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## ON A GENERALIZATION OF DE-RHAM LEMMA

by Kyoji SAITO

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In this short note, we give a proof of a theorem (cf. § 1) which is a generalization of a lemma due to de-Rham [1] and which was announced and used in [2].

As no proof of this theorem was available in the literature, Lê Dũng Tráng pushed me to publish it: I am grateful to him.

### 1. Notations and formulations of the theorem.

Let  $R$  be a noetherian commutative ring with unit. The profondeur of an ideal  $\mathfrak{A}$  of  $R$  is the maximal length  $q$  of sequences  $a_1, \dots, a_q \in \mathfrak{A}$  with:

i)  $a_1$  is a non-zero-divisor of  $R$ .

ii)  $a_i$  is a non-zero-divisor of  $R/a_1R + \dots + a_{i-1}R$ ,  $i=2, \dots, q$ .

Let  $M$  be a free  $R$ -module of finite rank  $n$ . We denote by  $\bigwedge^p M$  the  $p$ -th exterior product of  $M$  (with  $\bigwedge^0 M = R$  and  $\bigwedge^{-1} M = 0$ ).

Let  $\omega_1, \dots, \omega_k$  be given elements of  $M$ , and  $(e_1, \dots, e_n)$  be a free basis of  $M$ ,

$$\omega_1 \wedge \dots \wedge \omega_k = \sum_{1 \leq i_1 < \dots < i_k \leq n} a_{i_1 \dots i_k} e_{i_1} \wedge \dots \wedge e_{i_k}.$$

We call  $\mathfrak{A}$ : the ideal of  $R$  generated by the coefficients  $a_{i_1 \dots i_k}$ ,  $1 \leq i_1 < \dots < i_k \leq n$ . (We put  $\mathfrak{A} = R$ , when  $k = 0$ .)

Then we define :

$$Z^p := \{ \omega \in \bigwedge^p M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0 \} \quad p = 0, 1, 2, \dots$$

$$H^p := Z^p / \sum_{i=1}^k \omega_i \wedge \bigwedge^{p-1} M \quad p = 0, 1, 2, \dots$$

In the case when  $k = 0$ , we understand  $Z^p = 0$ ,  $H^p = 0$  for  $p = 0, 1, 2, \dots$

**THEOREM.** — i) *There exists an integer  $m \geq 0$  such that :*

$$\mathcal{A}^m H^p = 0 \quad \text{for } p = 0, 1, 2, \dots, n.$$

ii)  $H^p = 0$  for  $0 \leq p < \text{prof}(\mathcal{A})$ .

**2. Proof of the theorem.**

*Proof of i).* — Since  $R$  is noetherian, we have only to show for any  $\omega \in Z^p$  and any coefficients  $a_{i_1 \dots i_k}$ ,

$$1 \leq i_1 < \cdots < i_k \leq n,$$

there exists an integer  $m \geq 0$  such that

$$(a_{i_1 \dots i_k})^m \omega \in \sum_{i=1}^k \omega_i \wedge \bigwedge^{p-1} M.$$

If  $a_{i_1 \dots i_k}$  is nilpotent, then nothing is to show. Suppose  $a_{i_1 \dots i_k} = a$  is not nilpotent and let  $R_{(a)}$  be the localization of  $R$  by the powers of  $a = a_{i_1 \dots i_k}$ . There is a canonical morphism  $R \rightarrow R_{(a)}$  and we denote by  $[\omega]$  the image of  $\omega \in \bigwedge^p M$  in  $\left( \bigwedge^p M \right) \otimes_R R_{(a)} \left( = \bigwedge^p \left( M \otimes_R R_{(a)} \right) \right)$  because  $M$  is free over  $R$ .

Since the ideal in  $R_{(a)}$  generated by the coefficients of  $[\omega_1] \wedge \cdots \wedge [\omega_k]$  contains the image of  $a = a_{i_1 \dots i_k}$  in  $R_{(a)}$ , it coincides with  $R_{(a)}$  and we may consider

$$[\omega_1], \dots, [\omega_k]$$

as a part of free basis of  $M \otimes_R R_{(a)}$ . We add some other elements  $[e_1], \dots, [e_{n-k}]$  such that

$$[\omega_1], \dots, [\omega_k], [e_1], \dots, [e_{n-k}]$$

give a basis of  $M \otimes_{\mathbf{R}} R_{(a)}$ . Then any element

$$[\omega] \in \bigwedge^p (M \otimes_{\mathbf{R}} R_{(a)})$$

can be developed in the form :

$$[\omega] = \sum_{l+m=p} \sum_{\substack{1 \leq i_1 < \dots < i_l \leq k \\ 1 \leq j_1 < \dots < j_m \leq n-k}} a_{i_1 \dots i_l, j_1 \dots j_m} [\omega_{i_1}] \wedge \dots \wedge [\omega_{i_l}] \wedge [e_{j_1}] \wedge \dots \wedge [e_{j_m}].$$

Then the fact  $[\omega] \wedge [\omega_1] \wedge \dots \wedge [\omega_k] = 0$  is equivalent to the existence of some  $\eta'_i \in \bigwedge^{p-1} (M \otimes_{\mathbf{R}} R_{(a)})$   $i = 1, \dots, k$  with  $[\omega] = \sum_{i=1}^k \eta'_i \wedge [\omega_i]$ . Let us take  $\eta_i \in \bigwedge^{p-1} M$  and  $m_1 \geq 0$  with  $\eta'_i = a^{-m_1} [\eta_i]$   $i = 1, \dots, k$ .

Then we have :

$$\left[ a^{m_1} \omega - \sum_{i=1}^k \eta_i \wedge \omega_i \right] = a^{m_1} [\omega] - \sum_{i=1}^k [\eta_i] \wedge [\omega_i] = 0.$$

By the definition of  $R_{(a)}$ , there exists some  $m_2 \geq 0$  such that

$$a^{m_2} \left\{ a^{m_1} \omega - \sum_{i=1}^k \eta_i \wedge \omega_i \right\} = 0 \text{ in } \bigwedge^p M.$$

This completes the proof of i).

*Proof of ii).* We prove it by double induction on  $(p, k)$  for  $p, k \geq 0$ .

a) In the case  $k = 0$ , the assertion is trivially true by the definition of  $H^p$ .

b) Case  $p = 0$ .

Let  $\omega \in \bigwedge^0 M = R$  with  $\omega \wedge \omega_1 \wedge \dots \wedge \omega_k = 0$ . The fact  $p = 0 < \text{prof}(\mathcal{A})$  implies the existence of  $a \in \mathcal{A}$ , which is non-zero-divisor of  $R$ . Since  $a\omega = 0$ , we get  $\omega = 0$ .

c) Case  $0 < p < \text{prof}(\mathcal{A})$  and  $0 < k$ .

The induction hypothesis is then, that for  $(p - 1, k)$  and  $(p, k - 1)$  the assertion ii) of the theorem is true.

Let  $a \in \mathcal{A}$  be a non-zero-divisor of  $R$ . According to i), there exists an integer  $m > 0$  with  $a^m H^p = 0$ . Since  $a^m \in \mathcal{A}$  is again a non-zero-divisor of  $R$ , we may assume that  $m = 1$ .

We denote by  $\bar{\omega}$  the image of  $\omega \in \bigwedge^p M$  in

$$\left(\bigwedge^p M\right) \otimes_{\mathbb{R}} \mathbb{R}/a\mathbb{R} \simeq \bigwedge^p \left(M \otimes_{\mathbb{R}} \mathbb{R}/a\mathbb{R}\right).$$

For  $\omega \in Z^p$ , we have a presentation :

$$(*) \quad a\omega = \sum_{i=1}^k \eta_i \wedge \omega_i, \quad \text{with } \eta_i \in \bigwedge^{p-1} M.$$

We have then :  $0 = \sum_{i=1}^k \bar{\eta}_i \wedge \bar{\omega}_i$ .

For any  $1 \leq j \leq k$ , we get :

$$\begin{aligned} \bar{\eta}_j \wedge \bar{\omega}_1 \wedge \cdots \wedge \omega_k &= \left( \sum_{i=1}^k \bar{\eta}_i \wedge \bar{\omega}_i \right) \\ &\quad \wedge ((-1)^{j-1} \bar{\omega}_1 \wedge \cdots \wedge \hat{\bar{\omega}}_j \wedge \cdots \wedge \bar{\omega}_k) = 0. \end{aligned}$$

Here the symbol  $\hat{\phantom{x}}$  means, we omit the corresponding term. Since the ideal of  $\mathbb{R}/a\mathbb{R}$  generated by the coefficients of  $\bar{\omega}_1 \wedge \cdots \wedge \bar{\omega}_k$  is equal to  $\mathfrak{A}/a\mathbb{R}$  and

$$\text{prof } \mathfrak{A}/a\mathbb{R} = \text{prof } \mathfrak{A} - 1 \geq p - 1 \geq 0,$$

we can apply to  $\bar{\eta}_j$  the induction hypothesis for  $(p-1, k)$ ; there exist  $\xi_{ji} \in \bigwedge^{p-2} M$ ,  $j, i = 1, \dots, k$ , such that

$$\bar{\eta}_j = \sum_{i=1}^k \bar{\xi}_{ji} \wedge \bar{\omega}_i, \quad j = 1, \dots, k.$$

Lifting back this relation to  $\bigwedge^{p-1} M$ , we find some  $\zeta_j \in \bigwedge^{p-1} M$ ,  $j = 1, \dots, k$ , such that

$$\eta_j = \sum_{i=1}^k \xi_{ji} \wedge \omega_i + a\zeta_j \quad j = 1, \dots, k.$$

Replacing  $\eta_j$  in the presentation (\*) by this, we obtain :

$$a \left( \omega - \sum_{j=1}^k \zeta_j \wedge \omega_j \right) = \sum_{i,j=1}^k \xi_{ji} \wedge \omega_i \wedge \omega_j.$$

Multiplying by  $\omega_2 \wedge \cdots \wedge \omega_k$ , we have :

$$a \left( \omega - \sum_{i=1}^k \zeta_i \wedge \omega_i \right) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.$$

Since  $a$  is a non-zero-divisor of  $R$ , we have :

$$\left( \omega - \sum_{i=1}^k \zeta_i \wedge \omega_i \right) \wedge \omega_2 \wedge \cdots \wedge \omega_k = 0.$$

Now since the ideal  $\mathcal{A}'$  generated by the coefficients of  $\omega_2 \wedge \cdots \wedge \omega_k$  contains the ideal  $\mathcal{A}$ , we have  $\text{prof } \mathcal{A}' \geq \text{prof } \mathcal{A} > p$ . Again by the induction hypothesis for  $(p, k - 1)$ , we find some  $\theta_j \in \bigwedge^{p-1} M, j = 2, \dots, k$  with

$$\omega - \sum_{i=1}^k \zeta_i \wedge \omega_i = \sum_{j=2}^k \theta_j \wedge \omega_i.$$

This ends the proof of ii).

### 3. Remark.

We can formulate the theorem in § 2, for a more general class of modules  $M$  than the one of free modules, as follows.

Let  $M$  be a  $R$ -finite module with homological dimension  $hd_R(M) \leq 1$ , and  $\omega_1, \dots, \omega_k$  be elements of  $M$ . Since  $hd_R(M) \leq 1$ , we have a free resolution :

$$0 \rightarrow L_1 \rightarrow L_2 \rightarrow M \rightarrow 0.$$

Let  $\tilde{\omega}_1, \dots, \tilde{\omega}_k$  be some liftings of  $\omega_1, \dots, \omega_k$  in  $L_2$  and  $\tilde{e}_1, \dots, \tilde{e}_m$  be images in  $L_2$  of a free basis  $e_1, \dots, e_m$  of  $L_1$ . Let  $\mathcal{A}$  be the ideal of  $R$  generated by coefficients of  $\tilde{\omega}_1 \wedge \cdots \wedge \tilde{\omega}_k \wedge \tilde{e}_1 \wedge \cdots \wedge \tilde{e}_m$ .

Since  $\mathcal{A}$  can be considered as a Fitting ideal of the following resolution :

$$L_1 \oplus R^k \rightarrow L_2 \rightarrow M \Big/ \sum_{i=1}^k R\omega_i \rightarrow 0.$$

we obtain the following lemma.

LEMMA. —  $\mathcal{A}$  does only depend on  $M$  and  $\omega_1, \dots, \omega_k$  and does depend neither on the choice of  $\tilde{\omega}_1, \dots, \tilde{\omega}_k$  and  $e_1, \dots, e_m$  nor on the resolution of  $M$ , we have used.

Let us define again :

$$H^p = \left\{ \omega \in \bigwedge^p M : \omega \wedge \omega_1 \wedge \cdots \wedge \omega_k = 0 \right\} \Big/ \sum_{i=1}^k \omega_i \wedge \bigwedge^{p-1} M.$$

Then we obtain again: i)  $\mathcal{A}^m H^p = 0$ ,  $p = 0, 1, 2, \dots$  for some  $m > 0$  and ii)  $H^p = 0$  for  $0 \leq p < \text{prof } \mathcal{A}$ .

For the proof we have only to apply the theorem to  $L_2$  and  $\tilde{\omega}_1, \dots, \tilde{\omega}_k, \tilde{e}_1, \dots, \tilde{e}_m$ .

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