\mu.$$

Thus the σ-finiteness of μ_1 follows immediately from

THEOREM 4.2. — *Every scanty set Γ can be decomposed into a countable number of subsets Γ_i such that $\nu(\Gamma_i) < \infty$.*

This theorem will be proved in the Appendix.

Theorem 4.1 is a trivial consequence of Theorem 4.2 and the following.

THEOREM 4.3. — *The spectral measure of a continuous additive functional is σ-finite.*

Remaining part of section 4 is devoted to the demonstration of Theorem 4.3.

4.2. LEMMA 4.1. — *If* A *is a continuous additive functional and* $\eta \in \mathscr{F}_{>u}$, *then*

$$\text{(4.2)} \qquad \mathbf{P}\mathrm{A}(s, u)\eta \geqslant \mathbf{P} \int_s^u \mathbf{P}_{t, x_t}\eta \mathrm{A}\,(dt).$$

Proof. — Let $\zeta(t) = \mathbf{P}_{t, x_t}\eta$. Consider a finite subset $\Lambda = \{t_1 < \ldots < t_m\}$ and put $t_0 = s$, $t_{m+1} = u$ and

$$\delta_\Lambda(t) = t_k \quad \text{for} \quad t \in [t_{k-1}, \ t_k).$$

We have

$$\text{(4.3)} \quad \mathbf{P}\mathrm{A}(s, u)\eta = \Sigma\mathbf{P}\mathrm{A}(t_{k-1}, t_k)\eta = \Sigma\mathbf{P}\mathrm{A}(t_{k-1}, t_k)\zeta(t_k) = \mathbf{P} \int \zeta[\delta_\Lambda(t)]\mathrm{A}\,(dt).$$

Let now Λ_n be an increasing sequence of finite sets with the union dence in (s, u). Then $\delta_{\Lambda_n}(t) \downarrow t$ and, by 1.3.D,

$$\zeta[\delta_{\Lambda_n}(t)] \to \zeta(t) \text{ a.s.}$$

By Fatou's lemma (4.3) implies (4.2).

4.3. Now we prove Theorem 4.3. Since the measure $\mathrm{A}\,(du)$ is continuous, 1.1.α implies that $\mathrm{A}(t, u) < \infty$ a.s. for any $t < u \in \mathrm{T}$. Fix an integer m and consider an expression

$$\mathrm{J} = \int_m^\infty \exp\,[-\,\mathrm{A}(m, \ u)]\mathrm{A}\,(du).$$

Since the function $\mathrm{F}(u) = \mathrm{A}(m, u)$ is continuous, we have

$$\text{(4.4)} \quad \mathrm{J} = \int_m^\infty \exp\,[-\,\mathrm{F}(u)]\, d\mathrm{F}(u) = 1 - \exp\,[-\,\mathrm{A}(m, \ \infty)] \leqslant 1.$$

Put $\Delta_n^k = [(k-1)/2^n, \ k/2^n)$; $\delta_n(u) = k/2^n$ for $u \in \Delta_n^k$. We have

$$\mathrm{J} \geqslant \int_m^\infty \exp\,[-\,\mathrm{F}(\delta_n(u))]\mathrm{A}\,(du) \geqslant \Sigma \exp\,[-\,\mathrm{F}(k/2^n)]\mathrm{A}(\Delta_n^k)$$

with the sum extended over all $k \geqslant m.2^n + 1$. From here, by (4.2) and (4.4),

$$1 \geqslant \mathbf{P} \int_m^\infty \mathbf{P}_{u, x_u} \exp\,[-\,\mathrm{F}(\delta_n(u))]\mathrm{A}\,(du).$$

Since $\delta_n(u) \to u$ as $n \to \infty$, we have, by Fatou's lemma

$$1 \geqslant \mathbf{P} \int f_m(u, x_u) \mathrm{A}(du)$$

with

$$f_m(u, x) = \begin{cases} \mathbf{P}_{u,x} \exp[-\mathrm{A}(m, u)] & \text{for } u \geqslant m, \\ 0 & \text{for } u < m. \end{cases}$$

The sum $f(u, x) = \Sigma 2^m f_m(u, x)$ extended over all negative integers m is strictly positive and $\mathbf{P} \int f(u, x_u) \mathrm{A}(du) \leqslant 1$. Let μ be the spectral measure of A. Then $\int f \, d\mu \leqslant 1$. Hence μ is σ-finite.

5. The fundamental identity.

5.1. THEOREM 5.1. — *Let* μ *be the spectral measure of a normal additive functional* A. *For any solid function* ξ

$$(5.1) \qquad \mathbf{P} \int_{\mathrm{T}} \xi_t \mathrm{A}(dt) = \mathbf{P} \int_{\mathrm{T}} \Pi \xi \, \mathrm{A}(dt) = \int_{\mathrm{E}} \pi \xi \, d\mu.$$

In more precise terms, if one of three integrals has meaning, then two other have too, and equalities (5.1) *are fulfilled.*

Proof. — By Theorem 4.1, there exists a strictly positive f such that $\int f \, d\mu < \infty$. Formula $\hat{\mathrm{A}}(dt) = f(t, x_t) \mathrm{A}(dt)$ defines a normal additive functional with a finite spectral measure. The identity (5.1) is valid for A if it is valid for $\hat{\mathrm{A}}$. Therefore we have the right to assume that

$$\mathbf{P}\mathrm{A}(\mathrm{T}) = \mu(\mathscr{E}) < \infty.$$

The second equality (5.1) is obvious. By Theorem 2.1, the first one has to be proved only for the functions (2.1). Let $\Lambda = \{t_1 < \cdots < t_m\}$ be a finite subset of interval (s, u) such that $\mathbf{P}\mathrm{A}(\Lambda) = 0$. Put $t_0 = s$, $t_{m+1} = u$;

$$\gamma(t) = t_{k-1}, \ \delta(t) = t_k \quad \text{for} \quad t \in [t_{k-1}, t_k);$$
$$\varphi_t = \mathbf{P}^{t, x_t} \varphi, \quad \psi_t = \mathbf{P}_{t, x_t} \psi.$$

We have

$$\mathbf{P} \int \xi_t \mathrm{A}(dt) = \mathbf{P} \varphi \mathrm{A}(s, u) \psi = \Sigma \mathbf{P} \varphi \mathrm{A}(t_{k-1}, t_k) \psi$$
$$= \Sigma \mathbf{P} \varphi_{t_{k-1}} \mathrm{A}(t_{k-1}, t_k) \psi_{t_k} = \mathbf{P} \int_s^u \varphi_{\gamma(t)} \psi_{\delta(t)} \mathrm{A}(dt).$$

Consider a sequence $\Lambda_1 \subset \Lambda_2 \subset \cdots \subset \Lambda_n \subset \cdots$ with the union dence in (s, u). Evidently, $\gamma_n(t) \uparrow t$, $\delta_n(t) \downarrow t$. Using 1.3.D, we obtain by the passage to the limit

$$\mathbf{P} \int \xi_t \mathrm{A}\,(dt) = \mathbf{P} \int_s^u \varphi_t \psi_t \mathrm{A}\,(dt).$$

It remains to denote that, according to Theorem 3.1,

$$\Pi \xi = \varphi_t 1_{s<t<u} \psi_t.$$

5.2. Theorem 5.2. — *A normal additive functional is uniquely determined by its spectral measure.*

Proof. — Theorem 4.1 allows easily to reduce the general case to the case of the finite measure μ. Let φ be an independent of t solid function and let $\mathrm{B} \in \mathscr{B}_\mathrm{T}$. Applying (5.1) to $\xi_t = \varphi l_\beta$, we obtain

$$\mathrm{P}\varphi \mathrm{A}(\mathrm{B}) = \int_{\mathcal{E}(\mathrm{B})} \pi\varphi \, d\mu. \tag{5.2}$$

Let A and A′ be two normal additive functionals with the spectral measure μ. By 2.1.B, the function $\varphi = \mathrm{A}(\mathrm{B}) - \mathrm{A}'(\mathrm{B})$ is solid, and (5.2) implies that $\mathrm{P}\varphi^2 = \mathbf{P}\varphi\mathrm{A}(\mathrm{B}) - \mathbf{P}\varphi\mathrm{A}'(\mathrm{B}) = 0$. Thus $\mathrm{A}(\mathrm{B}) = \mathrm{A}'(\mathrm{B})$ a.s. But two finite measures coincide if they have equal values on all the intervals with the rational ends. Therefore A and A′ are indistinguishable.

5.3. Theorem 5.3. — *A normal additive functional is continuous if and only if its spectral measure vanishes on all scanty sets.*

Proof. — The necessity is obvious. We have seen in n° 4.1 that $\mathrm{A}\{t\} \equiv f(t, x_t)$ with a scanty set $\{f \neq 0\}$. By (5.1)

$$\mathbf{P} \int \mathrm{A}\{t\} \mathrm{A}\,(dt) = \int_{\mathcal{E}} f d\mu.$$

The right side is equal to 0 if μ charges no scanty set. The left side is equal to $\mathbf{P} \sum_t \mathrm{A}\{t\}^2$. Hence $\mathrm{A}\{t\} = 0$ for all t a.s.

6. Construction of an additive functional from a measure.

6.1. We show that any σ-finite measure μ charging no inaccessible set is a spectral measure of a normal additive functional. Clearly, it is sufficient to consider only the case when μ is finite.

For any $B \in \mathscr{B}_T$, denote by $\mathscr{G}(B)$ the set of all the solid elements of $\mathscr{F}(B)$. If f and f' are equivalent, then

$$\int f\, d\mu = \int f'\, d\mu.$$

Therefore for any$\mathscr{G}$ $\xi \in (B)$ the value

$$\Phi_B(\xi) = \int_{\mathcal{E}B} \pi\xi\, d\mu \tag{6.1}$$

is uniquely determined. By 3.2.B and 3.2.B*, Φ_B is a finite measure on $\mathscr{G}(B)$, and, by 3.2.A and 3.2.A* it is absolutely continuous with respect to $\mathbf{P}$. By the Radon-Nikodym theorem, there exists a $\mathscr{G}(B)$-measurable function $a(B)$ such that $\Phi_B\,(d\omega) = a(B)\mathbf{P}\,(d\omega)$ and hence

$$\mathbf{P}a(B)\xi = \int_{\mathcal{E}(B)} \pi\xi\, d\mu \quad \text{for} \quad \xi \in \mathscr{G}(B). \tag{6.2}$$

Let us prove that

$$\mathbf{P}a[s,\ t]\xi = \int_{\mathcal{E}[s,\, t]} \pi\xi\, d\mu \ \text{ for all } \ \xi \in \mathscr{G}(T). \tag{6.3}$$

In fact, let $B = [s,\ t]$. If $\xi \in \mathscr{G}(T)$, then, according to n° 2.4, the function $\xi' = \mathbf{P}\{\xi | \mathscr{F}(B)\}$ belongs to $\mathscr{G}(B)$ and (6.2) is applicable to ξ'. But $\mathbf{P}a(B)\xi = \mathbf{P}a(B)\xi'$ and, by Lemma 3.2, $\pi\xi = \pi\xi'$ on $\mathscr{E}(B)$ outside an inaccessible set.

6.2. Let R be the set of all rational numbers. It follows from (6.3) that, for almost all ω,

$$a[r_1,\ r_2] \leqslant a[r_1',\ r_2'] \quad \text{for all} \quad r_1' < r_1 < r_2 < r_2' \in R$$

and hence there exists limits

$$F_t(s) = \lim_{r \downarrow s,\, r \in R} a[r,\ t]. \tag{6.4}$$

for all $t \in R$. Clearly $F_t(s)$ is a non-increasing right continuous function on the half-line $(-\infty,\ t)$. A measure $A_t\,(du)$ on this line can be constructed such that $F_t(s) = A_t(s,\ t]$. It follows from (6.3) that

$$\mathbf{P}\xi A_t(s,\ t] = \int_{\mathcal{E}(s,\, t]} \pi\xi\, d\mu. \tag{6.5}$$

It is clear from here that, for any $s < t < u$,

$$A_t(s, t] + A_u(t, u] = A_u(s, u] \text{ a.s.}$$

which implies that $A_u(s, t] = A_t(s, t]$ a.s. Therefore for all ω, except for a set of measure 0, $A_u(s, t] = A_t(s, t]$ for all $s < t < u \in R$, hence $A_t = A_u$ on $(-\infty, t)$. Evidently there exists a measure A on the real line such that $A = A_t$ on $(-\infty, t)$ for any $t \in R$. It follows from (6.5) that

$$\text{(6.6)} \qquad P\xi A(s, t] = \int_{\mathcal{E}(s, t]} \pi\xi \, d\mu \quad \text{for all } s < t, \ \xi \in \mathcal{G}(T)$$

6.3. Let us show that A is a normal additive functional. Condition 1.1.α is obvious; conditions 1.1.β and 1.1.γ are valid since $A(s, t) \in \mathcal{G}(s, t) \subset \mathcal{F}(s, t)$ and $A\{t\} \in \mathcal{G}(T)$. Condition 1.1.$\delta$ is satisfied because $A(s, t) < \infty$ a.s.

It remains only to check that the spectral measure of A is equal to μ. We prove that

$$\text{(6.7)} \qquad \mathbf{P} \int \xi_u A\,(du) = \int_{\mathcal{E}} \pi\xi \, d\mu$$

for any solid $\xi \geqslant 0$. By Theorem 2.1, it is sufficient to check (6.7) for functions (2.1). For them, the left side of (6.7) is equal to $\mathbf{P}\varphi A(s, t)\psi$. On the other hand, $\pi\xi = 1_{s<t<u}\pi(\varphi\psi)$ hence the right side of (6.7) is equal to $\int_{\mathcal{E}(s, t)} \pi(\varphi\psi)\, d\mu$, and (6.7) follows from (6.6). Applying (6.7) to $\xi_u = 1_\Gamma(x_u)$, we obtain (1.3).

Appendix. Structure of scanty sets.

1. Our aim is to represent every scanty set as a countable union of subsets with the property: the expected number of times when x_t belongs to the subset is finite.

The sets which admit such a representation will be called *admissible*. After necessary preliminaries, the formulated statement will be proved in n° 5.

2. Let $\Gamma \in \mathcal{B}_{\mathcal{E}}$. Put $\sigma_\Gamma(t) = \inf\{u : u > t, \ x_u \in \Gamma\}$. A well-known theorem on debuts ([11], Chapter 4, T18 or [3], Chapter 3, T23) implies that $\sigma_\Gamma(\tau)$ is a Markov time for any Markov time τ. The function $\sigma_\Gamma(t)$ is right continuous in t and measurable with respect to $\mathscr{F}_{>t}$ for every fixed t. By strong Markov property ([8], n° 3.5),

$$(1) \qquad \mathbf{P}\{\exp[-\sigma_\Gamma(\tau)]|\mathscr{M}_\tau\} = q_\Gamma(\tau, x_\tau) \quad \text{a.s.}$$

where $q_\Gamma(t, x) = \mathbf{P}_{t,x} \exp[-\sigma_\Gamma(t)]$. It is easy to prove that q_Γ is a measurable function on $\mathscr{E}$ and

$$\mathbf{P}_{s,x} q_\Gamma(t, x_t) = \mathbf{P}_{s,x} \exp[-\sigma_\Gamma(t)]$$

for $s < t$. It follows from here that q_Γ is an excessive function and that $q_\Gamma(t, x_t)$ is right continuous a.s. (see [5], Theorem 5.1).

A point $x \in E_t$ is called *regular* for Γ if

$$P_{t,x}\{\sigma_\Gamma(t) = t\} = 1.$$

This is equivalent to the condition $q_\Gamma(t, x) = \exp(-t)$. Therefore the set of all regular points is measurable. We shall denote by $\hat{\Gamma}$ the set of all the points $x \in \Gamma$ which are not regular for Γ.

LEMMA 1 (12). — *The set* $\hat{\Gamma}$ *is admissible for any* $\Gamma \in \mathscr{B}_{\mathcal{E}}$.

Proof. — Since a countable union of admissible sets is admissible too, it is sufficient to prove that the set

$$C = \{(t, x) : x \in \Gamma,\ s \leqslant t < s+1,\ q_\Gamma(t, x) \leqslant \theta \exp(-t)\}$$

is admissible for any $s \in T$ and $0 < \theta < 1$. Form a sequence of Markov times $\tau_0 = s$, $\tau_{n+1} = \sigma_C(\tau_n)$ for $n \geqslant 0$. According to (1),

$$(2) \qquad \mathbf{P}\{\exp(-\tau_{n+1})|\mathscr{M}_{\tau_n}\} = q_C(\tau_n, x_{\tau_n}) \text{ a.s.}$$

It is clear that $q_C \leqslant q_\Gamma$. Therefore $q_C(t, x_t) \leqslant \theta \exp(-t)$ for $x_t \in C$. Since $q_C(t, x_t)$ is right continuous a.s., then

$$q_C(\tau_n, x_{\tau_n}) \leqslant \theta \exp(-\tau_n) \text{ a.s.}$$

Hence (2) implies that $\mathbf{P}\exp(-\tau_{n-1}) \leqslant c\mathbf{P}\exp(-\tau_n)$ and $\mathbf{P}\exp(-\tau_n) \leqslant \theta^n e^{-s}$. Denote by N the number of times

(12) Cf. [10], Chapter 15, T 30.

when x_t belongs to C. Then

$$\mathbf{P}\{N > n\} = \mathbf{P}\{\tau_n < s + 1\} = \mathbf{P}\ \{\exp(-\tau_n) > \exp(-s-1)\} \leqslant \exp(s+1)\mathbf{P}\exp(-\tau_n) \leqslant \exp s\,\theta^n.$$

Thus $\mathbf{P}N < \infty$.

3. Lemma 2. — *If* Γ *is a scanty set, then there exists at most countable family of Markov times* τ_n *such that*

$$1_\Gamma(x_t) \equiv \sum_n 1_{\tau_n = t}.$$

By virtue of Lemma 3.1, this proposition follows from [3] (Chapter 6, T33).

Lemma 3 [13]. — *Let* Γ *be a scanty set and let the closure of* $\Delta(\omega) = \{t : x_t(\omega) \in \Gamma\}$ *be at most countable with a positive probability. If* $\hat{\Gamma}$ *is inaccessible, then* Γ *is inaccessible too.*

Proof. — Let τ_n be Markov times described in Lemma 2. If $\hat{\Gamma}$ is inaccessible, then $q_\Gamma(\tau_n, x_{\tau_n}) = \exp(-\tau_n)$ a.s. From here and (1), it follows that

$$\mathbf{P}\{\exp(-\sigma(\tau_n)) | \mathscr{M}_{\tau_n}\} = \exp(-\tau_n) \text{ a.s.}$$

Hence $\sigma(\tau_n) = \tau_n$ a.s. and all the elements of $\Delta(\omega)$ are right-hand limit points of $\Delta(\omega)$. Thus the closure of $\Delta(\omega)$ is perfect. But a perfect set can not be countable.

4. Lemma 4. — *The space* $\mathscr{E}$ *can be imbedded into a compact metric space* $\mathscr{X}$ *in a such a way that:*

3.A. $\mathscr{B}_\mathscr{E}$ *coincides with the class of all Borel sets of* $\mathscr{X}$ *which are contained in* $\mathscr{E}$.

3.B. *There exists a set* Ω_0 *such that* $\mathbf{P}(\Omega_0) = 1$ *and, for all* $\omega \in \Omega_0$, $t \in \mathrm{T}$ *there exist right-hand limit* x_{t+} *and left-hand limit* x_{t-} *and moreover either* $x_{t+} \neq x_{t-}$ *or* $x_{t+} = x_{t-} = x_t$.

3.C. *The functions* x_{t+} *and* x_{t-} *are central.*

3.D. *For any* $\varepsilon > 0$ *and finite interval* I *the expected*

[13] Cf. [10], Chapter 15, T 68.

number of times $t \in \mathrm{I}$, *for which* $\rho(x_{t+}, x_{t-}) > \varepsilon$, *is finite* (*here* ρ *is a metric on* $\mathscr{X}$).

Proof. — By 1.1.A, it is possible to choose, for each u, a countable family $\mathscr{C}_u$ of measurable subsets of E_u with the property: if ν_1, ν_2 are finite measures on E_u and

$$\nu_1(\mathrm{C}) = \nu_2(\mathrm{C})$$

for all $\mathrm{C} \in \mathscr{C}_u$, then $\nu_1 = \nu_2$. Consider for every rational u and every $\mathrm{C} \in \mathscr{C}_u$ a pair of functions h_1, h_2 defined by (1.2) with $\xi = \eta = 1_{\mathrm{C}}(x_u)$. Denote the set of all these functions by $\mathscr{H}$. Associate with every $x \in \mathscr{E}$ the collection $h(x)$ of values of all functions belonging to $\mathscr{H}$ at the point x. We can interpret h as measurable mapping of $\mathscr{E}$ into the Cartesian product $\mathscr{X}$ of a countable number of closed unit intervals. It is easily seen from 1.3.B and 1.3.C that

$$h(x) \neq h(y) \quad \text{for} \quad x \neq y.$$

Therefore h determines an isomorphic imbedding of $\mathscr{E}$ into $\mathscr{X}$ (cf. [5], section 2).

Assertion 3.B follows from 1.3.D and 3.C can be proved exactly as Lemma 3.1. To demonstrate 3.D, we note that if h_i are defined by (1.2) and if $\xi_t^i = h_i(t, x_t)$, then $(\xi_t^1, \mathscr{M}_t, \mathbf{P})$ is a right continuous martingale on $(-\infty, u)$ and

$$(\xi_{-t}^2, \mathscr{M}^{-t}, \mathbf{P})$$

is a right continuous martingale on $(-\infty, -u)$. By a well-known theorem (see, for example, [11], Chapter 6, T1) for any bounded right continuous martingale, the expected number of jumps which are greater than a fixed constant is finite.

Corollary. — *There exists an admissible set* Q *such that*

$$\{t : x_{t+} \neq x_{t-}\} = \{t : x_t \in \mathrm{Q}\} \text{ a.s.} \tag{3}$$

In fact, according to 3.C and (3.7), for any integer m, there exists a function $f_m \in \mathscr{B}_{\mathscr{E}}$ such that

$$1_{|t|<m,\, \rho(x_{t+}, x_{t-})>1/m} = f_m(t, x_t).$$

Let $\mathrm{Q}_m = \{f_m \neq 0\}$. Then

$$\{t : |t| < m, \rho(x_{t+}, x_{t-}) > 1/m\} = \{t : x_t \in \mathrm{Q}_m\} \text{ a.s.}$$

By 3.D, the sets Q_m are admissible. So is their union Q. Clearly, Q satisfies (3).

5. We are able now to prove the statement formulated at the beginning of the Appendix.

Let Γ be a scanty set and let τ_n be the Markov times defined in Lemma 2. Put

$$\nu(C) = \mathbf{P} \sum_n 1/2^n 1_C(x_{\tau_n}).$$

It is easily seen that $\nu(C)$ attains its maximum over all the admissible sets C on a set $\Gamma_0 \subseteq \Gamma$. Put $\Gamma_1 = \Gamma \setminus \Gamma_0$. It is obvious that, for every admissible C, $\nu(C \cap \Gamma_1) = 0$, hence $C \cap \Gamma_1$ is inaccessible. Applying this remark to the set Q of n° 4, we see that $(t: x_t \in \Gamma_1, x_{t+} \neq x_{t-})$ is a.s. empty. From here $(t: x_t \in \Gamma_1) = (t: x_t = x_{t+} = x_{t-} \in \Gamma_1)$ a.s. To achieve our end, it is sufficient to demonstrate that the last set is empty a.s. This will be proved if we show that, for any closed $K \subseteq \Gamma_1$,

$$\Delta(\omega) = (t: x_t = x_{t+} = x_{t-} \in K) = \emptyset \quad \text{a.s.}$$

First, we show that the closure of $\Delta(\omega)$ is a.s. at most countable. If $t \bar{\in} \Delta(\omega)$ is a limit point of $\Delta(\omega)$, then there exists a sequence $t_n \in \Delta(\omega)$ such that either $t_n \downarrow t$ or $t_n \uparrow t$. Hence $x_{t+} \in K$ or $x_{t-} \in K$ and, by 3.B, $x_{t+} \neq x_{t-}$ if $\omega \in \Omega_0$. Therefore, except on an ω-set of measure 0, all limit points of $\Delta(\omega)$ belong either to $\Delta(\omega)$ or to the countable set

$$(t: x_{t+} \neq x_{t-}).$$

But the set $\Delta(\omega)$ is at most countable a.s. because $K \subseteq \Gamma$ and Γ is scanty.

By (3.7), Lemma 3.1 and 3.B, there exists a set $C \in \mathscr{B}_{\mathcal{E}}$ such that

$$\Delta(\omega) = (t: x_t(\omega) \in C) \text{ a.s.} \tag{4}$$

This relation remains valid for $C \cap K$. Thus we can assume that $C \subseteq K$. According to Lemma 1, the set $\hat{C}$ is admissible. Since $\hat{C} \subseteq C \subseteq \Gamma_1$, the set $\hat{C}$ is inaccessible. By Lemma 3, C is inaccessible too, and (4) implies that $\Delta(\omega)$ is empty a.s.

BIBLIOGRAPHY

[1] R. M. BLUMENTHAL and R. K. GETOOR, Markov Processes and Potential Theory, Academic Press, New York-London, 1968.

[2] C. DELLACHERIE, Ensembles aléatoires I, Séminaire de Probabilités III, Université de Strasbourg, *Lecture Notes in Math.*, 88, Springer-Verlag, Berlin-Heidelberg-New York, 1969, 97-114.

[3] C. DELLACHERIE, Capacités et processus stochastiques, Springer-Verlag, Berlin-Heidelberg-New York, 1972.

[4] E. B. DYNKIN, Markovskie processy, Fizmatgiz, Moskva, 1963, Markov Processes Vol. I-II, Springer-Verlag, Berlin-Göttingen-Heidelberg, 1965.

[5] E. B. DYNKIN, Regular Markov processes, *Uspehi Matem. Nauk*, 28, 2 (170), (1973) (English translation in *Russian Mathematical Surveys*, London).

[6] E. B. DYNKIN, Additive functionals of Markov processes and their spectral measures, *Dokl. Akad. Nauk SSSR*, 214 (1974), 1241-1244 (English translation in *Soviet Math. Dokl.*).

[7] E. B. DYNKIN, Markov representations of stochastic systems, *Dokl. Akad. Nauk SSSR*, 218 (1974), 1009-1012.

[8] E. B. DYNKIN, Markov representations of stochastic systems, *Uspehi Matem. Nauk*, 30, 1 (1975), 61-99.

[9] P. A. MEYER, Fonctionnelles multiplicatives et additives de Markov, *Ann. Inst. Fourier*, 12 (1962), 125-230.

[10] P. A. MEYER, Processus de Markov, *Lecture Notes in Math.*, 26, Springer-Verlag, Berlin-Heidelberg-New York, 1967.

[11] P. A. MEYER, Probability and Potentials, Blaisdell, Waltham Mass., 1966.

[12] M. METIVIER, Mesures vectorielles et intégrale stochastique II, Séminaire de probabilités, année 1972, Preprint.

[13] D. REVUZ, Mesures associées aux fonctionnelles additives de Markov I, *Trans. Amer. Math. Soc.*, 148 (1970), 501-531.

[14] M. ŠUR, Continuous additive functionals of Markov processes and excessive functions, *Dokl. Akad. Nauk SSSR*, 137 (1961), 800-803 [*Soviet Math.* 2 (1961), 365-368].

15] M. ŠUR, On approximation of additive functionals of Markov processes Uspehi Mat. Nauk, 29, 6 (1974), 183-184.

[16] A. D. WENTZELL, Nonnegative additive functionals of Markov processes, *Dokl. Akad. Nauk SSSR*, 137 (1961), 17-20 [*Soviet Math.*, 2 (1961), 218-221].

Manuscrit reçu le 24 Janvier 1975.

Eugen DYNKIN,
Academy of Sciences of USSR
Central Institute of Mathematics
and Economics
ul. Vavilova 44-2
Moscow V-333 (USSR).