ANNALES DE L'INSTITUT FOURIER

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Annales de l'institut Fourier, tome 25, n° 3-4 (1975), p. 207-233
<http://www.numdam.org/item?id=AIF_1975__25_3-4_207_0>

Ann. Inst. Fourier, Grenoble
25, 3 et 4 (1975), 207-233.

THE RAY SPACE OF A RIGHT PROCESS

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Dédié à Monsieur M. Brelot à l'occasion de son 70e anniversaire.

1. Introduction.

In [6] Meyer and Walsh created a beautiful theory based on earlier work of Knight [4], Ray [9], and Shih [10] to show that if X is a process with state space E satisfying Meyer's « hypothèses droites », then by changing the topology on E and enlarging E one may regard X as a Ray process, that is, roughly speaking, a Feller process except that branch points are allowed. In [3] the hypotheses of Meyer and Walsh were relaxed somewhat in that E was assumed only to be a U-space rather than Lusinien (a topological space is a U-space provided it is homeomorphic to a universally measurable subspace of a compact metric space), and the requirement that the excessive functions be nearly Borel was dropped. From a practical point of view these results were complete. However, there was one aesthetic gap : the new topology on E (which we call the Ray topology) and the enlargement of E seemed to depend upon a choice of an arbitrary uniformity on E and not just on the topology of E. The purpose of this paper is to close this gap.

In Section 2 we develop certain properties of U-spaces that will be used in the later sections. Allowing E to be a

(*) This research was supported in part by the National Science Foundation NSF Grant 41707 X.

U-space is not only a generalization but also leads to a pleasing symmetry because E in the Ray topology is again a U-space. In Section 3 we summarize the properties of right processes and the Ray-Knight compactification that are relevant to our discussion. In Section 4 we first show that the Ray topology on E is independent of the choice of the uniformity used in constructing the Ray-Knight compactification. We then go further and introduce a U-space R which contains E in the Ray topology as a dense universally measurable subspace and which has all of the properties of the Ray-Knight compactification that are relevant to the study of X. Although R is not compact it has the advantage of being independent of the choice of the original uniformity on E. We call R *the Ray space* associated with the process X. If we equip E with the Ray topology, X is still a right process, and hence we can apply the Ray-Knight procedure once again. The first result in Section 5 states that we obtain nothing new in this manner. Finally we characterize the Ray space up to a useless set.

2. U-Spaces.

Let E be a topological space. The Borel subsets of E are the elements of the smallest σ-algebra, $\mathcal{E}$, containing all the open subsets of E. The universally measurable subsets of E are the elements of the σ-algebra, $\mathcal{E}^*$, of universally measurable sets over $(E, \mathcal{E})$; that is, $B \in \mathcal{E}^*$ if and only if for each finite measure μ on $(E, \mathcal{E})$ there exist $B_1, B_2 \in \mathcal{E}$ with $B_1 \subset B \subset B_2$ and $\mu(B_1) = \mu(B_2)$. Each finite measure μ on $(E, \mathcal{E})$ has a unique extension to $(E, \mathcal{E}^*)$ which is denoted by μ again. Thus to give a finite measure μ on $(E, \mathcal{E})$ is exactly the same as giving a finite measure on $(E, \mathcal{E}^*)$.

2.1. Définition. — *A topological space* E *is a* U-*space provided it is homeomorphic to a universally measurable subspace of a compact metric space.*

Here a universally measurable subspace of a topological space means a universally measurable subset with the usual subspace topology. It is immediate that a U-space is metrizable and second countable. Let E be a U-space and let F be a compact metric space containing a homeomorphic image of E. For ease of exposition we identify E with a universally measurable subspace of F. Let d be a metric on F compatible with the topology of F so that the topology of the metric space (E, d) is precisely the topology of E. Let $C_b(E)$ denote the bounded (real valued) continuous functions on E and let $C_u(E, d)$ denote the bounded (real valued) d-uniformly continuous functions on E. Then $C_u(E, d)$ consists precisely of the restrictions to E of the functions in $C_b(F)$. In particular $C_u(E, d)$ is a separable closed (in the uniform norm) sub-algebra and sublattice of $C_b(E)$. Of course, $C_b(E)$ itself is not separable in general. Let $\mathscr{F}$ (resp. $\mathscr{F}^*$) denote the σ-algebra of Borel (resp. universally measurable) subsets of F. Recall that if $(X, \mathscr{B})$ is a measurable space and $Y \subset X$, then *the trace of* $\mathscr{B}$ *on* Y, denoted by $\mathscr{B}|_Y$, is the σ-algebra on Y which consists of those $A \subset Y$ such that there exists $B \in \mathscr{B}$ with $A = B \cap Y$. Of course, the representation $A = B \cap Y$ is not unique, but it is easy to check that $\mathscr{B}|_Y$ is, indeed, a σ-algebra on Y, and that if $Y \in \mathscr{B}$, then $A \in \mathscr{B}|_Y$ if and only if $A \in \mathscr{B}$ and $A \subset Y$. Using these definitions the following proposition collects some elementary properties of U-spaces. The routine proof is omitted. See [3].

2.2. Proposition.

(i) $\mathscr{E}$ *is the trace of* $\mathscr{F}$ *on* E.

(ii) $\mathscr{E}$ *is the σ-algebra generated by* $C_u(E, d)$.

(iii) $\mathscr{E}^*$ *is the trace of* $\mathscr{F}^*$ *on* E.

(iv) *A finite measure on* $(E, \mathscr{E})$ *is determined by its values on* $C_u(E, d)$.

An immediate consequence of (2.2-iii) is that a universally measurable subspace of a U-space is itself a U-space.

We next recall some facts about metric spaces and their completions. Let (E, d) be a metric space. Then there exists a complete metric space (M, ρ) and an isometry $i : E \to M$ such that $i(E)$ is dense in M. The complete metric space (M, ρ) is unique up to isometry and is called the completion of (E, d). The isometry i is called the injection of E into M. We shall sometimes identity E with the subspace $i(E)$ of M. Let E_1 and E_2 be metric spaces with completions M_1 and M_2. If $\varphi : E_1 \to E_2$ is uniformly continuous (resp. an isometry), then there exists a unique uniformly continuous map (resp. isometry) $\psi : M_1 \to M_2$ such that $i_2 \circ \varphi = \psi \circ i_1$ where i_1 and i_2 are the injections of E_1 and E_2 into M_1 and M_2 respectively. It is well known that (M, ρ) is compact if and only if (E, d) is totally bounded.

Let E be a non-void set and let d_1 and d_2 be metrics on E. Let E_1 and E_2 denote the completions of (E, d_1) and (E, d_2) respectively. If $d_1 \leqslant d_2$, then the identity map e on E is uniformly continuous from (E, d_2) to (E, d_1) and so there exists a unique uniformly continuous map $\varphi : E_2 \to E_1$ such that $\varphi \circ i_2 = i_1 \circ e = i_1$ where i_1 and i_2 are the injections of (E, d_1) and (E, d_2) into E_1 and E_2 respectively. The following lemma collects some facts that will be used repeatedly in the sequel.

2.3. Lemma. — *Let d_1 and d_2 be metrics on* E *with $d_1 \leqslant d_2$. Using the above notation:*

(i) *If E_2 is compact, then φ is a surjection of E_2 on E_1 and E_1 is compact.*

(ii) *If d_1 and d_2 induce the same topology on* E, *then $i_2(E) = \varphi^{-1}[i_1(E)]$.*

(iii) *If E_2 is compact and if $i_2(E) = \varphi^{-1}[i_1(E)]$, then d_1 and d_2 induce the same topology on* E.

Proof. — If E_2 is compact, then $\varphi(E_2)$ is compact, and hence closed, in E_1. But $\varphi(E_2) \supset \varphi[i_2(E)] = i_1(E)$ which is dense in E_1. Therefore $\varphi(E_2) = E_1$, proving (i).

For (ii) we shall first prove the following statement:

2.4. *Let d_1 and d_2 induce the same topology on* E. *If $z \in E_2$ and $x \in E$ with $\varphi(z) = \varphi \circ i_2(x)$, then $z = i_2(x)$.*

To establish this choose $(x_n) \subset E$ with $i_2(x_n) \to z$ in E_2. Since φ is continuous,

$$i_1(x_n) = \varphi \circ i_2(x_n) \to \varphi(z) = \varphi \circ i_2(x) = i_1(x).$$

Therefore $x_n \to x$ in (E, d_1), and since the topologies are the same, $x_n \to x$ in (E, d_2), or equivalently $i_2(x_n) \to i_2(x)$ in E_2. Consequently $z = i_2(x)$, proving (2.4).

To establish (2.3-ii) we observe that $\varphi \circ i_2(E) = i_1(E)$ implies $i_2(E) \subset \varphi^{-1}[i_1(E)]$. Conversely if $z \in \varphi^{-1}[i_1(E)]$, then there exists an $x \in E$ with $\varphi(z) = i_1(x) = \varphi \circ i_2(x)$. By (2.4), $z = i_2(x)$ and so $z \in i_2(E)$, proving (2.3-ii).

Coming to (iii), we first note that by (i), E_1 is compact and φ is a surjection. Since $d_1 \leqslant d_2$, if $(x_n) \subset E$ and $x_n \to x$ in (E, d_2), then $x_n \to x$ in (E, d_1). Therefore it suffices to show that if $(x_n) \subset E$ and $x_n \to x$ in (E, d_1), then $x_n \to x$ in (E, d_2). But this will follow if we show that each subsequence (x'_n) of (x_n) has a further subsequence (x''_n) with $x''_n \to x$ in (E, d_2), or equivalently, $i_2(x''_n) \to i_2(x)$ in E_2. Changing notation it suffices to show that if $x_n \to x$ in (E, d_1), then (x_n) has a subsequence (x'_n) with

$$i_2(x'_n) \to i_2(x)$$

in E_2. Since E_2 is compact (x_n) has a subsequence (x'_n) with $i_2(x'_n) \to z \in E_2$. Now $i_1(x'_n) \to i_1(x)$ by assumption while $i_1(x'_n) = \varphi \circ i_2(x'_n) \to \varphi(z)$. Therefore $i_1(x) = \varphi(z)$, or $z \in \varphi^{-1}[i_1(E)]$. Thus by hypothesis there exists $y \in E$ with $z = i_2(y)$. Consequently $i_1(y) = \varphi \circ i_2(y) = \varphi(z) = i_1(x)$ and so $x = y$. Hence $i_2(x'_n) \to z = i_2(x)$, proving (iii).

We come now to the key fact that is needed for characterizing U-spaces.

2.5. Proposition. — *Let d_1 and d_2 be metrics on a set* E *and let E_1 and E_2 be the completions of (E, d_1) and (E, d_2). If d_1 and d_2 induce the same second countable topology on* E, *then $i_1(E)$ is universally measurable in E_1 if and only if $i_2(E)$ is universally measurable in E_2.*

Proof. — Let $d = d_1 + d_2$. It is immediate that if d_1 and d_2 induce the same topology τ on E, then the topology induced by d on E is also τ. Thus it suffices to prove (2.5) when $d_1 \leqslant d_2$. Hence in the remainder of this proof we assume that $d_1 \leqslant d_2$, and as in the notation above (2.3), φ is the uniformly continuous map from E_2 to E_1 such that $\varphi \circ i_2 = i_1$.

We shall first show that if $i_1(E)$ is universally measurable in E_1, then $i_2(E)$ is universally measurable in E_2. *This part of the argument does not use the assumption that the metric spaces* (E, d_1) *and* (E, d_2) *are second countable.* Let μ be a finite measure on $(E_2, \mathscr{E}_2)$ and let $\nu = \varphi(\mu)$ be its image under φ on $(E_1, \mathscr{E}_1)$. Of course, φ, being continuous, is a measurable map from $(E_2, \mathscr{E}_2)$ to $(E_1, \mathscr{E}_1)$. Since $i_1(E) \in \mathscr{E}_1^*$, there exists sets $A, B \in \mathscr{E}_1$ with $A \subset i_1(E) \subset B$ and $\nu(B) = \nu(A)$. Using (2.3-ii) this yields

$$\varphi^{-1}(A) \subset \varphi^{-1}[i_1(E)] = i_2(E) \subset \varphi^{-1}(B)$$

and since $\varphi^{-1}(A), \varphi^{-1}(B) \in \mathscr{E}_2$ and

$$\mu[\varphi^{-1}(A)] = \nu(A) = \nu(B) = \mu[\varphi^{-1}(B)],$$

it follows that $i_2(E) \in \mathscr{E}_2^*$.

Next assume $i_2(E) \in \mathscr{E}_2^*$. Then we must show that

$$i_1(E) = \varphi[i_2(E)] \in \mathscr{E}_1^*.$$

In light of (2.4) this is an immediate consequence of the following lemma.

2.6. Lemma. — *Let* E *and* F *be complete separable metric spaces and let* φ *be a measurable map from* $(E, \mathscr{E})$ *to* $(F, \mathscr{F})$. *Suppose that* $A \in \mathscr{E}^*$ *has the property that if* $x \in A$ *and* $z \in E$ *with* $\varphi(x) = \varphi(z)$, *then* $x = z$. *Then* $\varphi(A) \in \mathscr{F}^*$.

Proof. — The hypothesis implies that φ restricted to A is injective. Let $B = \varphi(A)$ and let $\psi = \varphi^{-1} : B \to A$. Thus ψ is a bijection of B on A. Let $\mathcal{A}$ be the trace of $\mathscr{E}$ on A and $\mathscr{B}^*$ the trace of $\mathscr{F}^*$ on B. We assert that ψ is a measurable map from $(B, \mathscr{B}^*)$ to $(A, \mathcal{A})$. This amounts to showing that if $A_1 \in \mathcal{A}$, then

$$\psi^{-1}(A_1) = \varphi(A_1) \in \mathscr{B}^*.$$

Now $A_1 = A_0 \cap A$ with $A_0 \in \mathscr{E}$, and $\varphi(A_1) \subset \varphi(A_0) \cap \varphi(A)$. However, if $y \in \varphi(A_0) \cap \varphi(A)$, then there exist $z \in A_0$ and $x \in A$ with $y = \varphi(z) = \varphi(x)$. Thus by hypothesis $x = z$, and so $y \in \varphi(A_0 \cap A)$. As a result

$$\varphi(A_1) = \varphi(A_0) \cap \varphi(A) = \varphi(A_0) \cap B.$$

But $A_0 \in \mathcal{E}$ and so by III-T13 of [5], $\varphi(A_0)$ is $\mathcal{F}$-analytic in F, and consequently $\varphi(A_0) \in \mathcal{F}^*$ (see III-24 of [5]). Therefore $\varphi(A_1) \in \mathcal{B}^*$ which proves that ψ is a measurable map from $(B, \mathcal{B}^*)$ to $(A, \mathcal{A})$.

Let μ be a finite measure on $(F, \mathcal{F}^*)$ and choose

$$B_0 \in \{C \in \mathcal{F} : C \supset B\}$$

of minimal μ measure. If $B_1 \in \mathcal{B}^*$, then $B_1 = B_1^* \cap B$ with $B_1^* \in \mathcal{F}^*$. Define $\tilde{\mu}(B_1) = \mu(B_1^* \cap B_0)$. It is easy to check that $\tilde{\mu}$ is well defined (that is, does not depend on the particular choice of B_1^*, or B_0 for that matter) and that $\tilde{\mu}$ is a measure on $\mathcal{B}^*$. See [3], for example. The measure $\tilde{\mu}$ is called *the trace of* μ *on* $\mathcal{B}^* = \mathcal{F}^*|_B$. Now $\nu = \psi(\tilde{\mu})$ is a finite measure on $(A, \mathcal{A})$. It is straightforward to verify that $\mathcal{A}^* = \mathcal{E}^*|_A$, and so ν may be regarded as a measure on $(E, \mathcal{E}^*)$ that is carried by A. (See [3].) Therefore there exists $A_1 \subset A$, $A_1 \in \mathcal{E}$ with $\nu(A_1) = \nu(A)$. Let

$$B_1 = \varphi(A_1) = \psi^{-1}(A_1) \subset B.$$

Moreover φ is injective on $A_1 \subset A$, and so by Lusin's theorem (Cor. 3.3, p. 22 of [8]), $B_1 \in \mathcal{F}$. Now

$$\tilde{\mu}(B_1) = \tilde{\mu}\psi^{-1}(A_1) = \nu(A_1) = \nu(A) = \tilde{\mu}(B),$$

while $\tilde{\mu}(B) = \mu(B_0)$ and $\tilde{\mu}(B_1) = \mu(B_1 \cap B_0) = \mu(B_1)$. Thus $B_1, B_0 \in \mathcal{F}$ with $B_1 \subset B \subset B_0$ and $\mu(B_1) = \mu(B_0)$. Therefore $B \in \mathcal{F}^*$, proving (2.6).

2.7. *Remark.* — Exercising a bit more care in the argument, one can show that (2.6) is valid when F is assumed only to be metrizable. It is natural to ask if (2.6) is valid when one only assumes that φ is injective on A. We would guess that the answer to this question is « no ».

2.8. Theorem. — *Let* E *be a second countable metrizable space. Then the following are equivalent:*

(i) E *is a* U-*space.*

(ii) *For each metric* d *on* E *compatible with the topology of* E, E *is universally measurable in its* d-*completion.*

(iii) *There exists a metric* d *on* E *compatible with the topology of* E *such that* E *is universally measurable in its* d*-completion.*

Proof. — Since (ii) implies (iii), it suffices to show that (i) implies (ii) and that (iii) implies (i). Suppose E is a U-space. Then there exists a compact metric space $(\hat{E}, d)$ which contains (a homeomorphic image of) E as a universally measurable subspace. Since the d-completion of E is just the closure of E in $\hat{E}$, it follows that E is universally measurable in its d-completion. Consequently (ii) is an immediate consequence of (2.5). Let d be a metric on E compatible with the topology of E and let F be the d-completion of E. Assume that $i(E)$ is universally measurable in F where i is the injection of E in F. Now F is a Polish space since (E, d) is separable, and consequently F is homeomorphic to a G_δ subset of a compact metric space K. (See Cor. 1, p. 197 of [1].) Let $h : F \to K$ be the homeomorphism. Then $h(F)$, being a G_δ, is Borel in K. Since $i(E) \in \mathscr{F}^*$, it follows that $h \circ i(E)$ is universally measurable in $h(F)$, and hence in K. But $h \circ i$ is clearly a homeomorphism of E on $h \circ i(E)$, and so E is a U-space. This establishes (2.8).

Recall that if E is a metrizable space, then a finite measure μ on $(E, \mathscr{E})$ is *tight* if $\mu(E) = \sup \{\mu(K) : K \text{ compact}\}$. Moreover if μ is a tight measure on $(E, \mathscr{E})$, then for each $B \in \mathscr{E}$, $\mu(B)$ is the supremum of $\mu(K)$ as K ranges over the compact subsets of B. See [8]. Clearly the same statement is valid for $B \in \mathscr{E}^*$.

2.9. Theorem. — *Let* E *be a second countable metrizable space. Then* E *is a* U*-space if and only if every finite measure on* $(E, \mathscr{E})$ *is tight.*

Proof. — Let E be a U-space and let F be a compact metric space containing E as a universally measurable subspace. Let μ be a finite measure on $(E, \mathscr{E})$. Then we may regard μ as a measure on $(F, \mathscr{F}^*)$ that is carried by E. Since every measure on a compact metric space is tight, there exists an increasing sequence (K_n) of compact subsets of F with each $K_n \subset E$ and $\mu(E) = \sup \mu(K_n)$. But from the definition of the subspace topology each K_n is compact in E and so μ is tight as a measure on $(E, \mathscr{E})$.

Conversely suppose each finite measure on $(E, \mathscr{E})$ is tight. Let d be a metric on E compatible with the topology and let F be the d-completion of E. For simplicity we identify E with a (dense) subset of the complete separable metric space F. In order to show that E is a U-space if suffices to show that E is universally measurable in F. Let μ be a finite measure on $(F, \mathscr{F})$. It is immediate that $\mathscr{E} = \mathscr{F}|_E$ and so we let $\tilde{\mu}$ be the trace of μ on E as in the proof of (2.6). Let E_0 be a Borel set in F containing E of minimal μ measure such that $\tilde{\mu}(A) = \mu(A_0 \cap E_0)$ whenever $A \in \mathscr{E}$ is of the form $A = A_0 \cap E$ with $A_0 \in \mathscr{F}$. By hypothesis there exists an increasing sequence (K_n) of compact subsets of E such that $K = \cup K_n \subset E$ and $\tilde{\mu}(K) = \tilde{\mu}(E)$. But each K_n is compact in F and so $K \in \mathscr{F}$. Thus $K \subset E \subset E_0$ with K, $E_0 \in \mathscr{F}$ and $\mu(E_0) = \tilde{\mu}(E) = \tilde{\mu}(K) = \mu(K \cap E_0) = \mu(K)$. Consequently $E \in \mathscr{F}^*$ proving (2.9).

3. Right Processes.

In this section we summarize the basic properties of right processes that we shall need in the sequel. Most of these facts are contained in [6], and all of them are in [3]. We fix a U-space E once and for all. Let $(P_t)_{t \geq 0}$ be a semigroup of Markov kernels on $(E, \mathscr{E}^*)$. Our first assumption is precisely HD1 of Meyer and Walsh (except that E is assumed to be Lusinien in [6]) :

HD 1: *For each probability μ on $(E, \mathscr{E}^*)$ there exists a right continuous Markov process admitting (P_t) as transition semigroup and μ as initial measure.*

One then constructs the *canonical right continuous realization* of the semigroup (P_t) : Ω, $\mathscr{F}^0$, $\mathscr{F}^0_t$, X_t, P^μ, etc. Here our notation is exactly as in [6]. It follows from HD1 that $t \to P_t f(x)$ is right continuous on $[0, \infty)$ for each $f \in C_b(E)$ and one can define its resolvent U^α by $U^\alpha = \int e^{-\alpha t} P_t \, dt$. Our second assumption is the following:

HD 2: *Let* f *be* α*-excessive. Then for each probability* μ *on* E, $t \to f \circ X_t$ *is* P^μ *almost surely right continuous on* $[0, \infty)$.

In [6] it is also assumed that f is nearly Borel, but we do *not* make that assumption here. One now introduces the σ-algebras $\mathscr{F}^\mu$, $\mathscr{F}$, $\mathscr{F}_t^\mu$, and $\mathscr{F}_t$. The hypothesis HD2 implies that the canonical realization X is strong Markov and that for each μ the system $(\Omega, \mathscr{F}_t^\mu, \mathscr{F}^\mu, P^\mu)$ satisfies the usual hypotheses of the general theory of processes, in particular the family $(\mathscr{F}_t^\mu)$ is right continuous. See [2] for all terminology regarding the general theory of processes. We shall call a semigroup satisfying HD1 and HD2 a *right* semigroup, and the corresponding process a *right* process.

We next choose a compact metric space $\hat{E}$ which contains a homeomorphic image of E as a universally measurable subspace, and, for simplicity, we identify E with this universally measurable subspace of $\hat{E}$. In light of (2.8) this amounts to choosing a totally bounded metric d on E compatible with the topology of E. Then without loss of generality we may assume that $\hat{E}$ is the completion of (E, d). The Ray cone $\mathbf{R} = \mathbf{R}(d)$ is defined to be the smallest convex cone $\mathbf{R}$ of positive bounded universally measurable functions on E with the following properties:

(i) $U^\alpha C_u^+(E, d) \subset \mathbf{R}$ for each $\alpha > 0$;

(ii) $U^\alpha \mathbf{R} \subset \mathbf{R}$ for each $\alpha > 0$; and

(iii) $\mathbf{R}$ is closed under pointwise minima.

[Recall from Section 2 that $C_u(E, d)$ is the space of bounded d-uniformly continuous functions on E and $C_u^+(E, d)$ is the set of positive functions in $C_u(E, d)$.] It turns out that $\mathbf{R}$ is separable in the uniform norm of functions on E, and we shall need the following explicit construction of $\mathbf{R}$ in the next section. If $\mathscr{H}$ is any convex cone contained in $b\mathscr{E}_+^*$ (the positive bounded universally measurable functions on E), define

$$(3.1)\quad \mathscr{U}(\mathscr{H}) = \{U^{\alpha_1}f_1 + \cdots + U^{\alpha_n}f_n;\ \alpha_j > 0,\ f_j \in \mathscr{H},\ 1 \leqslant j \leqslant n,\ n \geqslant 1\}$$

$$\Lambda(\mathscr{H}) = \{f_1 \wedge \cdots \wedge f_n;\ f_j \in \mathscr{H},\ 1 \leqslant j \leqslant n,\ n \geqslant 1\}.$$

Define $\mathbf{R}_0 = \mathscr{U}C_u^+(E, d), \ldots, \mathbf{R}_{n+1} = \Lambda(\mathbf{R}_n + \mathscr{U}\mathbf{R}_n), \ldots$ Then each $\mathbf{R}_n$, $n \geqslant 0$ is a convex cone and $\mathbf{R} = \cup \mathbf{R}_n$. Of course, $\mathbf{R}$ depends on the particular metric d chosen (or, equivalently the choice of the compact metric space $\hat{E}$) via $C_u(E, d)$. When we want to emphasize this dependence we shall write $\mathbf{R}(d)$ for $\mathbf{R}$. But for the moment d is fixed and we suppress it in our notation. Let (g_n) be a sequence dense in $\mathbf{R}$. Since $\mathbf{R}$, and hence (g_n), separates the points of E,

$$\rho(x, y) = \Sigma 2^{-n} \frac{|g_n(x) - g_n(y)|}{1 + |g_n(x) - g_n(y)|} \tag{3.2}$$

defines a metric on E. Let F be the completion of (E, ρ). Then F is a compact metric space and we regard E as a subset of F. Of course, F and the topology ρ induces on E do not depend on the particular sequence (g_n) chosen, but only on the cone $\mathbf{R}(d)$. We shall call the topology that ρ induces on E the *Ray topology*. *A priori* it would appear to depend on the choice of d, but we shall show in the next section that it does not. This should cause no difficulty in this section since we regard d as fixed in the present discussion. In general the original topology on E, that is the topology induced by d, and the Ray topology are not comparable.

It is evident that each element $f \in \mathbf{R}$ has a unique continuous extension $\bar{f}$ to F and we denote the set of these extensions by $\overline{\mathbf{R}}$. It turns out that $\overline{\mathbf{R}} - \overline{\mathbf{R}}$ is dense in C(F), and using this one shows that there exists a unique Ray resolvent $(\overline{U}^\alpha)_{\alpha > 0}$ on $(F, \mathscr{F})$ such that for each $f \in C(F)$, $(\overline{U}^\alpha f)|_E = U^\alpha(f|_E)$, where $g|_E$ denotes the restriction of g to E whenever g is a function defined on F. In particular, $(\overline{U}^\alpha)$ maps C(F) into C(F) and there exists a unique right continuous semigroup $(\overline{P}_t)$ on $(F, \mathscr{F})$ whose resolvent is $(\overline{U}^\alpha)$. In general $\overline{P}_0$ is not the identity.

So far this is all relatively elementary. We come now to the key points. First of all one shows that E is universally measurable in F. In other words E equipped with the Ray topology is a U-space. Secondly if $x \in E$, then

$$\overline{U}^\alpha(x, .) = U^\alpha(x, .) \quad \text{and} \quad \overline{P}_t(x, .) = P_t(x, .).$$

In particular, $\overline{U}^\alpha(x, .)$ and $\overline{P}_t(x, .)$ are measures on F that are carried by E whenever x is in E. Finally for each probability μ on E, P^μ almost surely the map $t \to X_t$ from $[0, \infty)$ to E is right continuous when E is given the Ray topology and has left limits in F. We denote this left limit by X_{t-} if $t > 0$ and set $X_{0-} = X_0$. We also write X^- for the process $(X_{t-})_{t \geq 0}$. We emphasize that X_{t-} denotes the left limit of $s \to X_s$ at t in the space F. If A is a subset of F, then $\{X \in A\} = \{(t, \omega) : X_t(\omega) \in A\}$ and

$$\{X^- \in A\} = \{(t, \omega) : X_{t-}(\omega) \in A\}.$$

Although the Ray topology and the original topology are not comparable, there is a close relationship between their Borel and universal structures. Let $\mathcal{E}_r$ (resp. $\mathcal{E}_r^*$) denote the Borel (resp. universally measurable) subsets of E equipped with the Ray topology. It is shown in Sections 10 and 12 of [3] that $\mathcal{E} \subset \mathcal{E}_r$ and $\mathcal{E}^* = \mathcal{E}_r^*$. Also each excessive function is nearly Ray Borel. (See (12.3) of [3] for the obvious definition.) Since $\overline{U}^\alpha : C(F) \to C(F)$ it follows that U^α sends $C_u(E, \rho)$ into $C_u(E, \rho)$ — here $C_u(E, \rho)$ is the space of bounded ρ-uniformly continuous functions on E. It then follows that each P_t is a kernel on $(E, \mathcal{E}_r)$. See Section 12 of [3].

Recall that $C_b(E)$ is the space of bounded continuous functions on E with its original topology. Similarly let $C_b(E, r)$ denote the set of bounded continuous functions on E with the Ray topology. Also we shall denote the Ray topology by r whenever convenient.

3.3. PROPOSITION. — *For each* $\alpha > 0$, $U^\alpha C_b(E) \subset C_b(E, r)$ *and* $U^\alpha C_b(E, r) \subset C_b(E, r)$.

Proof. — The proof of (3.3) depends on the following well known fact. See [7], for example. *Let* (Z, δ) *be a metric space and let* f *be a lower bounded, lower semicontinuous, function on* Z. *Then there exists an increasing sequence* (f_n) *of finite* δ*-uniformly continuous functions on* E *with* $f_n \uparrow f$. An immediate consequence of this is that if $f \in C_b(Z)$ then there exist monotone sequences (f_n) and (g_n) in $C_u(Z, \delta)$ with $f_n \uparrow f$ and $g_n \downarrow f$. Now by construction $U^\alpha C_u(E, d) \subset \mathbf{R} \subset C_b(E, r)$.

Thus if $f \in C_b(E)$, choose sequences (f_n) and (g_n) in $C_u(E, d)$ with $f_n \uparrow f$ and $g_n \downarrow f$. Since $(U^\alpha f_n)$ and $(U^\alpha g_n)$ are contained in $C_b(E, r)$ and $U^\alpha f_n \uparrow U^\alpha f$ while $U^\alpha g_n \downarrow U^\alpha f$, it follows that $U^\alpha f \in C_b(E, r)$. Similarly starting with the fact that $U^\alpha C_u(E, \rho) \subset C_u(E, \rho)$, it follows that $U^\alpha C_b(E, r) \subset C_b(E, r)$ proving (3.3).

4. The Ray Topology and Ray Space.

We shall use the notation of Section 3 without special mention. Our first result shows that the Ray topology on E does not depend on the choice of the metric d.

4.1. PROPOSITION. — *Let d_1 and d_2 be totally bounded metrics on E compatible with the topology of E. Then the corresponding Ray topologies r_1 and r_2 are the same.*

Proof. — Let $\mathbf{R}(d_1)$ and $\mathbf{R}(d_2)$ be the Ray cones constructed from d_1 and d_2 respectively. Since r_1 (resp. r_2) is the weakest topology on E relative to which the elements of $\mathbf{R}(d_1)$ [resp. $\mathbf{R}(d_2)$] are continuous, the desired conclusion will follow in view of the symmetry between d_1 and d_2 once we show $\mathbf{R}(d_1) \subset C_b(E, r_2)$. By (3.3) for each $\alpha > 0$, $U^\alpha C_u(E, d_1) \subset U^\alpha C_b(E) \subset C_b(E, r_2)$. Also $C_b^+(E, r_2)$ is a convex cone closed under « $\wedge$ » and, by (3.3.) again

$$U^\alpha C_b^+(E, r_2) \subset C_b^+(E, r_2)$$

for each $\alpha > 0$. Consequently by the very definition of $\mathbf{R}(d_1)$ one has $\mathbf{R}(d_1) \subset C_b^+(E, r_2)$. This establishes (4.1).

We shall denote the Ray topology on E by r. As we have shown it depends only on the original topology of E and, of course, the resolvent (U^α). In particular E with the Ray topology is a U-space.

4.2. COROLLARY.

(i) *The Ray topology is the weakest topology τ on E satisfying $U^\alpha C_b(E) \subset C_b(E, \tau)$ and $U^\alpha C_b(E, \tau) \subset C_b(E, \tau)$ for each $\alpha > 0$.*

(ii) *Using the notation* (3.1) *let* $\mathbf{Q}_0 = \mathscr{U}C_b^+(E), \ldots,$

$$\mathbf{Q}_{n+1} = \Lambda(\mathbf{Q}_n + \mathscr{U}\mathbf{Q}_n), \quad \ldots, \quad \textit{and} \quad \mathbf{Q} = \cup \mathbf{Q}_n.$$

Then the Ray topology is the weakest topology on E *relative to which the elements of* $\mathbf{Q}$ *are continuous.*

Proof. — By (3.3) the Ray topology has the two properties in (i). Let τ be a topology on E having these properties and let d be a totally bounded metric on E compatible with the original topology of E. Then just as in the proof of (4.1) one finds $\mathbf{R}(d) \subset C_b^+(E, \tau)$. Consequently the Ray topology is weaker than τ. Coming to (ii) let d be as above. Then $\mathbf{R}(d) \subset \mathbf{Q}$ and so if τ is the topology generated by $\mathbf{Q}$, it is clear that r is weaker than τ. But by (3.3),

$$U^\alpha C_b(E) \subset C_b(E, r) \quad \text{and so} \quad \mathbf{Q}_0 \subset C_b^+(E, r).$$

Suppose $\mathbf{Q}_n \subset C_b^+(E, r)$. Then using (3.3), $\mathbf{Q}_n + \mathscr{U}\mathbf{Q}_n \subset C_b^+(E, r)$ and so $\mathbf{Q}_{n+1} \subset C_b^+(E, r)$. As a result $\mathbf{Q} \subset C_b^+(E, r)$ and so τ is weaker than r, establishing (ii).

Remark. — In general $\mathbf{Q}$ is not separable.

For the moment fix a totally bounded metric d on E and let $\mathbf{R}(d)$, ρ, F, $(\overline{U}^\alpha)$, and $(\overline{P}_t)$ be as in section 3. Recall that a set $A \subset F$ is *useless* if for each μ on E the sets $\{X \in A\}$ and $\{X^- \in A\}$ are both P^μ evanescent. It is shown in [6] or [3] that

4.3. $$N = \{x \in F : \overline{P}_0(x, F - E) > 0\}$$

is useless. Since $\overline{P}_0(x, .) = P_0(x, .)$ if x is in E, one has $N \subset F - E$. It is evident that $N \in \mathscr{F}^*$.

4.4. Lemma. — *The set* $M = \{x \in F : \overline{U}^\alpha(x, F - E) > 0\}$ *is independent of* $\alpha > 0$ *and* $M \in \mathscr{F}^*$. *Moreover* $M \subset N$ *and, hence,* M *is useless.*

Proof. — For the moment let

$$M_\alpha = \{x \in F : \overline{U}^\alpha(x, F - E) > 0\}.$$

Obviously $M_\alpha \in \mathscr{F}^*$. Fix $x \in F - M_\alpha$. Then $\overline{U}^\alpha(x, .)$

is carried by E and so

$$\overline{U}^{\beta}(x, .) = \overline{U}^{\alpha}(x, .) + (\alpha - \beta) \int_E \overline{U}^{\alpha}(x, dy)\overline{U}^{\beta}(y, .).$$

But if $y \in E$, $\overline{U}^{\beta}(y, .) = U^{\beta}(y, .)$ is carried by E, and so $\overline{U}^{\beta}(x, .)$ is carried by E. But $\alpha, \beta > 0$ are arbitrary and so $M_\alpha = M_\beta$. A similar argument using $\overline{U}^\alpha = \overline{P}_0\overline{U}^\alpha$ shows that $M \subset N$. See [6] or [3].

Remark. — Simple examples show that it is not true that $M = N$ in general.

4.5. DEFINITION. — *The Ray space of the semigroup* (P_t) *is the set* $R = F - M$ *with the subspace topology it inherits from* F.

At first glance it appears that R depends on the metric d, or more precisely the uniformity generated by d, via F. However, the main purpose of this section is to show that R is unique up to homeomorphism. Note that $E \subset R \subset F$, that R is a U-space, that E is universally measurable and dense in R, and that the topology R induces on E is the Ray topology. The next proposition lists some important properties of R.

4.6. PROPOSITION.

(i) *For each* μ *on* E, $t \to X_t$ *has left limits in* R *almost surely* P^μ.

(ii) *If* $(\mu_t)_{t>0}$ *is a bounded entrance law for* (P_t), *then there exists a unique finite measure* μ *carried by* R *with* $\mu_t = \mu\overline{P}_t$ *for each* $t > 0$.

(iii) *If* $x \in R$, $\overline{P}_0(x, .)$ *is carried by* R *and* $\overline{P}_t(x, .)$ *is carried by* E *for all* $t > 0$.

Proof. — Since $F - R = M$ is useless, (i) is clear. For (ii), Meyer and Walsh (Theorem 9 of [6]) showed that there exists a unique finite measure μ carried by

$$D = \{x \in F : \overline{P}_0(x, .) = \varepsilon_x\},$$

the set of non-branch points, with $\mu_t = \mu\overline{P}_t$. Taking Laplace transforms yields

$$\int_0^\infty e^{-\alpha t}\mu_t(f)\,dt = \int \mu\,(dx)\,\overline{U}^\alpha f(x)$$

for each $\alpha > 0$ and $f \in b\mathscr{F}^*$. But each μ_t is carried by E and so $\int \mu\,(dx)\overline{U}^\alpha(x,\ F - E) = 0$. Therefore μ is carried by R. If $x \in R$, then

$$0 = \overline{U}^\alpha(x,\ F - E) = \overline{P}_0\overline{U}^\alpha(x,\ F - E) \\ = \int \overline{P}_0(x,\ dy)\overline{U}^\alpha(y,\ F - E)$$

and so $\overline{P}_0(x,\ .)$ is carried by R. Since $\overline{U}^\alpha(x,\ F - E) = 0$ there exists a sequence (t_n), depending on x, of strictly positive numbers decreasing to zero with $\overline{P}_{t_n}(x,\ F - E) = 0$ for each n. If $t > t_n$ for some n, then

$$\overline{P}_t(x,\ .) = \int_E \overline{P}_{t_n}(x,\ dy)\overline{P}_{t-t_n}(y,\ .)$$

and since $\overline{P}_s(y,\ .) = P_s(y,\ .)$ for $y \in E$ and $s > 0$ it follows that $\overline{P}_t(x,\ .)$ is carried by E for all $t > 0$ if $x \in R$. This completes the proof of (4.6).

In view of (4.6-i) and (4.6-ii) the space R has all of the properties of F that are of interest in studying X. Of course, we give up the compactness of F, but, as we shall see, we gain the fact that R depends only on the topology of E and not on the particular choice of the metric d.

To this end let d_1 and d_2 be two totally bounded metrics on E compatible with the topology. With the obvious notation we want to show that there is a homeomorphism of R_2 onto R_1 that leaves E fixed. Let $d = d_1 + d_2$. Then d is a totally bounded metric on E inducing the same topology as d_1 and d_2. Therefore, in light of (2.5), there is no loss of generality in supposing that $d_1 \leqslant d_2$ in our discussion. Then $C_u(E,\ d_1) \subset C_u(E,\ d_2)$ and so $\mathbf{R}(d_1) \subset \mathbf{R}(d_2)$. Therefore one may choose the metrics ρ_1 and ρ_2 in (3.2) to satisfy $\rho_1 \leqslant \rho_2$. Let F_1 and F_2 be the ρ_1 and ρ_2 completions of $(E,\ \rho_1)$ and $(E,\ \rho_2)$ respectively. Let i_1 and i_2 be the injections of E into F_1 and F_2 respectively. Since we are going to deal with these two completions simultaneously we shall not identify E with $i_1(E)$ and $i_2(E)$, at least temporarily. Thus we shall preserve the distinction among E, $i_1(E)$, and $i_2(E)$. This causes certain obvious changes in what we wrote before. For example, using the subscripts 1 and 2 in an obvious manner, if $(\overline{U}_1^\alpha)$ is the Ray resolvent on F_1, then for $x \in E$,

$$\overline{U}_1^\alpha(i_1(x), .) = U^\alpha(x, i_1^{-1}(.)) = i_1 U^\alpha(x, .),$$

$M_1 = \{x \in F_1 : \overline{U}_1^\alpha(x, F_1 - i_1(E)) > 0\}$, and $R_1 = F_1 - i_1(E)$ with similar relations for $(\overline{U}_2^\alpha)$, M_2, and R_2.

Since $\rho_1 \leqslant \rho_2$ there exists by (2.3-i) a continuous surjection φ of F_2 onto F_1 such that $\varphi \circ i_2 = i_1$. Of course, both ρ_1 and ρ_2 induce the Ray topology on E and so by (2.3-ii), $i_2(E) = \varphi^{-1} i_1(E)$. Since φ is continuous $\varphi^{-1}(\mathscr{F}_1) \subset \mathscr{F}_2$ and $\varphi^{-1}(\mathscr{F}_1^*) \subset \mathscr{F}_2^*$. We come now to the key technical lemma of this section.

4.7. Lemma. — *For each* $\alpha > 0$ *and* $x \in F_2$ *one has* $\varphi\overline{U}_2^\alpha(x, .) = \overline{U}_1^\alpha(\varphi(x), .)$.

Proof. — Of course, $\varphi\overline{U}_2^\alpha(x, .)$ is the measure on F_1 given by $A \to \overline{U}_2^\alpha(x, \varphi^{-1}(A))$ for all $A \in \mathscr{F}_1$ (or $\mathscr{F}_1^*$). Suppose first that $x \in E$. Then by definition $\overline{U}_2^\alpha(i_2(x), .) = i_2 U^\alpha(x, .)$ and $\overline{U}_1^\alpha(i_1(x), .) = i_1 U^\alpha(x, .)$. Thus for $x \in E$,

$$\begin{aligned}\varphi\overline{U}_2^\alpha(i_2(x), .) &= (\varphi \circ i_2)U^\alpha(x, .) = i_1 U^\alpha(x, .)\\ &= \overline{U}_1^\alpha(i_1(x), .) = \overline{U}_1^\alpha(\varphi \circ i_2(x), .).\end{aligned}$$

Thus (4.7) is true for $x \in i_2(E)$. Since $\overline{U}_j^\alpha$ sends $C(F_j)$ into itself, it is immediate that the map $x \to \overline{U}_j^\alpha(x, .)$ from F_j to the bounded measures on F_j with the usual weak topology for measures is continuous for $j = 1, 2$. Given $x \in F_2$ choose $(x_n) \subset E$ with $i_2(x_n) \to x$. Then since φ is continuous one has $\varphi\overline{U}_2^\alpha(i_2(x_n), .) \to \varphi\overline{U}_2^\alpha(x, .)$, and

$$\varphi\overline{U}_2^\alpha(i_2(x_n), .) = \overline{U}_1^\alpha(\varphi \circ i_2(x_n), .) \to \overline{U}_1^\alpha(\varphi(x), .).$$

Therefore $\varphi\overline{U}_2^\alpha((x, .) = \overline{U}_1^\alpha(\varphi(x), .)$ proving (4.7).

4.8. Proposition. — $\varphi(M_2) = M_1$ *and* $\varphi(R_2) = R_1$.

Proof. — For each $x \in F$ it follows from (4.7) and from $i_2(E) = \varphi^{-1} i_1(E)$ that

$$\overline{U}_1^\alpha[\varphi(x), F_1 - i_1(E)] = \overline{U}_2^\alpha[x, \varphi^{-1}(F_1 - i_1(E))]$$
$$= \overline{U}_2^\alpha[x, F_2 - i_2(E)].$$

Consequently $x \in R_2$ (resp. M_2) if and only if $\varphi(x) \in R_1$ (resp. M_1), proving (4.8).

4.9. Proposition. — φ *is a homeomorphism of* R_2 *onto* R_1 *that is natural in the sense that* $\varphi \circ i_2(x) = i_1(x)$ *for all* $x \in E$.

Proof. — We already know that φ is a continuous surjection of R_2 on R_1. Thus we must show that φ is injective on R_2 and that φ^{-1} is continuous. We first show that φ is injective on R_2. To this end fix x and y in R_2 with $\varphi(x) = \varphi(y)$. Let g be a bounded universally measurable function defined on $i_2(E)$. Define f on $i_1(E)$ by $f(i_1(z)) = g(i_2(z))$ for $z \in E$. Then $g = f \circ \varphi$ on $i_2(E)$, and since $f = g \circ i_2 \circ i_1^{-1}$ on $i_1(E)$, f is universally measurable on $i_1(E)$. Let f be defined to be zero on $F_1 - i_1(E)$. Since x and y are in R_2 the measures $\overline{U}_2^\alpha(x, .)$ and $\overline{U}_2^\alpha(y, .)$ are carried by $i_2(E)$, and so

$$\int_{i_2(E)} \overline{U}_2^\alpha(x, dz)\, g(z)$$
$$= \int_{i_2(E)} \overline{U}_2^\alpha(x, dz)\, f \circ \varphi(z) = \overline{U}_2^\alpha(x, f \circ \varphi) = \overline{U}_1^\alpha(\varphi(x), f)$$
$$= \overline{U}_1^\alpha(\varphi(y), f) = \int_{i_2(E)} \overline{U}_2^\alpha(y, dz)\, g(z).$$

Therefore $\overline{U}_2^\alpha(x, .) = \overline{U}_2^\alpha(y, .)$ for each $\alpha > 0$. It is shown in the proof of Lemma 1 of [6] (see also [3]), that this implies that $x = y$. Consequently φ is a bijection of R_2 on R_1.

In order to show φ^{-1} from R_1 to R_2 is continuous it suffices to show φ from R_2 to R_1 is closed. Accordingly we shall show that if A is closed in F_2, then $\varphi(A \cap R_2)$ is closed in R_1. Since $\varphi(A)$ is compact, and hence closed, in F_1 this will follow if $\varphi(A \cap R_2) = \varphi(A) \cap R_1$. But

$$\varphi(A \cap R_2) \subset \varphi(A) \cap \varphi(R_2) = \varphi(A) \cap R_1.$$

Conversely if $y \in \varphi(A) \cap R_1$, then there exists $x \in A$ with

$y = \varphi(x)$. Now $\varphi(x) = y \in R_1$ and so by (4.8), $x \in R_2$. Thus $y \in \varphi(A \cap R_2)$ and therefore $\varphi(A \cap R_2) = \varphi(A) \cap R_1$, completing the proof of (4.9).

Remark. — Simple examples show that φ is not injective on all of F_2 in general. In the course of our discussion we have assumed that $d_1 \leqslant d_2$, but as mentioned earlier this is not a restriction. Thus in the general case there exists a natural homeomorphism φ of R_2 on R_1 such that

$$\varphi\overline{U}_2^\alpha(x, .) = \overline{U}_1^\alpha(\varphi(x), .)$$

for all $x \in R_2$. However, this last identity no longer makes sense for arbitrary $x \in F_2$ since φ is only defined on R_2.

We close this section with a few more results about the homeomorphism φ and the Ray space R.

4.10. Lemma. — *If* $d_1 \leqslant d_2$ *let* φ *be the surjection of* F_2 *on* F_1 *constructed above. Then* $\varphi\overline{P}_t^2(x, .) = \overline{P}_t^1(\varphi(x), .)$ *for all* $t \geqslant 0$ *and* $x \in F_2$. *In the general case let* φ *be the natural homeomorphism of* R_2 *on* R_1 *constructed above. Then*

$$\varphi\overline{P}_t^2(x, .) = \overline{P}_t^1(\varphi(x), .)$$

for all $t \geqslant 0$ *and* $x \in R_2$.

Proof. — Suppose $d_1 \leqslant d_2$. It follows from (4.7) that

$$\int_0^\infty e^{-\alpha t}\overline{P}_t^2(x, f \circ \varphi)\, dt = \int_0^\infty e^{-\alpha t}\overline{P}_t^1(\varphi(x), f)\, dt$$

for all $f \in C(F_1)$ and $x \in F_2$. Using the right continuity of $t \to \overline{P}_t^j(x, f_j)$ for $f_j \in C(F_j)$ and the uniqueness theorem for Laplace transforms gives the first assertion in (4.10). The second is an immediate consequence of the first.

The next proposition says that the natural homeomorphism we have constructed preserves the branch points. In its statement φ is the natural homeomorphism of R_2 on R_1.

4.11. Proposition. — *Let* B_1 *and* B_2 *be the set of branch points in* F_1 *and* F_2 *respectively. Then*

$$\varphi(B_2 \cap R_2) = B_1 \cap R_1 \quad \textit{and} \quad \varphi(B_2^c \cap R_2) = B_1^c \cap R_1.$$

Proof. — Recall that $x \in B_j$ provided $\overline{P}_0^j(x, .) \neq \varepsilon_x$, and according to (4.6-iii), $\overline{P}_0^j(x, .)$ is carried by R_j if $x \in R_j$,

$j = 1, 2,$. Let $x \in R_2$. If $x \in B_2^c$, then by (4.10),

$$\overline{P}_0^1(\varphi(x), .) = \varphi \overline{P}_0^2(x, .) = \varphi \varepsilon_x = \varepsilon_{\varphi(x)}$$

and so $\varphi(x) \in B_1^c$. Conversely if $\varphi(x) \in B_1^c$ and g is a bounded Borel function on R_2, then $f = g \circ \varphi^{-1}$ is a bounded Borel function on R_1 and

$$\overline{P}_0^2(x, g) = \overline{P}_0^2(x, f \circ \varphi) = \overline{P}_0^1(\varphi(x), f) = f \circ \varphi(x) = g(x).$$

Since $\overline{P}_0^2(x, .)$ is carried by R_2 this implies that $x \in B_2^c$. Combining these facts yields (4.11).

4.12. Remark. — Recall the definition of N_1 and N_2. Then an argument similar to the proof of (4.11) shows that $\varphi(N_2 \cap R_2) = N_1 \cap R_1$ and $\varphi(N_2^c \cap R_2) = N_1^c \cap R_1$.

It follows from (4.7) and (4.10) that for each $x \in R$, $\overline{U}^\alpha(x, .)$ and $\overline{P}_t(x, .)$ are uniquely defined independently of the particular metric d used in the construction. Moreover it is clear that $(\overline{U}^\alpha)$, resp. $(\overline{P}_t)$, is a resolvent, resp. a semigroup, of kernels on $(R, \mathscr{R}^*)$. For each x in R the very definition of R implies that $\overline{U}^\alpha(x, .)$ is carried by E, while according to (4.6-iii), $\overline{P}_t(x, .)$ is carried by E if $t > 0$ and by R if $t = 0$. Since $\mathscr{R}$, the σ-algebra of Borel subsets of R, is just the trace of $\mathscr{F}$ on R it is easy to see that $\overline{U}^\alpha$ and $\overline{P}_t$ send $b\mathscr{R}$ into $b\mathscr{R}$ for each $\alpha > 0$ and $t \geq 0$. See the proof of (12.1) in [3]. In other words $\overline{U}^\alpha$ and $\overline{P}_t$ are kernels on $(R, \mathscr{R})$.

4.13. Proposition.

(i) *For each* $\alpha > 0$, $\overline{U}^\alpha$ *maps* $C_b(E)$ *and* $C_b(E, r)$ *into* $C_b(R)$.

(ii) *For each* $f \in C_b(R)$ *and* $x \in R$, $t \to \overline{P}_t f(x)$ *is right continuous on* $[0, \infty)$.

Proof. — For $x \in R$, $\overline{U}^\alpha(x, .)$ is carried by E and so $f \to \overline{U}^\alpha f$ may be regarded as a map from bounded functions on E to bounded functions on R. Statement (i) now follows by the same argument as that used to prove (3.3). Fix a metric d and the corresponding ρ and F. If $f \in C_u(R, \rho)$, then there exists a unique $g \in C(F)$ with $f = g|_R$ and so for each $t \geqslant 0$, $\overline{P}_t f = \overline{P}_t g$ on R. Thus $t \to \overline{P}_t f(x)$ is right continuous on $[0, \infty)$ for each $x \in R$ and $f \in C_u(R, \rho)$. But if $f \in C_b(R)$ there exists sequences (f_n) and (g_n) in $C_u(R, \rho)$ such that $f_n \downarrow f$ and $g_n \uparrow f$. Consequently $t \to \overline{P}_t f(x)$ is right continuous on $[0, \infty)$ for $x \in R$, proving (4.13-ii).

The next proposition shows exactly how the topology of the metrizable space R is determined by the resolvent $(\overline{U}^\alpha)$ on R. Recall that a sequence of bounded measures (ν_n) on a metrizable space Z converges to a bounded measure ν provided $\nu_n(f) \to \nu(f)$ for all $f \in C_b(Z)$.

4.14. Proposition. — *Let $(x_n) \subset R$ and $x \in R$. Then $x_n \to x$ if and only if for each $\alpha > 0$, $\overline{U}^\alpha(x_n, .) \to \overline{U}^\alpha(x, .)$ as measures on E with the Ray topology.*

Proof. — By (4.13-i), $\overline{U}^\alpha$ maps $C_b(E, r)$ into $C_b(R)$ and so if $x_n \to x$ in R, then $\overline{U}^\alpha(x_n, .) \to \overline{U}^\alpha(x, .)$. For the converse fix a metric d and the corresponding ρ and F. In order to show that $x_n \to x$ it suffices to show that every subsequence of (x_n) contains a further subsequence which converges to x. Changing notation it suffices to show that (x_n) contains a subsequence converging to x whenever $\overline{U}^\alpha(x_n, .) \to \overline{U}^\alpha(x, .)$ for all $\alpha > 0$. But F is compact and so (x_n) has a subsequence (x'_n) that converges to some $y \in F$. Since $\overline{U}^\alpha$ maps C(F) into C(F) it follows that $\overline{U}^\alpha(x'_n, .) \to \overline{U}^\alpha(y, .)$ as measures on F. But for each n, $\overline{U}^\alpha(x'_n, .)$ is carried by E as is $\overline{U}^\alpha(x, .)$ and since the restriction to E of any function in C(F) is in $C_u(E, \rho) \subset C_b(E, r)$ it follows that $\overline{U}^\alpha(x'_n, .) \to \overline{U}^\alpha(x, .)$ as measures on F. Consequently $\overline{U}^\alpha(x, .) = \overline{U}^\alpha(y, .)$ as measures on F for all $\alpha > 0$, and this implies that $x = y$ completing the proof of (4.14).

5. Further Properties of the Ray Space.

We know that E equipped with the Ray topology is a U-space and that X as an (E, r) valued process is a right process with resolvent (U^α). The first thing that we shall show in this section is that if we apply the Ray-Knight procedure to (E, r) we get nothing new as far as the Ray space is concerned. After that we shall characterize the Ray space up to a useless set. Simple examples show that this is as much as one can expect.

Let d be a fixed totally bounded metric on E compatible with the original topology of E. Starting from d we construct ρ, F, R, and the Ray topology r as before. Let δ be a totally bounded metric on E compatible with the Ray topology of E. As in Section 3 we construct the Ray cone $\mathbf{R}(\delta)$ with respect to (E, δ). To be explicit using the notation of (3.1), we have

$$\mathbf{R}_0(\delta) = \mathscr{U}C_u^+(E, \delta), \ \ldots, \ \mathbf{R}_{n+1}(\delta) = \Lambda(\mathbf{R}_n(\delta) + \mathscr{U}\mathbf{R}_n(\delta)),$$

and $\mathbf{R}(\delta) = \cup \mathbf{R}_n(\delta)$. From $\mathbf{R}(\delta)$ we construct a metric ρ' as in (3.2) and we let F$'$ be the (compact) completion of (E, ρ'). We let $(\tilde{U}^\alpha)_{\alpha>0}$ be the corresponding Ray resolvent on F$'$ and R$'$ the corresponding Ray space. Finally r' will denote the topology induced by ρ' on E, or equivalently, the subspace topology E inherits from F$'$.

5.1. Proposition. — *The r and r' topologies on* E *are the same.*

Proof. — In the present context (3.3) states that

$$U^\alpha C_b(E, r) \subset C_b(E, r')$$

and $U^\alpha C_b(E, r') \subset C_b(E, r')$ for each $\alpha > 0$. Moreover from (3.3) itself $U^\alpha C_b(E, r) \subset C_b(E, r)$. Since $C_u(E, \delta) \subset C_b(E, r)$, it follows that $\mathbf{R}(\delta) \subset C_b^+(E, r)$. This in turn implies that $C_b(E, r') \subset C_b(E, r)$, or, in other words, that r' is weaker than r. Since both r and r' are metrizable in order to complete the proof of (5.1) it suffices to show that if $x_n \to x$ in r', then $x_n \to x$ in r whenever $(x_n) \subset E$ and $x \in E$. Let $f \in C_b(E, r)$. Then $U^\alpha f \in C_b(E, r')$ and so if $x_n \to x$ in r', $U^\alpha f(x_n) \to U^\alpha f(x)$. This says that $U^\alpha(x_n, .) \to U^\alpha(x, .)$ as measures on (E, r), and hence by (4.14), $x_n \to x$ in r establishing (5.1).

The next result shows that R and R$'$ are the same. The notation is that given above (5.1). For simplicity of notation we identify E with a subspace of R and of R′ simultaneously. Also, recall that $\overline{U}^\alpha(x, .)$ and $\tilde{U}^\alpha(x', .)$ are carried by E for $x \in R$ and $x' \in R'$.

5.2. PROPOSITION. — *There exists a homeomorphism* ψ *from* R *onto* R′ *preserving* E *such that*

$$\overline{U}^\alpha(x, .) = \tilde{U}^\alpha(\psi(x), .)$$

for each $x \in R$ *and* $\alpha > 0$.

Proof. — Since E is dense in R, given $x \in R$ there exists a sequence $(x_n) \subset E$ with $x_n \to x$ in R. Hence by (4.14), $U^\alpha(x_n, .) \to \overline{U}^\alpha(x, .)$ as measures on (E, r). But $(x_n) \subset E \subset R' \subset F'$ and so (x_n) has a subsequence, call it (x_n) again for notational simplicity, converging to some point $x' \in F'$. Let $f' \in C(F')$. Then

$$U^\alpha(f'|_E)(x_n) = \tilde{U}^\alpha f'(x_n) \to \tilde{U}^\alpha f'(x').$$

But $f'|_E$ is r', and hence r, continuous and so

$$U^\alpha(f'|_E)(x_n) \to \overline{U}^\alpha(f'|_E)(x).$$

That is, for all $f' \in C(F')$, $\tilde{U}^\alpha(x', f') = \overline{U}^\alpha(x, f'|_E)$. Since for fixed $\alpha > 0$ both expressions are measures on F′, it follows that $\tilde{U}^\alpha(x', .)$ is carried by E, that is, $x' \in R'$, and that $\tilde{U}^\alpha(x', .) = \overline{U}^\alpha(x, .)$. Also x' depends only on x and not on the particular construction, because if a second construction leads to $x'' \in R'$, then $\tilde{U}^\alpha(x', .) = \tilde{U}^\alpha(x'', .)$ for all $\alpha > 0$ and so $x' = x''$. We now define $\psi(x) = x'$. Using (4.14) and the fact that $(\overline{U}^\alpha)$ and $(\tilde{U}^\alpha)$ separate the points of R and R′ respectively, it is easy to check that ψ is a homeomorphism of R onto R′. Of course, by construction,

$$\overline{U}^\alpha(x, .) = \tilde{U}^\alpha(\psi(x), .)$$

for all x in R. In particular $\psi(x) = x$ for all x in E, completing the proof of (5.2).

Remark. — Since ψ is the identity on E and $\overline{U}^\alpha(x, .)$ is carried by E for x in R, it is immediate that

$$\psi\overline{U}^\alpha(x, .) = \overline{U}^\alpha(x, .).$$

Thus the relationship between the resolvents in (5.2) may be written $\psi\overline{U}^\alpha(x, .) = \tilde{U}^\alpha(\psi(x), .)$ for $x \in R$ more in keeping with Section 4. The added simplicity in (5.2) is due to the fact that we are now identifying E with a subspace of R and R′ simultaneously.

Proposition (5.1) and (5.2) give a precise meaning to the statement that if one applies the Ray-Knight procedure to (E, r), then one obtains nothing new.

We turn next to a characterization of R. Let H be a U-space that contains a homeomorphic image of E with the Ray topology as a universally measurable subspace. We identify E with its image in H so that the subspace topology E inherits from H is the Ray topology. In what follows all topological statements about E refer to the Ray topology on E. However, we shall repeat this occasionally for emphasis. We now make the following assumptions on H:

5.3. E *is dense in* H.

5.4. *There exists a Markov resolvent* $(V^\alpha)_{\alpha>0}$ *on* $(H, \mathscr{H}^*)$ *satisfying for each* $\alpha > 0$:

(i) *For each* $x \in H$, $V^\alpha(x, .)$ *is carried by* E.

(ii) *If* $x \in E$, $U^\alpha(x, .) = V^\alpha(x, .)$.

(iii) $V^\alpha C_b(E, r) \subset C_b(H)$.

(iv) *If for each* $\alpha > 0$, $V^\alpha(x_n, .) \to V^\alpha(x, .)$ *as measures on* (E, r), *then* $x_n \to x$ *in* H.

Remark. — *It is immediate from* (5.4-iv) that if

$$V^\alpha(x, .) = V^\alpha(y, .)$$

for all $\alpha > 0$, then $x = y$.

5.5. Proposition. — *Let* H *satisfy* (5.3) *and* (5.4). *Then there exists a homeomorphism* ψ *of* H *into* R *which preserves* E *and satisfies* $\psi V^\alpha(x, .) = \overline{U}^\alpha(\psi(x), .)$ *for each* $\alpha > 0$ *and* $x \in H$.

Proof. — In order to prove (5.5) we shall first construct a convenient realization of R starting from the Ray topology on E. To this end let $(\tilde{H}, \delta)$ be a compact metric space that contains H as a dense universally measurable subspace. Of course, the topology induced by δ on E is just the Ray topology and E is a dense [because of (5.3)] universally measurable subspace of $\tilde{H}$. If $f \in C_b(E, r)$, then $U^\alpha f \in C_b(E, r)$ and by (5.4), $V^\alpha f$ is a (unique) bounded continuous extension of $U^\alpha f$ to H. Let C(E, H) denote the restrictions to E of the elements in $C_b(H)$. Then C(E, H) consists precisely of those bounded continuous functions on E that have a (unique) bounded continuous extension to H. By the above remark for each $\alpha > 0$, $U^\alpha C_b(E, r) \subset C(E, H)$. Since

$$C_u^+(E, \delta) \subset C_b^+(E, r) \quad \text{and} \quad C^+(E, H) \subset C_b^+(E, r)$$

we see that

$$U^\alpha C_u^+(E, \delta) \subset C^+(E, H) \quad \text{and} \quad U^\alpha C^+(E, H) \subset C^+(E, H)$$

for each $\alpha > 0$. Moreover $C^+(E, H)$ is a convex cone closed under « $\wedge$ » and so by the definition of the Ray cone $\mathbf{R}(\delta)$ one has $\mathbf{R}(\delta) \subset C^+(E, H)$. In view of (5.1) if (g_n) is a dense sequence in $\mathbf{R}(\delta)$, then ρ defined by (3.2) induces the Ray topology on E. But each g_n has a unique bounded continuous extension to H which we again denote by g_n since

$$\mathbf{R}(\delta) \subset C^+(E, H).$$

Observe that $\mathbf{R}(\delta)$ separates the points of H because if $V^\alpha f(x) = V^\alpha f(y)$ for all $f \in C_u(E, \delta)$, then this holds for all $f \in C_b(E, r)$ and consequently $x = y$. Therefore (g_n) also separates H and so ρ is a metric on H as well as on E. Since each g_n is δ-continuous on H, the fact that E is dense in H in the topology induced by δ implies that E is dense in H in the topology induced by ρ. Of course, the topology that ρ induces on E is the Ray topology.

Next let F be the (compact) completion of (H, ρ). Since E is ρ-dense in H we may identify the ρ-completion of E with F also, and in view of (5.2), we may construct the extended resolvent $\overline{U}^\alpha$ and the Ray space R from ρ and F. Thus $E \subset R \subset F$ and $E \subset H \subset F$. Since each g_n is δ

continuous it is immediate that if $(x_k) \subset H$ and $x \in H$ with $\delta(x_k, x) \to 0$, then $\rho(x_k, x) \to 0$.

Next we shall show that $H \subset R$ and that ρ and δ induce the same topology on H (as well as on E). To this end let $\bar{f} \in C(F)$ and $f = \bar{f}|_E$. Then

$$f \in C_b(E, r) \quad \text{and} \quad \bar{U}^\alpha \bar{f} = U^\alpha f$$

on R by the very definition of R. If $x \in H$ and $(x_k) \subset E$ with $\delta(x_k, x) \to 0$, then from (5.4), $U^\alpha f(x_k) \to V^\alpha f(x)$. But $\delta(x_k, x) \to 0$ implies $\rho(x_k, x) \to 0$ and so

$$U^\alpha f(x_k) = \bar{U}^\alpha \bar{f}(x_k) \to \bar{U}^\alpha \bar{f}(x).$$

Hence for each $\bar{f} \in C(F)$, $x \in H$, and $\alpha > 0$,

$$\bar{U}^\alpha \bar{f}(x) = V^\alpha(\bar{f}|_E)(x),$$

and since both sides are measures this relationship holds for all $f \in b\mathscr{F}^*$. As a result $H \subset R$ and $\bar{U}^\alpha(x, .) = V^\alpha(x, .)$ for all x in H and $\alpha > 0$. Finally if $x_k \to x$ in (H, ρ), then by (4.14), $\bar{U}^\alpha(x_k, .) \to \bar{U}^\alpha(x, .)$ as measures on (E, r) and combining this with (5.4-iv) and the equality of V^α and $\bar{U}^\alpha$ on H we see that $x_k \to x$ in the original topology of H. Therefore ρ and δ induce the same topology on H. It is now obvious that the identity map ψ of H into R is a homeomorphism with the desired properties. This establishes (5.5).

The next result is an immediate corollary of (5.5) and is our promised characterization of R up to a useless set.

5.6. Proposition. — *Let* H *satisfy* (5.3) *and* (5.4) *and suppose, in addition, that for each probability* μ *on* E, $t \to X_t$ *has left limits in* H *almost surely* P^μ. *Then* $R - \psi(H)$ *is useless where* ψ *is the homeomorphism in* (5.5).

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Manuscrit reçu le 23 décembre 1974.

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