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ANALYTIC FUNCTIONS IN A LACUNARY END OF A RIEMANN SURFACE

by Zenjiro KURAMOCHI

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Let R be a Riemann surface and $\{R_n\}(n = 0, 1, 2, \dots)$ be its exhaustion. We suppose Kerékjártó-Stoilow's topology S is defined on $R + B$, where B is the set of all ideal boundary components. Also we suppose Martin's topology M is defined over $R - R_0 + \Delta$ as follows:

$$\operatorname{dist}(p_1, p_2) = \sup_{z \in R_1 - R_0} \left| \frac{K(z, p_1)}{1 + K(z, p_1)} - \frac{K(z, p_2)}{1 + K(z, p_2)} \right| :$$

$p_1, p_2 \in R - R_0 + \Delta$, where $K(p_0, p) = 1$, $p_0 \in R_1 - R_0$ and Δ is the set of the ideal boundary points. We denote by Δ_1 the set of all minimal boundary points in Δ. Let $\mathfrak{p}$ be a boundary component. If there exists a sequence $\{z_i\}$ in $R - R_0$ such that $z_i \xrightarrow{S} \mathfrak{p}$ (convergence relative to S) and $z_i \xrightarrow{M} p$ (relative to M), we say p lies over $\mathfrak{p}$. We denote by $\nabla(\mathfrak{p})$ the set of Martin's points over $\mathfrak{p}$. Let G be an end of a Riemann surface R with null boundary. Let

$$F_i (i = 1, 2, \dots)(F_i \cap F_j = 0 \text{ for } i \neq j)$$

be a compact continuum in G such that $\{F_i\}$ clusters nowhere in R and $G' = G - F\left(F = \sum_i F_i\right)$ is connected. We call G' a lacunary end. Let $\mathfrak{p}$ be a boundary component of G. If there exists a determining sequence $\mathfrak{v}_n(\mathfrak{p})$ of $\mathfrak{p}$ such that $\partial\mathfrak{v}_n(\mathfrak{p})$ is a dividing cut and

$$\overline{\lim_n} \min_{z \in \partial \mathfrak{v}_n(\mathfrak{p})} G'(z, q_0) > 0 : q_0 \in G',$$

we say F is *completely thin* at $\mathfrak{p}$, where $G'(z, q_0)$ is a Green's function of G'. We proved.

THEOREM 1 [1]. — *Let* G *be an end of a Riemann surface* R *with null boundary. Let* F *be a completely thin set at a boundary component* $\mathfrak{p}$. *If there exists an analytic function*

$$w = f(z) : z \in G' = G - F$$

such that its value falls on the w-sphere and

$$\sup_w n(w) = n_0 < \infty,$$

then $\Delta_1 \cap \nabla(\mathfrak{p})$ *consists of at most* n_0 *number of points, where* $n(w)$ *is the number of times* w *is covered by* G'. The purpose of the present paper is to extend Theorem 1.

Let $U(w)$ be a$_n$ lower semicontinuous function and

$$U(w) \leqslant \frac{1}{2\pi}\int_{\partial C} U(\zeta)\frac{\partial}{\partial n} G(\zeta, w)\, ds$$

for any circle C in D, we call $U(w)$ a *quasisubharmonic in* D, where $G(\zeta, w)$ is a Green's function of C.

LEMMA 1. — *Let* Ω *be a domain in* R *with a relative boundary* $\partial\Omega$ *consisting of at most countably infinite number of analytic curves. Let* E *be a compact set on* $\overline{\Omega}$ *of positive capacity.* Let $w = f(z)$ *be an analytic function in* $\Omega + E$ *and* $\{\Omega_n\}(n = 0, 1, 2, \ldots)$ *be an exhaustion* i.e. $\overline{\Omega}_n$ *is compact in* Ω *and* $\partial\Omega_n$ *consists of a finite number of analytic curves. Let* $n^0_m(w)$ *be the number of points in* $f^{-1}(w) \cap \mathring{E}_m : \mathring{E}_m = \left\{z : \text{dist}\,(E, z) \leqslant \frac{1}{m}\right\}$. *Suppose there exists a number* m_0 *such that* $\sup_w n^0_{m_0}(w) = N < \infty$ *and* $\text{dist}\,(f(\Omega_n - \mathring{E}_{m_0}), f(E)) > 0$ *for any* n. *Let* $u(z)$ *be the harmonic measure of* E *relative to* Ω *and put* $U(w) = \sum_i u(z_i) : f(z_i) = w$. *Then* $U(w)$ *is quasi-subharmonic in* $C(f(\mathring{E}))$ *and* $U(w) \leqslant N$.

Proof. — Let $E_m : m = 1, 2, \ldots,$ be a closed set such that $E_m \subset \mathring{E}_m$, dist $(\partial E_m, E) > 0$, $E_m \downarrow E$ and ∂E_m consists of a finite number of analytic curves. Let $\{\Omega'_n\} : n = 1, 2, \ldots$ be an exhaustion of $\Omega - E$ in the direction of $\partial\Omega - E$ satisfying conditions.

a) $\partial\Omega'_n - E_m$ is compact in Ω and $(\Omega'_n \cap E_m)$ consists of a finite number of components for any n and m and $\Omega'_n \nearrow \Omega - E$ as $n \to \infty$.

b)
$$\partial\Omega'_n \cap \partial\Omega = \partial\Omega \cap E$$
and $\partial\Omega'_n \cap \Omega \cap E = \Omega \cap E$ for any n.

c)
$$\partial\Omega'_n - \mathring{E}^0_{m_0} = \partial\Omega_n - \mathring{E}^0_{m_0}$$
for any n. Since $f(z)$ is analytic on $\overline{\Omega'_n - E_m}$ by *a*)
$$\infty > \overset{*}{n} = \sup n(w)$$
in $f(\Omega'_n) - f(E_m)$, where $n(w)$ is the number of points in $\Omega'_n - E_m$ lying over w. Let $u_n(z)$ be harmonic measure of E relative to Ω'_n. Then $u'_n(z) \nearrow u(z)$. Let Ω' be a domain such that $\Omega' \subset \Omega'_n$, $\Omega' \cap E_m = \Omega'_n \cap E_m$. Let $\Omega' \nearrow \Omega'_n$. Then $\sup |u_n(z) - u'(z)|$ on $\Omega' - E_m \leqslant \sup u_n(z)$ on $\partial\Omega' - E_m \to 0$, where $u'(z)$ is a harmonic measure of E relative to Ω'. Hence for any $\varepsilon > 0$ we can find a domain $\Omega' \subset \Omega'_n$ such that *a'*) $\Omega' \cap E_m = \Omega'_n \cap E_m$. *b'*) $f(\partial\Omega' - E_m)$ intersects itself at a finite number of points $w_1, w_2, \ldots, w_k$ *c'*). Any subarc of $f(\partial\Omega' - E_m)$ is covered only once by $\partial\Omega' - E_m$ except $\sum_1^k w_i$ and
$$|u_n(z) - u'(z)| < \frac{\varepsilon}{n^*} \qquad \text{in} \qquad \Omega' - E_m.$$

Hence by *a*), *b'*), *c'*) $f(\partial\Omega' - E_m)$ divides $C(f(E_m))$ into a finite number of domains $\omega_1, \omega_2, \ldots, \omega_l$. Let $n(w)$ be the number of points in $\Omega' - E_m$ lying over w. Then $n(w) = n_i$ in ω_i and $n(w)$ jumps 1 in crossing $f(\partial\Omega')$. Let D be a circle in $C(f(E_m))$.

Case 1. D is contained in ω_i, then
$$U'(w) = \Sigma u'(z_i) \qquad (w) = f(z_i)$$
is harmonic in D.

Case 2. $D = (\omega_i + \omega_{i+1} + f(\partial\Omega')) \cap D$. Suppose

$$n_{i+1} \geqslant n_i.$$

Then $n_{i+1} = n_i + 1$. There exists a domain G' (or sum of domains denoted by G' also) of n_i leaves of disks and another domain G'' of one leaf such that $f(G'') = \omega_{i+1} \cap D$. Put $U''(w) = \Sigma u'(z_i) : f(z_i) = w : z_i \in G'$. Then $U''(w)$ is harmonic. Put $U'''(w) = u'(z) : w = f(z) : z \in G''$ and $U'''(w) = 0$ on $D - \omega_{i+1}$. Then since $u'(z) = 0$ on $\partial\Omega'$ and $U'''(w) = 0$ on $\partial\omega_i \cap D$ by $(\partial\omega_{i+1} \cap D) \subset f(\partial\Omega' - E_m)$ and

$$\partial\omega_i \cap f(E_m) = 0.$$

$U'''(w)$ is continuous and subharmonic in D and

$$U'(w) = U''(w) + U'''(w)$$

is continuous and subharmonic in D.

Case 3. $D - f(\Omega')$ consists of a finite number of domains, in this case similarly as before $U'(w)$ is continuous and subharmonic. Put $U'(w) = 0$ in $C(f(E_m))$. Then $U'(w)$ is continuous and subharmonic in $C(f(E_m))$. Now

$$0 \leqslant U_n(w) - U'(w) \leqslant \Sigma u_n(z_i) - u'(z_i) \leqslant n^* \cdot \frac{\varepsilon}{n^*} = \varepsilon$$

in $f(\Omega') - f(E_m)$. Let $\varepsilon \to 0$. Then $U'(w)$ uniformly $\to U_n(w)$ in $f(\Omega'_n) - f(E_m)$ and $U_n(w)$ is continuous and subharmonic in $C(f(E_m))$ by putting $U_n(w) = 0$ in $C(f(E_m))$. For any given number n by (c) there exists a number $m(n)$ such that

$$\text{dist}\,(f(\partial E_m) - f(E)) < \text{dist}\,((f(\Omega_n - \mathring{E}_{m_0}), f(E)) : m > m(n).$$

Then $f^{-1}(w) \cap \Omega'_n$ (for $w \in \overline{f(E_m)}$ and $m > m(n)$) consists of only points in $\mathring{E}_{m_0}$, whence $f^{-1}(w) : w \in \overline{f(E_m)} : m > m(n)$ consists of at most N number of points. By $u(z) \leqslant 1$, $U(w) \leqslant N$ in $\overline{f(E_m)}$. Since $U_n(w)$ is continuous and subharmonic in $C(f(E_m))$, by the maximum principle $U_n(w) \leqslant N$ in $C(f(E_m))$. Let $m \to \infty$. Then $U_n(w)$ is continuous and subharmonic and $\leqslant N$ in $C(f(E))$. Let $n \to \infty$. Then

$U_n(w) \nearrow U(w)$ by a) and $U(w)$ is lower semicontinuous and $\leqslant N$ in $C(f(E))$. Evidently

$$U(w) \leqslant \frac{1}{2\pi}\int_{\partial C} U(\zeta)\frac{\partial}{\partial n}G(\zeta, w)\,ds$$

for any circle C in $C(f(E))$. Hence $U(w)$ is quasisubharmonic and $\leqslant N$ in $C(f(E))$.

Lemma 1 is very simple but interesting. For example we apply it to the theory of value distribution. Then we have

Proposition. — *Let $w = f(z)$ be an analytic function in $(0 <)r < |z| \leqslant 1$ such that $|f(z)| \leqslant 1$ and $f(e^{i\theta})$ covers $|w| = 1$ n_0 times as θ varies from 0 to 2π. Then $f^{-1}(w)$: $|w| < 1$ consists at most $m\left(\leqslant \frac{n_0}{\alpha}\right)$ number of points in $r^{1-\alpha} < |z| < 1$.*

In fact, let $\Omega = \{1 > |z| > r\}$ and $E = \{|z| = 1\}$. Then $u(z) = 1 - \dfrac{\log|z|}{\log r}$. Let $z_i(i = 1, 2, \ldots)$ be a point in

$$r^{1-\alpha} < |z| < 1$$

such that $w = f(z_i)$. Then $u(z_i) > \alpha$ and

$$U(w) = \Sigma u(z_i) \leqslant n_0.$$

Hence we have the proposition.

Lemma 2. — *Let Ω be a domain in $R - R_0$ and let*

$$F_i(i = 1, 2, \ldots)$$

be a compact set clustering nowhere in R. Put

$$\Omega' = \Omega - F : F = \Sigma F_i.$$

Let $\varphi(\zeta)$ be a non negative continuous function on $\partial\Omega - F$. Let $U(z)$ be the least positive harmonic function in Ω' such that $U(\zeta) = \varphi(\zeta)$ on $\partial\Omega - F$. Then

$$U(z) = \frac{1}{2\pi}\int_{\partial\Omega - F}\varphi(\zeta)\frac{\partial}{\partial n}G(\zeta, z)\,ds,$$

where $G(\zeta, z)$ is a Green's function of Ω'.

Put $\Omega'_n = (\Omega' \cap R_n) - F$. Let $U_n(z)$ be a harmonic function in Ω'_n such that $U_n(z) = \varphi(\zeta)$ on $(\partial\Omega - F) \cap R_n$, $= 0$ on $(F \cap \partial\Omega) + (\partial R_n \cap \Omega) + (F \cap \Omega)$. Then $U_n(z) \nearrow U(z)$. Let $G_n(\zeta, z)$ be a Green's function of Ω'_n. Then

$$U_n(z) = \frac{1}{2\pi} \int_{(\partial\Omega - F) \cap R_n} \varphi(\zeta) \frac{\partial}{\partial n} G_n(\zeta, z)\, ds.$$

Since $\frac{\partial}{\partial n} G_n(\zeta, z) \nearrow \frac{\partial}{\partial n} G(\zeta, z)$ on $\partial\Omega$, we have Lemma 2.

Let $D_1 \supset D_2$ be two domains. Let U be a positive harmonic function in D_1. We denote by rU the greatest subharmonic function in D_2 vanishing on ∂D_2 not larger than U. Let V be a positive harmonic function in D_2 vanishing on ∂D_2 except at most a set of capacity zero. We denote by sV the least positive superharmonic function in D_1 larger than V. Then the following are well known.

rU *and* sV (*for* $sV < \infty$) *are harmonic and* $rsrU = rU$ *and* $srsV = sV$.

Let U *be minimal in* D_1. *Then if* $rU > 0$, $srU = U$ *and* rU *is minimal in* D_2.

Let V *be minimal in* D_2. *If* $sV < \infty$, $rsV = V$ *and* sV *is minimal in* D_1.

If $U_n \nearrow U$, $rU = \lim rU_n$.

In the sequel we suppose R is a Riemann surface with null boundary and Martin's topology M is defined over $R - R_0 + \Delta$ by $K(z, p) = \frac{G(z, p)}{G(p, p_0)}$, where p_0 is a fixed point in $R_1 - R_0$ and $G(z, p)$ is a Green's function of $R - R_0$. We remark there exist consts. M and N such that $M > G(z, p_0) > N > 0$ in $R - R_2$. Let G be an end in $R - R_0$ and let G' be a lacunary end such that

$$G' = G - F : F = \Sigma F_i.$$

Degree of irregularity $\delta(p) (p \in G + \Delta_1)$ at p. Let $G'(z, q_0)$ (q_0 is a fixed point in $G' \cap R_{n_0}$) be a Green's function of G'. We define $\delta(p)$ as follows:

$$\delta(p) = \varlimsup_{z \xrightarrow{M} p} G'(z, q_0) : q_0 \in G' \cap R_{n_0}.$$

We see at once $\delta(p) > 0$ for $p \in G'$ and $\delta(p) > 0 (p \in \Delta_1)$

if and only if there exists a sequence $\{z_i\}$ such that $G'(z, z_i) \to$ a positive harmonic function. Let $\delta'(p)$ the one defined with respect to $G'(z, q_0')$. Then since there exists an exhaustion $\{R_n\}$ such that $\partial R_n \cap F = 0$, there exists a const. K such that

$$\frac{1}{K}\,\delta(p) \leqslant \delta'(p) \leqslant K\,\delta(p) \qquad \text{for} \qquad p \in \Delta_1.$$

Let p^1 and p^2 in Δ_1. If there exists a sequence of curves $\{\gamma_i\}$ $(i = 1, 2, \ldots)$ with two end points p_i^1 and p_i^2 such that $p_i^1 \xrightarrow{M} p^1$, $p_i^2 \xrightarrow{M} p^2$ and γ_i tends to the ideal boundary of R and

$$\varlimsup_{i=\infty} \min_{z \in \gamma_i} G'(z, q_0) > 0,$$

we say p^1 and p^2 are *chained.* Let p^0 and p^{n_0}. If there exists $p^1, p^2, \ldots, p^{n_0-1}$ such that p^j and p^{j+1} are chained: $j = 0, 1, 2, \ldots, n_0 - 1$, we say p^0 and p^{n_0} are *kindred.* It is clear if p^i and p^j are kindred, p^i and p^j lie over the same boundary component. Kindredness does not depend on the choice of q_0.

Definition of $G(z, p)$ *and* $G'(z, p) : p \in G + \Delta_1$.

LEMMA 3. — *Suppose Martin's topology is defined on*

$$R - R_0 + \Delta,$$

G *is an end in* $R - R_0$ *and* $G' = G - F$ *be a lacunary end. Let* $G(z, z_i)$ *and* $G'(z, z_i)$ *be Green's functions of* $R - R_0$ *and* G' *respectively. Then:*

1) *Let* $\{z_i\}$ *be any sequence such that* $z_i \xrightarrow{M} p \in G + \Delta_1$. *Then* $G(z, z_i)$ *converges to a uniquely determined positive minimal harmonic function in* $R - R_0$ *denoted by* $G(z, p)$ *and* $G(z, p) = \alpha K(z, p)$, *where* $\alpha = 2\pi \Big/ \int_{\partial R_0} \frac{\partial}{\partial n} K(z, p)\, ds$ *and* $N' = \min_{z \in \partial R_1} G(z, p_0) < \alpha < M' = \max_{z \in \partial R_1} G(z, p_0)$.

2) *Let* $\{z_i\}$ *be a sequence such that* $z_i \xrightarrow{M} p \in G + \Delta_1$ *and*

$$G'(z, z_i) \to G'(z, \{z_i\}).$$

Then $G'(z, \{z_i\}) = \beta r G(z, p)$, *with* $0 \leqslant \beta < 1$ *and where the operation* r *concerns domains* $R - R_0$ *and* G'. *Let* $\{z_i\}$

be a sequence such that $z_i \xrightarrow{M} p \in G + \Delta_1$ *and*

$$G'(z_i, q_0) \to \delta(p) > 0.$$

Then $G'(z, z_i) \to$ *a uniquely determined positive minimal harmonic function denoted by* $G'(z, p)$ *and*

$$G'(z, p) \geqslant \frac{\delta(p)}{M} rG(z, p),$$

where $M = \max_{z \in \partial R_{n_0} + 1} G(z, q_0)$

Proof of 1. — Let $\{z'_i\}$ be a subsequence of $\{z_i\}$ such that $G(z, z'_i) \to$ a harmonic function $G(z)$. Then

$$G(z)/M' \leqslant K(z, p).$$

By the minimality of $K(z, p)$, $G(z) = \alpha K(z, p)$. On the other hand, by the compactness of $\partial R_0 \int_{\partial R_0} \frac{\partial}{\partial n} G(z)\, ds = 2\pi$. Hence $\alpha = 2\pi \Big/ \int_{\partial R_0} \frac{\partial}{\partial n} K(z, p)\, ds$. Now $\{z'_i\}$ is an arbitrary sequence for which $G(z, z_i)$ converges. Hence $G(z, z_i) \to$ a uniquely determined harmonic function denoted by $G(z, p)$.

Proof of 2. — Let $\{z_i\}$ be a sequence such that $G'(z, z_i) \to$ a harmonic function $G'(z, \{z_i\})$. Then by 1)

$$G'(z, \{z_i\}) \leqslant G(z, p)$$

and we have $G'(z, \{z_i\}) \leqq rG(z, p)$. By the minimality of $rG(z, p)$ $G'(z, \{z_i\}) = \beta rG(z, p) : 0 \leqslant \beta < 1$. Let $\{z'_i\}$ be a subsequence of $\{z_i\}$ such that $G'(z, z'_i)$ converges and $\lim_i G'(z'_i, q_0) = \delta(p)$. In this case β attains the greatest value β^* given by

$$\delta(p)/rG(q_0, p)$$

Now $\{z'_i\}$ is an arbitrary subsequence with

$$\lim G'(z'_i, q_0) = \delta(p).$$

Hence $G'(z, z_i) \to$ a uniquely determined positive minimal harmonic function in $G' - p$ denoted by $G'(z, p)$. Now

$\lim\limits_{i} G'(z_i, q_0) \geqslant \dfrac{\delta(p)}{M} G(q_0, p)$. Hence

$$G'(q_0, p) \geqslant \frac{\delta(p)}{M} G(q_0, p) \geqslant \frac{\delta(p)}{M} rG(q_0, p) \quad \text{and} \quad \beta^* \geqslant \frac{\delta(p)}{M}.$$

Thus we have 2).

We shall discuss the behaviour of Green's functions of a planar domain. Let Ω be a domain in the z-sphere such that Ω has a Green's function $G(z, p)$. Let t_0 be a fixed point in Ω and $v(t_0)$ be a neighbourhood of t_0 in Ω and put $\delta(p) = \overline{\lim\limits_{z \to p}} G(z, t_0) : p \in \overline{\Omega}$. Then $\delta(p)$ *is upper semicontinuous in* $\overline{\Omega}$ and $\delta(p) \leqslant \max\limits_{z \in \partial v(t_0)} G(z, t_0)$. We see $\delta(p) > 0 : p \in \partial\Omega$ if and only if p is irregular. We introduce Martin's topology over $\Omega + \Delta$ by $K(z, p^M) : p^M \in \Omega + \Delta$ with $K(t_0, p^M) = 1$. By Brelot's theorem [2] there exists only one point p^M on p for $\delta(p) > 0$ and p^M is minimal. We denote by $p^M = \varphi(p)$. Then also this implies $\varphi(p)$ is continuous at p with $\delta(p) > 0$. Clearly $K(z, p^M)$ is continuous with respect to p^M. Hence $K(z, \varphi(p))$ is continuous at p with $\delta(p) > 0$ and we denote p^M by p simply in the following. Let $\{z_i\}$ be a sequence such that $z_i \to p$, $G(z_i, t_0) \geqslant \varepsilon_0 > 0$. Then there exists a subsequence $\{z_i'\}$ with $G(z, z_i') \to$ a harmonic function $G(z)$. Then $G(z) \leqslant \dfrac{M}{\varepsilon_0} K(z, p) : M = \max\limits_{z \in \partial v(t_0)} G(z, t_0)$. By the minimality of $K(z, p)$ $G(z) = \alpha K(z, p) : 0 < \alpha < \infty$. Let $\{z_i\}$ be a sequence such that $z_i \to p$, $G(z_i, t_0) \to \delta(p)$. Then we see easily $G(z, z_i) \to$ a harmonic function $\tilde{G}(z) = \bar{\alpha} K(z, p)$ and $\bar{\alpha}$ is the maximal value and $\tilde{G}(z)$ is the limit of $\{G(z, z_i)\}$ such that $z_i \to p$ and

$$G(z_i, z) \to \overline{\lim_{w \to p}} G(w, z)$$

for any z. We make $\tilde{G}(z)$ correspond to p and denote it by $G(z, p) : p \in \partial\Omega : \delta(p) > 0$. Thus the domain of definition of p of $G(z, p)$ is extended to $\Omega + \{p \in \partial\Omega : \delta(p) > 0\}$: This fact means $G(z, p)$ *is upper semicontinuous with respect to* p. Let $p \in \overline{\Omega}$ with $\delta(p) > 0$. Then by $K(t_0, p) = 1'$ we have $G(z, p) = \delta(p) K(z, p)$. Let μ be a positive mass distribution over $\Omega + \{p \in \partial\Omega : \delta(p) > 0\}$: Then a potential

$$\int G(z, q)\, d\mu\,(q) \quad \text{and} \quad \delta(q)\mu(q)$$

are defined well. Then we have

LEMMA 4. — 1) *Let* $\{z_i\}$ *be a sequence such that* $z_i \to p$ *and* $G(z, z_i) \to$ *a harmonic function* $G(z, \{z_i\})$. *Then*

$$G(z, \{z_i\}) \leqslant G(z, p).$$

2) *Let* $v(p)$ *be a neighbourhood of* p, *then there exists a const.* L *such that* $G(z, p) < L$ *on* $C\, v(p)$.

3) *Let* $U(z)$ *be a potential* $U(z) = \int_{\overline{\Omega}} G(z, q)\, d\mu\,(q)$ *and* $\int d\mu(q) < \infty$.

If $G(z, p) \leqslant U(z)$, $\lim\limits_n \int_{v_n(p)} d\mu(q) \geqslant 1$:

$$v_n(p) = \left\{|z - p| < \frac{1}{n}\right\}.$$

Proof. — 1) is evident. We shall prove 2). Let $\{z_i\}$ be a sequence such that $G(z_i, t_0) \to \delta(p)$ and $z_i \to p$. Then $G(z, z_i) \to G(z, p)$. Let $v'(t_0) = \{z : |z - t_0| < r'\}$ such that $v'(t_0) \in \Omega$. Let $G'(z, z_i)$ be a Green's function in $\Omega - v'(t_0)$. Let $H_i(z)$ be the least positive harmonic function in

$$\Omega - v'(t_0)$$

such that $H_i(z) = G(z, z_i)$ on $\partial v'(t_0)$. Then

$$G(z, z_i) - H_i(z) = G'(z, z_i).$$

Since $\partial v'(t_0)$ is compact, $H_i(z) \to H(z)$, where $H(z)$ is the least positive harmonic function in $\Omega - v'(t_0)$ such that $H(z) = G(z, p)$. Whence $G'(z, z_i) \to$ a uniquely determined function denoted by $G'(z, p)$. On the other hand, there exists no singular minimal point on planar domains (this is equivalent to there exists no bounded minimal positive harmonic function). Whence $\sup\limits_z G(z, p) = \infty$. But

$$\sup_z H(z) \leqslant \max_{z \in \partial v(t_0)} G(z, p).$$

Hence $G'(z, p) > 0$. Let $v(t_0) = \{|z - t_0| < r\}$ such that $r > r'$ and $v(t_0) \subset \Omega$. Now

$$\min_{z \in \partial v(t_0)} G'(z, z_i) = N'_i \to N' = \min_{z \in \partial v(t_0)} G'(z, p)$$

and $\max_{z \in \partial v(t_0)} G(z, z_i) = M_i \to M = \max_{z \in \partial v(t_0)} G(z, p)$. Clearly

$$M_i \geqslant N'_i,\ G'(z, z_i)/N'_i \quad \text{and} \quad G(z, z_i)/M_i$$

have log singularities with coefficients $1/N'_i$ and $1/M_i$ respectively and $G'(z, z_i)/N'_i \geqslant G(z, z_i)/M_i$ on $\partial v(t_0)$. Hence by the maximum principle and by letting $i \to \infty$ we have $G'(z, p)/N' \geqslant G(z, p)/M$ in $Cv(t_0)$. Let $\tilde{\Omega} = z\text{-sphere} - v'(t_0)$ and let $\tilde{G}(z, p)$ be a Green's function of $\tilde{\Omega}$. Then

$$\tilde{G}(z, p) \geqslant G'(z, p).$$

Clearly there exists a const. L such that

$$LN'/M \geqslant \tilde{G}(z, p) \geqslant G'(z, p)$$

on $Cv(p)$. Hence $L \geqslant G(z, p)$ on $Cv(p)$ for any neighbourhood $v(p)$. Hence we have 2).

Proof of 3). *Case* 1. $p \in \Omega$. Let $v(p) = \{z : |z - p| < 1/n_0\}$ such that $v(p) \subset \Omega$. Then $G(z, q) = G'(z, q) + H(z, q)$ or $G(z, q)$ according as $q \in v(p)$ or $q \notin v(p)$, where $G'(z, q)$ is a Green's function of $v(p)$ and $H(z, q)$ and $G(z, q)$: $q \notin v(p)$ are least positive harmonic functions in $v(p)$ such that $H(z, q) = G(z, q)$ and $G(z, q) = G(z, q)$ on $\partial v(p)$. Since for any q and any neighbourhood $v(q)$ there exists a const. $L(q, v(q))$ such that $G(z, q) < L(q, v(q))$ on $Cv(q)$ and since $\partial v(p)$ is compact, there exists a const. L such that $H(z, q) \leqslant L$ and $G(z, q)(q \notin v(p)) \leqslant L$ on

$$v_{n_1}(p) \subset v(p) : n_1 > n_0.$$

Hence:

$$\begin{aligned} G'(z, p) \leqslant G(z, p) \leqslant U(z) &= \int_{v(p)} G'(z, q)\, d\mu(q) \\ &+ \int_{v(p)} H(z, q)\, d\mu\,(q) + \int_{cv(p)} G(z, q)\, d\mu\,(q) \\ &\leqslant \int_{v(p)} G'(z, q)\, d\mu\,(q) + L \int d\mu \quad \text{in } v_{n_1}(p). \end{aligned}$$

Let $H_n(z): n > n_0$, be a harmonic function in $v(p) - v_n(p)$ such that $H_n(z) = 0$ on $\partial v(p)$ and $= L'$ on $\partial v_n(p)$: $L' = L \int d\mu$. Then $U(z) \leqslant \int_{v(p)} G'(z, q)\, d\mu\,(q) + H_n(z)$ on $\partial v_n(p)$. Since $G'(z, p) = 0$ on $\partial v(p)$,

$$G'(z, p) \leqslant \int_{v(p)} G'(z, q)\, d\mu\,(q) + H_n(z)$$

on $\partial v(p) + \partial v_n(p)$. $G'(z, p)$ is harmonic and

$$\int_{v(p)} G'(z, q)\, d\mu\,(q) + H_n(z)$$

is superharmonic in $v(p) - v_n(p)$. By the maximum principle and by letting $n \to \infty$ $G'(z, p) \leqslant \int_{v(p)} G'(z, q)\, d\mu\,(q)$. This implies $1 \leqslant \int_{v(p)} d\mu\,(q)$. Now $v(p)$ is arbitrary and

$$\lim_n \int_{v_n(p)} d\mu \geqslant 1.$$

Case 2. $p \in \partial\Omega$. In this case p is irregular. Let $q \in \partial\Omega$ and $G(z, q) > 0$. Then by definition

$$G(z, q) = \lim_i G(z, q_i),$$

where $q_i \in \Omega$, $q_i \to q$, $G(q_i, t_0) \to \delta(q) = \overline{\lim_{z \to q}}\, G(z, t_0)$. Hence $G(z, q) = \delta(q)K(z, q)$: $K(t_0, q) = 1$. Since the greatest subharmonic minorant of Green's potential (in ordinary sense) $= 0$, we have by

$$G(z, p) \leqslant \int_{\partial\Omega} G(z, q)\, d\mu(q) + \int_{\Omega} G(z, q)\, d\mu\,(q)$$
$$G(z, p) \leqslant \int_{\partial\Omega} G(z, q)\, d\mu(q) \text{ i.e.}$$
$$\delta(p)K(z, p) \leqslant \int_{\Delta_1} \delta(q)K(z, q)\, d\mu(q).$$

Now $\delta(q)\mu(q)$ is a canonical representation, hence

$$\delta(p) \leqslant \delta(q)\mu(q)$$

and $\delta(p) \leqslant \int_{v_n(p)} d(\delta(q)\mu(q))$. On the other hand, $\delta(q)$ is upper semicontinuous. Hence $\lim_n \int_{v_n(p)} d\mu(q) \geqslant 1$.

Suppose an analytic function in a lacunary end G':

$$G' = G - F$$

of a Riemann surface R with null boundary such that

$$w = f(z) : z \in G'$$

falls on the w-sphere. We investigate the behaviour of $f(z)$ and the structure of Δ_1. Then

THEOREM 2. — *Suppose $f(G')$ does not cover a set E of positive capacity. Then:*

1) *Let $\{z_i\}$ be a sequence in G' such that $z_i \xrightarrow{M} p \in \Delta_1$ and $\underline{\lim}_i G(z_i, q_0) > 0$. Then $f(z_i) \to$ a point not depending on the choice of $\{z_i\}$. We denote it by $f(p)$.*

2) *If $\Delta_1 \cap \nabla(\mathfrak{p})$ consists of at most countably infinite number of points $\{p_i\}$ and $\delta(p_i) \geqslant \delta > 0$ for any i. Then $f(p_i) = f(p_j)$ for two kindred points p_i and p_j in $\Delta_1 \cap \nabla(\mathfrak{p})$.*

Proof of 1). — The complementary set of E relative to the w-sphere consists of domains. Let Ω be the one containing $f(G')$. Let $G^w(w, w_0)$ be a Green's function of Ω. Then $G^w(f(z), f(z_i)) \geqslant G'(z, z_i)$. Consider a sequence $\{z_i\}$ such that $z_i \xrightarrow{M} p$ and $\underline{\lim}_i G'(z_i, q_0) > 0$. Assume $f(z_i)$ has two limiting points t_l: $l = 1, 2, t_1 \neq t_2$ on $\Omega + \partial\Omega$. Then we can find two subsequences $\{z_i^{l'}\}$ of $\{z_i^l\}$ such that $z_i^{l'} \xrightarrow{M} p$, $f(z_i^{l'}) \to t_l$, $G'(z, z_i^{l'}) \to$ a harmonic function $G^{l'}(z)$ in G' and $G^w(w, f(z_i^{l'})) \to$ a positive harmonic function $G(w, \{f(z_i^{l'})\})$ in $\Omega - t_l$. Then by Lemma 4

$$(1) \quad 0 < G^{l'}(z) \leqslant G^w(w, \{f(z_i^{l'})\}) \leqslant G^w(w, t_l) : w = f(z),$$

where $G^w(w, t_l)$ is the function defined in Lemma 4. Now by Lemma 3.2, $G^{l'}(z) = \alpha_l i_{G'}^{R-R_0} G(z, p) : 0 < \alpha_l < \infty$, whence

$$(2) \qquad G^{1'}(z) = bG^{2'}(z) : 0 < b < \infty.$$

Let $v(t_l)$ be a neighbourhood of t_l such that

$$\overline{v(t_1)} \cap \overline{v(t_2)} = 0.$$

Then by Lemma 4 $G^w(w, t_l)$ is bounded in $Cv(t_l)$. Whence by (2) and (1) $G^{1'}(z)$ and $G^{2'}(z)$ are bounded in G' and $G^{1'}(z) = G^{2'}(z) = 0$. This is a contradiction. Hence we have 1). Because the following fact is well known: *Let ω be a domain* *tn a Riemann surface with null boundary and let* $U(z)$ *be a bounded harmonic function in* ω *with* $U(z)=0$ *on* $\partial\omega$, *then* $U(z)=0$.

For the proof of 2) we use following:

Proposition 1. — *Let* $\{z_i\}$ *be a sequence in* G' *such that* $z_i \xrightarrow{S} \mathfrak{p}$, $f(z_i) \to w^* \in \Omega + \partial\Omega$ *satisfying.*

a) $G'(z, z_i) \to$ *a positive harmonic function* $G'(z, \{z_i\})$ *in* G'.

b) $G^w(w, f(z_i)) \to$ *a positive harmonic function* $G^w(w, \{f(z_i)\})$ *in* $\Omega - w^*$. *Let* $\overset{*}{E}G'(z, \{z_i\})$ *be the least positive superharmonic function in* Ω *larger than* $G'(z, \{z_i\})$. *Then*

$$\overset{*}{E}G'(z, \{z_i\})$$

is minimal in $\Omega - w^*$ *and* $= \alpha G^w(w, w^*) : 0 < \alpha \leqslant 1$.

In fact by Lemma 4

$$G'(z, \{z_i\}) \leqslant G^w(f(z), \{f(z_i)\}) \leqslant G^w(f(z), w^*)$$

and by the minimality of $G^w(w, w^*)$ $\overset{*}{E}G'(z, \{z_i\}) = \alpha G^w(w, w^*)$ and $\overset{*}{E}G'(z, \{z_i\})$ is minimal in $\Omega - w^*$.

Proposition 2. — *Let* $\{z_i\}$ *be a sequence* $z_i \xrightarrow{M} p \in G' + \Delta_1$ *such that* $G'(z_i, q_0) \to \delta(p) > 0$. *Then by Lemma* 3

$$G'(z, z_i) \to G'(z, p),\ G(z, z_i) \to G(z, p).$$

By 1) *of Theorem* 2 $f(z_i) \to f(p)$ *and by Lemma* 4

$$G'(z, p) \leqslant \overline{\lim_i}\, G^w(w, f(z_i)) \leqslant G^w(w, f(p)) :$$
$$p \in \Delta_1 + G',\ \delta(p) > 0,\ f(z) = w.$$

Proof of 2) — By 1) of Theorem 2 $f(p_i) : p_i \in \Delta_1 \cap \nabla(\mathfrak{p})$ is defined. Let p_1 and p_2 be two points chained. Assume $f(p_1) \neq f(p_2)$. We can find a circle

$$C = \{w : |w - f(p_1)| < \frac{1}{2}|f(p_1) - f(p_2)|\}$$

and $C \cap \sum_{i=1}^{\infty} f(p_i) = 0$. By the definition there exists a sequence of curves γ_n with endpoints z_n^1 and z_n^2 such that

$z_n^1 \xrightarrow{M} p_1$, $z_n^2 \xrightarrow{M} p_2$ and $G'(z, q_0) \geqslant \delta_0 > 0$ on γ_n. Whence $f(z_n^1) \to f(p_1)$, $f(z_n^2) \to f(p_2)$. Consider $f(\gamma_n)$. Then $f(\gamma_n)$ intersects C for $n > n_0$. Let w_n be one of intersecting points with $f(z_n) = w_n$ and $z_n \in \gamma_n$. Since C is compact, there exists a limiting point w^* of $\{w_n\}$:

$$w^* \in \Omega + \partial\Omega, \; w^* \cap \sum_{i=1} f(p_i) = 0.$$

Hence we can find a subsequence $\{z_m\}$ of $\{z_n\}$ such that $z_m \subset \gamma_m$, $z_m \xrightarrow{S} \mathfrak{p}$, $f(z_m) \to w^*$ and $G'(z, z_m) \to$ a positive harmonic function $G'(z) \leqslant G^w(f(z), w^*)$ by Proposition 1) and $G(z, \{z_m\}) \to$ a positive harmonic function $G(z)$ in $R - R_0$ by $\int_{\partial R_0} \frac{\partial}{\partial n} G(z)\, ds = 2\pi$.

Suppose Martin's topology is defined over $R - R_0 + \Delta$ by $K(z, p)$: $K(p_0, p) = 1$ and $p_0 \in R_1 - R_0$. Then $G(z)$ is represented by a canonical distribution over $\Delta_1 \cap \nabla(\mathfrak{p})$, i.e. $G(z) = \sum_i a_i K(z, p_i)$, where $\sum_i a_i = G(p_0) < \infty$. By Lemma 3 $K(z, p_i) = G(z, p_i)/\alpha(p_i)$ and

$$G'(z) \leqslant G(z) = \sum_i (a_i/\alpha(p_i))G(z, p_i) \leqslant N'^{-1} \sum_i a_i G(z, p_i),$$

where $N' \leqslant \alpha(p_i)$ and $N' = \min_{z \in \partial R_1} G(z, p_0)$. Put

$$U_n(z) = \sum_{i=1}^{n} (a_i/\alpha(p_i))G(z, p_i).$$

Then $U_n(z) \nearrow G(z)$. By proposition 2

$$\begin{aligned} 0 < G'(z) \leqslant rG(z) &\leqq \sum_i \frac{a_i}{N'} rG(z, p_i) \\ &\leqslant (M/N') \sum_i (a_i/\delta(p_i))G'(z, p_i) \leqslant (M/N'\delta) \sum_i a_i G'(z, p_i) \\ &\leqslant (M/N'\delta) \sum_i a_i G^w(w, f(p_i)), \end{aligned}$$

where $M = \max_{z \in \partial R_{n_0+1}} G(z, q_0) : w = f(z)$.

By proposition 1

$$\overset{*}{E}G'(z) = \alpha G^w(w, w^*) \leqslant (M/N'\delta) \sum a_i G^w(z, f(p_i)) < \infty.$$

Now $w^* \in C$ and $C \cap \sum_i f(p_i) = 0$. $G^w(w, w^*)$ has mass at

w^*, on the other hand, the term on the right hand has no mass at w^*. This contradicts 3) of Lemma 4. Hence

$$f(p_1) = f(p_2) \quad \text{and} \quad f(p_i) = f(p_j)$$

for kindred points p_i and p_j. Thus we have 2).

THEOREM 3. — *If the spherical area of* $f(G') < \infty$, *then:*

1) *Let* $\{z_i\}$ *be a sequence in* G' *such that* $z_i \xrightarrow{M} p$ *and* $\underline{\lim}\, G'(z_i, q_0) > 0$. *Then* $f(z_i) \to a$ *point not depending on the choice of* $\{z_i\}$. *We denote it by* $f(p)$.

2) *Let* p^1 *and* p^2 *be two chained points. Then* $f(p^1) = f(p^2)$. *Hence* $f(p^i) = f(p^j)$ *for two kindred points* p^i *and* p^j.

At first we define $\tilde{R}$ and $\tilde{G}$ as follows. Since spherical area of $f(G') < \infty$, we can find a number n_0 such that spherical area of $f(G' \cap (R - R_{n_0})) < \frac{\pi}{4}$ and $\partial R_{n_0} \cap F = 0$. Evidently $f(G' \cap (R - R_{n_0}))$ does not cover a set E of positive capacity. Now $f(z)$ is analytic on ∂R_{n_0},

$$G' \cap (R - R_{n_0})$$

consists of a finite number of components $G'_1, \ldots, G'_k$ and ∂R_{n_0} consists of $\partial R^1_{n_0}, \ldots, \partial R^k_{n_0}$. We can find an arc Γ_j on $\partial R^j_{n_0}$ such that $f(\Gamma_j)$ is a simple arc,

$$f(\Gamma_i) \cap f(\Gamma_j) = 0 : i \neq j$$

and $\sum_j f(\Gamma_j) \cap E = 0$. Let $\mathscr{F}$ be the whole w-sphere. Put $\mathscr{F}' = \mathscr{F} - \sum f(\Gamma_j)$. Connect $\mathscr{F}'$ with $G'_1, \ldots, G'_k$ at an adequate side of $f(\Gamma_j)$ with Γ_j of G'_j so that

$$\mathscr{F}' + \sum^k f(G'_j)$$

may be a connected covering surface. By deforming R_{n_0}

$$\mathscr{F}' + \sum^k G'_j$$

can be considered a domain and $\mathscr{F}' + G \cap (R - R_{n_0})$ can be considered an end $\tilde{G}$ of another Riemann surface $\tilde{R}$ with null boundary. Now $\partial\tilde{G}$ consists of $\partial R_{n_0} - \sum^k \Gamma_j$

and the other side of $f(\Gamma_j)(j = 1, 2, \ldots, k)$ where $\mathcal{F}'$ and G'_j are not connected. Put $\tilde{G}' = \mathcal{F}' + (G' \cap (R - R_{n_0}))$. Then $f(z)$ can be continued analytically into $\mathcal{F}'$ by putting $f(z) =$ projection of z over w-sphere, which we also denote by $f(z) : z \in \tilde{G}'$. Let $\tilde{p}_0$ and $\tilde{q}_0$ be points in $G' \cap (R - R_{n_0})$. Then Martin's topology M will be defined over $\tilde{G}$. Then $\tilde{M}$-top. and M—top (given originally on $R - R_0 + \Delta$) are isomorphic on $(R - R_{n_0}) + \Delta$ and the minimality does not change. Also let $\tilde{G}'(z, q_0)$ be a Green's function of $\tilde{G}'$: $\partial R_n \cap F = 0$, there exists a const. K such that

$$G'(z, q_0)/K \leqslant \tilde{G}'(z, \tilde{q}_0) < KG'(z, q_0)$$

in $G' \cap (R - R_{n_0})$ and $k^{-1}\delta(p) < \tilde{\delta}(p) < K\delta(p)$ for $p \in \Delta_1$, where $\tilde{\delta}(p)$ is defined in $\tilde{G}'$ relative to $\tilde{q}_0$. Put

$$^*\tilde{G}' = \tilde{G}' - E_{\mathcal{F}},$$

then $f(^*\tilde{G}')$ does not cover a set E of positive capacity, where $E_{\mathcal{F}}$ is the set of $\mathcal{F}$ over E.

Proof of 1). — So long as we investigate $f(z)$ in a neighbourhood of the ideal boundary of R, we can consider $^*\tilde{G}'$ instead of G'. Then we have at once 1) by 1) of Theorem 2.

Proof of 2). — For the purpose we consider only

$$G' \cap (R - R_{n_0})$$

such that spherical area of $f(G' \cap (R - R_{n_0})) < \frac{\pi}{4}$.

$$G \cap (R - R_{n_0})$$

consists of a finite number of ends. Let $^\bullet G$ be one of them and put $^\bullet G' = {}^\bullet G - F$. Let $^\bullet G'(z, q'_0)$ be Greens function of $^\bullet G'$. Then there exists a const. K such that

$$(3) \qquad \frac{1}{K} G'(z, q_0) \leqslant {}^\bullet G'(z, q'_0) \leqslant KG'(z, q_0)$$

in $G' \cap (R - R_{m_0})$, where q_0 and $q'_0 \in G' \cap R_{m_0-1}$.

Hence $\overline{\lim_n} \min_{z \in \gamma_n} {}^{\bullet}G'(z, q_0') > 0$ for $\{\gamma_n\}$ defining chainedness of points. Hence for simplicity we denote ${}^{\bullet}G$, ${}^{\bullet}G'$, ${}^{\bullet}G'(z, q_0')$ by G, G' and $G'(z, q_0')$.

By Evans's [5] theorem there exists a positive harmonic function $U(z)$ in $G' = G - F$ such that

1) $U(z) = 0$ on $\partial G + F$, $D(\min(M, U(z)) = 2\pi M$,

$$\int_{\partial\Omega_L} \frac{\partial}{\partial n} U(z)\, ds = 2\pi$$

for almost $L < \infty$, where $\Omega_L = \{z \in G' : U(z) > L\}$.

2) $U(z) \to \infty$ as $z \to B$ in any

$$G_\delta = \{z \in G' : G'(z, q_0) > \delta\} :$$

$\delta > 0$. Ω_L consists of at most countably number of domains. Let Ω_L' be one component of Ω_L. Then Ω_L' is a domain in a surface with null boundary, whence $\sup_{z \in \Omega_L} U(z) = \infty$. Since spherical area of $f(G') < \frac{\pi}{4}$, by 1) we see by length and area's method there exists a sequence $L_i : i = 1, 2, \ldots$ such that $L_i \nearrow \infty$ and spherical length of

$$f(\partial\Omega_{L_i}) = \varepsilon_i \to 0 \quad \text{as} \quad i \to \infty.$$

Let $\{\gamma_n\}$ be a sequence of curves defining the chainedness of p_1 and p_2. Then $\overline{\lim_n} \min_{z \in \gamma_n} G'(z, q_0) > 0$, and there exists a subsequence $\{\gamma_m\}$ of $\{\gamma_n\}$ such that

$$\min_{z \in \gamma_m} G'(z, q_0) > \delta > 0.$$

Since $\gamma_m \to$ boundary of R, by 2) for any given L_i, there exists a number $m(L_i)$ such that $U(z) > L_i$ on

$$\gamma_m : m > m(L_i).$$

Hence for any L_i there exists $m(L_i)$ such that

$$\Omega_{L_i} \supset \gamma_m : m > m(L_i)$$

and there exists only one component $\Omega_i'(\gamma_m)$ of Ω_{L_i} such that $\Omega_i'(\gamma_m) \supset \gamma_m$ where $\Omega_i'(\gamma_m)$ depends on γ_m.

By Evans's theorem there exists a harmonic function $V(z)$ in G such that

1) $V(z) = 0$ on ∂G, $D(\min(M, V(z)) = 2\pi M$,

$$\int_{\partial D_M} \frac{\partial}{\partial n} V(z)\, ds = 2\pi$$

for M, where $D_M = \{z \in G : V(z) < M\}$.

2) $V(z) \to \infty$ as $z \to$ boundary of R. Similarly as $U(z)$, there exists a sequence M_j such that spherical length of

$$f(\partial D_{M_j} \cap G) = \varepsilon_j \to 0$$

as $j \to \infty$. Since $\Omega'_i(\gamma_m) = \lim\limits_j \Omega'_i(\gamma_m) \cap D_{M_j}$, there exists a number M_j such that $\Omega'_i(\gamma_m) \cap D_{M_j} \supset \gamma_m$. Put

$$\Omega'_{i,j}(\gamma_m) = \Omega'_i(\gamma_m) \cap D_{M_j}.$$

Since $\Omega'_{i,j}(\gamma_m)$ is compact boundary of

$$f(\Omega'_{ij}(\gamma_m)) \subset f(\partial \Omega'_{ij}(\gamma_m))$$

and the spherical length of $f(\partial\Omega'_{ij}(\gamma_m)) < \varepsilon_i + \varepsilon_j$. $f(\partial\Omega'_{ij}(\gamma_m))$ divides the w-sphere into a number of domains G^w_1, G^w_2, ... Since the spherical length of $f(\partial\Omega'_{ij}(\gamma_m)) < \varepsilon_i + \varepsilon_j < \frac{1}{4}$, there exists only one domain with spherical area

$$\geqslant 4\pi - (\varepsilon_i + \varepsilon_j)^2.$$

We denote such domain by $\overset{*}{G}$. Then since spherical area of $f(\Omega'_{ij}(\gamma_m)) < \frac{\pi}{4}$, $f(\Omega'_{ij}(\gamma_m)) \cap \overset{*}{G} = 0$ and $f(\Omega'_{ij}(\gamma_m))$ is contained in a semisphere and the spherical diameter of $f(\gamma_m) \leqslant$ spherical diameter of $f(\Omega'_{ij}(\gamma_m)) < \varepsilon_i + \varepsilon_j$.

$$\text{Let } j \to \infty.$$

Then spherical diameter of $f(\gamma_m) < \varepsilon_i$. Let z^1_m and z^2_m be endpoints of γ_m. Then $fz^1_m \to f(p_1)$ and $fz^2_m \to f(p_2)$ as $m \to \infty$. Let $m \to \infty$ and then $i \to \infty$. Then $f(p_1) = f(p_2)$ for chained points p_1 and p_2. Thus we have 2).

Theorem 4. — *Let $\tilde{G} \supset G$ be two ends of a Riemann surface R with null boundary and let $G' = G - F$ be a* *lacunary domain. Let* E_z *be a compact set of positive capacity in* $\tilde{G}-G$. *Suppose an analytic function* $f(z)$ *in* $\tilde{G}-F$ *and there exists a neighbourhood* $v(E_z)$ *of* E_z *with the property:* $f(z)$ *is univalent in* $v(E_z)$, $f(\tilde{G}-F-v(E_z))$ *does not cover* $E=f(E_z)$ *(clearly* E *is of positive capacity). Let* $u(z)$ *be a harmonic mesure of* E_z *with respect to* $\tilde{G}-F-E_z$. *Suppose Martin's topology* M *is defined over* $R-R_0+\Delta$. *Let* $G'(z, q_0)$ *be a Green's function of* G' *and let*

$$\delta(p): p \in G'+\Delta_1 = \overline{\lim_{z \xrightarrow{M} p}}\, G'(z, q_0): q_0 \in G' \cap R_{n_0}.$$

Then by theorem 2 $f(z_i) \to f(p)$ *for* $z_i \xrightarrow{M} p$ *and*

$$\underline{\lim}\, G'(z_i, q_0) > 0.$$

Then

1) *Let* $\{z_i\}$ *be a sequence such that* $z_i \xrightarrow{M} p \in \Delta_1$ *and* $\underline{\lim}\, G'(z_i, q_0) > 0$. *Then* $f(z_i) \to f(p)$ *and there exists a uniquely determined connected piece* ω *over* $|w-f(p)|<r$ *such that* $f(z_i) \in \omega$ *for* $i \geqslant i_0$ *and* $f(z_i) \to f(p)$.

2) *Let* $u(p) = \overline{\lim_{z \xrightarrow{M} p}}\, u(z)$ $p \in \Delta_1$. *Let* $\{z_i\}$ *be a sequence such that* $z_i \xrightarrow{M} p$ *and* $u(z_i) \to u(p) > 0$. *Let* $G^{\omega}(z, z_i)$ *be a Green's function of* ω. *Then* $G^{\omega}(z, z_i) \to$ *a unique positive minimal harmonic function* $G^{\omega}(z, p)$ *and*

$$u(p) = \int_{\partial^1\omega} u(\zeta) \frac{\partial}{\partial n} G^{\omega}(\zeta, p)\, ds,$$

where $\partial^1\omega$ *is the part of* $\partial\omega$ *such that*

$$f(\partial^1\omega) \subset \{w: |w-f(p)| = r\}.$$

3) *Let* w *be a point and let* $p_i \in \Delta_1$ *with* $\delta(p_i) > 0$ *and* $f(p_i) = w$ *and let* $q_j \in \tilde{G}' = \tilde{G}-F-E_z$ *with* $f(q_j) = w$. *Then*

$$\Sigma u(p_i) + \Sigma u(q_j) \leqslant 1 \quad \text{for any} \quad w.$$

Case 1. $w_0 \in E$. Let $0 < r < \operatorname{dist}(w_0, f(\partial v(E_z))$ $(>0$ by the univalency of $f(z)$ in $v(E_z))$. The part of $\tilde{G}-F-E_z$ over $|w-w_0|<r$ consists of a most countably infinite

number of domains (connected pieces). Let $\{\omega'\}$ be the set of connected pieces contained in $\varphi(E_z)$ and let $\omega_i : i = 1, 2, \ldots$ be pieces except $\{\omega'\}$. Then $\omega_i \cap \varphi(E_z) = 0$. By the assumption, there exists no point z in $\tilde{G} - E_z - F$ such that $f(z) = w_0$. Further let $p \in \Delta_1$, then for any sequence $\{z_i\}$ with $z_i \to p$, there exists a number i_0 such that $z_i \notin \varphi(E_z)$ for $i > i_0$. If there exists a point $p \in \Delta_1$ such that $\delta(p) > 0$, $f(p) = w_0$, there exists a certain ω_j containing z_i (in this case clearly w_0 is an irregular point of the domain $=$ w-sphere $- \overline{f(\tilde{G} - F - E_z)}$. Let ω be one of $\{\omega_i\}$. Then by $\overline{\varphi(E_z)} \cap \overline{f^{-1}(\omega)} = 0$ it is proved similarly as Lemma 1 $U^{\omega}(w) = \Sigma u(z_i) : f(z_i) = w,\ z_i \in \overline{\omega}$.

Case 2. $w_0 \notin E$. The part of $\tilde{G} - E_z - F$ over

$$|w - w_0| < \text{dist}\,(E, w_0)$$

consists of connected pieces $\omega_i (i = 1, 2, \ldots)$. In this case ω_i does not tend to E_z by the univalency of $f(z)$ in $\varphi(E_z)$. In both cases it is sufficient to consider only $\omega_i : i = 1, 2, \ldots$ Let ω be one of $\{\omega_i\}$. Then ω is compact or non compact in $\tilde{G} - E_z - F$ and $\partial\omega$ consists of $\partial^1\omega$ and $\partial^2\omega$ such that $f(\partial^1\omega) \subset \{w : |w - w_0| = r\}$ and $\partial^2\omega = \partial\omega - \partial^1\omega$. Then $u(z)$ is harmonic on $\partial^1\omega$ and > 0 on $\partial^1\omega - F$ and $u(z) = 0$ on $\partial^2\omega$ and $U^{\omega}(w)$ is quasisubharmonic in $|w - w_0| < r$ and by Lemma 1.

$$\sum_i U_i^{\omega}(w) \leqslant U(w) \leqslant 1.$$

Proof of 1). — There exists a const. K such that

$$KG'(z, q_0) > u(z) > \frac{1}{K} G'(z, q_0)$$

in $G' \cap (R - R_l) : R_l \to q_0$.

Hence without loss of generality we can suppose

$$u(z_i) \geqslant \delta > 0$$

and $|f(z_i) - w_0| < \frac{r}{2}$. Suppose a connected piece $\omega \to z_i$.

Then by Lemma 2

$$u(z_i)=\frac{1}{2\pi}\int_{\partial^1\omega}u(\zeta)\frac{\partial}{\partial n}G^{\omega}(\zeta,z_i)\,ds,$$

where $G^{\omega}(\zeta, z)$ is a Green's function of ω.

Let $G^{w}(w, \eta)$ be Green's function of $|w-w_0|<r$. Then by $G^{w}(f(z), f(z_i)) \geqslant G^{\omega}(z, z_i)$

$$\text{(4)}\qquad \frac{\partial}{\partial n}G^{w}(f(\zeta),f(z_i)) \geqslant \frac{\partial}{\partial n}G^{\omega}(\zeta,z_i)\geqslant 0$$

on $\partial^1\omega$. Let $\omega_n=\omega\cap R_n$, then $\omega_n\nearrow\omega$. Hence by considering ω_n we have similarly as Lemma 2

$$\int_{\partial^1\omega}u(\zeta)\frac{\partial}{\partial n}G^{w}(f(\zeta),f(z_i))\,ds$$
$$=\int_{|w-w_0|=r}U^{\omega}(\eta)\frac{\partial}{\partial n}G^{w}(\eta,f(z_i))\,ds.$$

On the other hand, there exists a const. K' such that

$$\text{(5)}\qquad \frac{\partial}{\partial n}G^{w}(\eta,w)\leqslant K'\frac{\partial}{\partial n}G^{w}(\eta,w_0)$$

on

$$|\eta-w_0|=r\quad\text{for}\quad|w-w_0|<\frac{r}{2}.$$

Hence

$$\text{(6)}\qquad \delta<u(z_i)=\frac{1}{2\pi}\int_{\omega'\partial}u(\zeta)\frac{\partial}{\partial n}G^{\omega}(\zeta,z_i)\,ds$$
$$\leqslant\frac{K'}{2\pi}\int U^{\omega}(w_0+re^{i\theta})\,d\theta\leqslant K'$$

Assume there exist $m\left(>\frac{K'}{\delta}\right)$ number of connected pieces $\omega_i:\ i=1,2,\dots,m$ containing at least one z_i of $\{z_i\}$. Then by (6) and $1\geqslant\sum_i U^{\omega_i}(w)$

$$m\delta\leqslant\frac{1}{2\pi}K'\sum_i\int U^{\omega_i}(w_0+re^{i\theta})\,d\theta\leqslant K'.$$

This is a contradiction. Hence there exists at least one ω containing a subsequence $\{z_i'\}$ of $\{z_i\}$. Let $\{z_i''\}$ be a subsequence of $\{z_i'\}$ such that $u(z_i'') \to a(>0)$, $G^{\omega}(z, z_i'') \to$ a harmonic function $G^{\omega}(z, \{z_i''\})$.

Then by (4), (5), (6) and by Lebesgue's theorem

$$(7) \quad 0 < \delta \leqslant \lim_i u(z_i'') = \lim_i \frac{1}{2\pi} \int_{\partial\omega} u(\zeta) \frac{\partial}{\partial n} G^{\omega}(\zeta, z_i'')\, ds$$
$$= \frac{1}{2\pi} \int_{\partial\omega} u(\zeta) \frac{\partial}{\partial n} G(\zeta, \{z_i''\})\, ds$$

and

$$G^{\omega}(z, \{z_i''\}) > 0.$$

Put $\omega' = \omega \cap (R - R_0)$. Then $\omega' \subset \omega$. Since $\omega - \omega'$ is compact and $G^{\omega}(z, z_i'') \leqslant \tilde{G}(z, z_i'')$ is uniformly bounded on $\omega - \omega'$ for $i \geqslant i_0$, the convergence of $\{G^{\omega}(z, z_i'')\}$ implies $G^{\omega'}(z, z_i'') \to$ a positive harmonic function $G^{\omega'}(z, \{z_i''\})$ and

$$G^{\omega}(z, \{z_i''\}) = \overset{\omega}{\underset{\omega'}{S}}\, G^{\omega'}(z, \{z_i''\}) > 0,$$

where $G^{\omega}(z, z_i)$ and $\tilde{G}(z, z_i)$ are Green's function of ω' and $\tilde{G}$ respectively. We suppose Martin's is defined over $\overline{R} - R_0 \supset G$. Now $\omega' \subset R - R_0$ and by Lemma 4

$$G^{\omega'}(z, \{z_i''\}) = \alpha r K(z, p) : 0 < \alpha < 1.$$

Hence $0 < G^{\omega}(z, \{z_i''\}) = \alpha s r K(z, p)$ and $rK(z, p)$ and $G^{w}(z, \{z_i''\})$ is minimal (where r is relative to $R - R_0$, ω'; s relative to ω, ω') (8). Assume there exists another connected piece ω^* containing a subsequence $\{z_j\}$ of $\{z_i\}$. Then as above we can find a subsequence $\{z_j'\}$ of $\{z_j\}$ such that (for r, s relative to $R - R_0$, $\omega^{*\prime}$, ω^*, with

$$\omega^{*\prime} = \omega^* \cap (R - R_0))$$
$$(9) \quad 0 < G^{\omega^*}(z, \{z_j'\}) = srK(z, p) \text{ and } rK(z, p) > 0,$$

It is well known for minimal function $V(z)$ in $R - R_0$ if $rV > 0$ (relative to $R - R_0$ and D) $rV(z) = 0$ (relative to $R - R_0$ and CD for any domain D in $R - R_0$). Hence (8) contradicts (9). Thus there exists only one connected piece ω contains z_i for $i \geqslant i_0$.

Proof of 2). — Let z_i be a sequence such that $z_i \xrightarrow{M} p$, $\lim_i u(z_i) = u(p)$. Then $\varliminf_i G(z_i, q_0) \geqslant \dfrac{u(p)}{K} > 0$. Hence by

(1) of this theorem z_i is contained in the only one connected piece ω and by (8)

$$(10) \qquad u(p) = \frac{1}{2\pi} \int_{\partial'\omega} u(\zeta) \frac{\partial}{\partial n} G^{w}(z, \{z_i\})\, ds$$

and $G^{\omega}(z, \{z_i\})$ is the function when the value

$$G^{\omega}(z, \{z_i\})/srK(z, p),$$

(with r relative to $R - R_0$, ω' and s relative to ω, ω') attains the maximal value and the function $G'(z, \{z_i\})$ is uniquely determined. We denote it by $G^{\omega}(z, p)$. Thus we have 2).

Proof of 3. — For $p \in \Delta_1$ and $u(p) > 0$, there exists a uniquely determined connected piece ω(over $|w - f(p)| < r$) containing a sequence $z_i \xrightarrow{M} p$ and $\underline{\lim}\, u(z_i) > 0$. In this case we say ω contains p.

Case 2. $w_0 \notin E$. Let ω be a connected piece over

$$|w - w_0| < \frac{1}{2} \operatorname{dist}(w_0, E).$$

Let $p_i \in \Delta_1 : f(p_i) = w_0$ be a point contained in ω. Then $G^{\omega}(z, p_i)$ is minimal and $\leqslant G^{w}(f(z), w_0) = \lim\limits_{i} G^{w}(f(z), f(z_i))$. Let q_j be a point in ω such that $f(q_j) = w_0$. Then $G^{\omega}(z, q_j)$ is minimal in ω and $\leqslant G^{w}(f(z), w_0)$. Hence

$$G^{w}(f(\zeta), w_0) \geqslant \sum_i G^{\omega}(\zeta, p_i) + \sum_j G^{\omega}(\zeta, q_j).$$

Clearly $u(q_j) = \dfrac{1}{2\pi} \displaystyle\int u(\zeta) \frac{\partial}{\partial n} G^{\omega}(\zeta, q_j)\, ds.$ Hence by (10)

$$(11) \qquad \int_{|W - W_0| = r} U^{\omega}(\zeta) \frac{\partial}{\partial n} G^{w}(\zeta, w_0)\, ds \geqslant \sum_i u(p_i) + \sum_j u(q_j).$$

Summing up over all connected pieces over $|w - w_0| < r$. Then by $\sum\limits_i U^{\omega_i}(w) \leqslant U(w) \leqslant 1$ we have

$$(12) \qquad 1 \geqslant \sum_i u(p_i) + \sum_j u(q_j).$$

Case 1. $w_0 \in E$. In this case there exists no point q_j in ω such that $f(q_j) = w_0$. It is sufficient to consider $\omega_1, \omega_2, \ldots$, remarked at the top of Theorem 4. Hence similarly as case 2) we have

$$(12) \qquad 1 \geqslant \sum_i u(p_i).$$

Thus we have 3).

THEOREM 5. — *Let* G *be an end of a Riemann surface with null boundary. Let* G′ *be a lacunary end:* $G' = G - F$. *Let* $f(z)$ *be an analytic function in* G′ *and on* ∂G. *If* $f(G')$ *does not cover a set* E *of positive capacity, or spherical area of* $f(G') < \infty$, *then there exists a const.* K *not depending on* w *such that*

$$\Sigma \delta(p_i) \leqslant K,$$

where $f(p_i) = w$ *and* $p_i \in \Delta_1$.

Proof. — Suppose $f(G')$ does not cover a set E of positive capacity. Let $\mathcal{S}$ be w-sphere. Let Γ be an arc on ∂G such that $f(\Gamma)$ is a simple arc and $f(\Gamma) \cap E = 0$. Put $\mathcal{S}' = \mathcal{S} - f(\Gamma)$. Then we can connect $\mathcal{S}'$ with G′ at Γ (at adequate side of $f(\Gamma)$) so that we may have a prolonged surface $\tilde{G} = \mathcal{S}' + (G - F)$ and $G + \mathcal{S}'$ may be an end of another Riemann surface $\tilde{R}$ with null boundary. Let $E_{\mathcal{S}}$ be the set of $\mathcal{S}$ over E and put

$$\tilde{G}' = \tilde{G} - E_{\mathcal{S}} : \ E_{\mathcal{S}} \subset \mathcal{S}'.$$

For the case [spherical area of $f(G')$] $< \infty$, we can define $\tilde{G}$, $\tilde{G}'$ and $\tilde{R}$ as above (see the proof of Theorem 3). Hence $f(z)$ can be continued analytically into $\tilde{G}$. Then since $f(z)$ is univalent in neighbourhood $v(E_{\mathcal{S}})$, $1 \geqslant U(w) = \sum_i u(z_i)$:

$$f(z_i) = w : z_i \in \tilde{G}',$$

where $u(z)$ is a harmonic measure of $E_{\mathcal{S}}$ relative to $\tilde{G}'$. Since $\partial R_n \cap F = 0$, there exists a const. K such that $\frac{1}{K} G'(z, q_0) \leqslant u(z) \leqslant KG'(z, q_0)$ in $(R - R_{m_0}) \cap G'$:

$$G' \cap R_{m_0} \to q_0,$$

where $G'(z, q_0)$ is a Green's function of $G' \subset \tilde{G}'$. Now $f(\tilde{G}')$ does not cover E. Hence by Theorem 4 we have Theorem 5.

Corollary 1. — *Suppose spherical area of $f(G') < \infty$.*

1) *Let $p_1, p_2, \ldots$ be kindred points of p_1. Then there exists a const.* K(*defined in Theorem* 5) *such that* $\sum_i \delta(p_i) \leqslant K$.

2) *If* $F = \Sigma F_i$ *is completely thin at* $\mathfrak{p}$, *then* $\Delta_1 \cap \nabla(\mathfrak{p})$ *consists of at most* m *points* (*with:*

$$m \leqq \frac{K}{\delta} : \quad \delta = \overline{\lim_n} \min G'(z, q_0)$$

on $\partial \mathfrak{v}_n(p) \leqslant K$).

1) Is evident by Theorem 3 and 4.

Proof of 2). — Let $p \in \Delta_1 \cap \nabla(\mathfrak{p})$. Then there exists a path Γ tending to p. Γ must intersect $\partial \mathfrak{r}_n(\mathfrak{p})$, where $\mathfrak{r}_n(\mathfrak{p})$ is a determining sequence of $\mathfrak{p}$ and $\partial \mathfrak{r}_n(\mathfrak{p})$ is a dividing cut such that $\overline{\lim_n} \min_{z \in \partial \mathfrak{r}_n(\mathfrak{p})} G'(z, q_0) > \delta_0 > 0$. Whence $\delta(p) \geqslant \delta_0$. Also any two points in $\Delta_1 \cap \nabla(\mathfrak{p})$ are clearly chained. Hence by Theorem 3 and 1) of corollary 1 we have 2).

Corollary 2. — *Suppose $f(G')$ does not cover set* E *of positive capacity and* $\Delta_1 \cap \nabla(\mathfrak{p})$ *consists of at most countably infinite number of points* p_i *with* $\delta(p_i) > \delta_0 > 0$.

1) *Let $p_1, p_2, \ldots, p_m$ be a set of kindred points. Then there exists a const.* K *such that* $m\left(\leqslant \dfrac{K}{\delta_0}\right)$.

2) *If* F *is completely thin at* $\mathfrak{p}$, *then* $\Delta_1 \cap \nabla(\mathfrak{p})$ *consists of at most* m *points, where* m *is the integer given in Corollary* 1.

By corollary 1 *and* 2, *we have at once:*

Corollary 3. — *Let* G *be an end of a Riemann surface with null boundary. Suppose* F *is completely thin at* $\mathfrak{p}$.

1) *If the harmonic dimension of* $\mathfrak{p}$ *is countably infinite (this is equivalent* $\Delta_1 \cap \nabla(\mathfrak{p})$ *consists of countably infinite number of points) and* $\delta(p_i) \geqslant \delta_0 > 0$, *then there exist no*

analytic functions in $G' = G - F$ *such that* $f(G')$ *does not cover a set of positive capacity.*

2) *If the harmonic dimension of* $\mathfrak{p}$ *is infinite, then there exist no analytic functions in* G' *with spherical area of*

$$f(G') < \infty.$$

Remark. — The ameliorations of this paper appear [6].

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