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INVARIANT SUBSPACES ON OPEN RIEMANN SURFACES. II

by Morisuke HASUMI

The purpose of the present paper is to describe a few remarks on our previous work [4]. In the paper [4] we determined the closed invariant subspaces of the L^p spaces with respect to a harmonic measure on the Martin boundary of a hyperbolic Riemann surface R under three hypotheses (A), (B) and (C). These hypotheses will be stated later for ease of reference. In the course of getting the main result we showed in [4] that almost all Green lines on our surface R are convergent in its Martin compactification, which implies coincidence of fine limits and limits along Green lines for certain classes of analytic or harmonic functions on R, and we also obtained an extended version of Cauchy-Read theorems. The first thing we shall mention in this paper is an observation of Harold Widom which says that the hypothesis (C) is a consequence of the hypothesis (B) or of a much weaker one, which we shall call (B′). Next we shall show that the hypothesis (A) is redundant, so that most results in [4] hold under a single hypothesis at most as strong as (B). As we shall see later, every Riemann surface R satisfying (B′) can be completed to a regular Riemann surface R^+ satisfying (B′) by filling up an at most countable number of isolated holes of R. Once we get this, the redundance of the hypothesis (A) becomes quite clear. In fact, problems concerning the Hardy classes on R can be reduced to corresponding ones for the extended surface R^+. In this way, we see that our invariant subspace theorems in [4] continue to hold for any Riemann surface satisfying (B). We shall add a little more information to Widom's observation mentioned above. In section 4, we shall state a proof of Lemma 5.5, (*a*) in [4], because the proof suggested there is rather misleading. In the final section, we shall collect miscellaneous remarks, inclusing corrections of the paper [4].

We wish to thank Professor H. Widom for his permission to incorporate his result in this paper, Professor L. Carleson for helpful suggestions, and Mr. M. Hayashi for simplification of some arguments.

Notations and basic hypotheses.

We shall denote by R a hyperbolic Riemann surface and by $G(a, z)$ the Green function for R with pole at a point $a \in R$. For a positive number α and a point $a \in R$, we put $R(\alpha; a) = \{z \in R: G(a, z) > \alpha\}$ and

$$\mathrm{Cl}\, R(\alpha; a) = \{z \in R: G(a, z) \geqslant \alpha\}.$$

We shall denote by $B(\alpha; a)$ the first Betti number of the region $R(\alpha; a)$. For a point $a \in R$, let

$$Z(a; R) = \{z_j = z_j(a): j = 1, 2, \ldots\}$$

be an enumeration of the critical points of the function $G(a, .)$ which we repeat according to multiplicity. For $1 \leqslant p \leqslant \infty$, $H^p(R)$ will denote the Hardy classes of analytic functions on R in the sense of Rudin [9]. We finally state the hypotheses mentioned above.

(A) R is regular in the sense that $\mathrm{Cl}\, R(\alpha; a)$ is compact for any $\alpha > 0$.

(B) Let $H_1(R; \mathbf{Z})$ be the integral first singular homology group of R and $\Pi(R)$ the group of multiplicative characters of the group $H_1(R; \mathbf{Z})$. Then, there exists a family of outer l.a.m.'s (l.a.m. stands for locally analytic modulus; cf. [4]) $\{\delta(\theta): \theta \in \Pi(R)\}$ such that (*a*) $\delta(1) = 1$; (*b*) $\delta(\theta)$ has character θ for each $\theta \in \Pi(R)$; (*c*) $0 < \delta(\theta) \leqslant 1$ for each $\theta \in \Pi(R)$; (*d*) if a sequence of the form

$$\{\delta(\theta_n): \theta_n \in \Pi(R),\ n = 1, 2, \ldots\}$$

is pointwise convergent to a function of the form $|f|$ with $f \in H^\infty(R)$, then f is β exterior in the sense that $fH^\infty(R)$ is dense in $H^\infty(R)$ with respect to the strict (or β) topology.

(B′) For each $\theta \in \Pi(R)$, there exists an l.a.m. $\delta(\theta)$ such that $\delta(\theta) \leqslant 1$, $\delta(\theta) \not\equiv 0$, and $\delta(\theta)$ has character θ.

(C) There exists a point $a \in R$ for which

$$\Sigma\{G(z_j, z) : z_j \in Z(a; R)\} < \infty \quad \text{on} \quad R \sim Z(a; R).$$

1. Widom's result.

We shall begin with stating Widom's observation, together with his proof, concerning the hypotheses (B′) and (C).

Theorem 1 (H. Widom). — *For any hyperbolic Riemann surface* R, *the hypothesis* (B′) *implies the hypothesis* (C).

Proof (H. Widom). — As is easily seen, the hypothesis (B′) is equivalent to the fact that every flat complex line bundle over R has a non-trivial bounded holomorphic section. And the latter is equivalent, by the main theorem of Widom [11], to the hypothesis (B″) : $\int_0^\infty B(\lambda; a)\, d\lambda < \infty$ for some $a \in R$ (and hence for all $a \in R$). Let $\{S_n : n = 1, 2, \ldots\}$ be a sequence of Jordan subregions of R such that (i) each S_n is bounded by a finite number of analytic Jordan curves; (ii) $R \sim S_n$ has no compact components; (iii) $\mathrm{Cl}\, S_n \subseteq S_{n+1}$ for each n; and (iv) $R = \bigcup_{n=1}^{\infty} S_n$. Let $a \in S_1$ be fixed and let $G_n(a, z)$ be the Green function for the region S_n with pole at the point a. Let

$$Z(a; S_n) = \{z_{j,n} : j = 1, \ldots, l(n)\}$$

be an enumeration of the critical points of $G_n(a, .)$ which we repeat according to multiplicity. Let

$$S_n(\alpha; a) = \{z \in S_n : G_n(a, z) > \alpha\}$$

for each $\alpha > 0$ and $B_n(\alpha; a)$ the first Betti number of

$S_n(\alpha; a)$. Then, by the formula (7) of [11], we have

$$\Sigma\{G_n(z_{j,n}, a) : z_{j,n} \in Z(a; S_n)\} = \int_0^\infty B_n(\lambda; a)\, d\lambda.$$

Since $G_n(a, z)$ and $\delta G_n(a, z) = \partial_z G_n(a, z)\, dz$ converge almost uniformly on R to $G(a, z)$ and $\delta G(a, z)$, respectively, the argument principle shows that every critical point of $G(a, .)$ is a limit of critical points of $G_n(a, .)$ and, in fact, that the total order of the critical points of $G(a, .)$ lying in a given relatively compact region in R is equal to that of $G_n(a, .)$ for all sufficiently large n. Thus,

$$\begin{aligned}\Sigma\{G(z_j, a) : z_j \in Z(a; R)\} &\leqslant \liminf_{n\to\infty} \sum_{j=1}^{l(n)} G_n(z_{j,n}, a)\\ &= \liminf_{n\to\infty} \int_0^\infty B_n(\lambda; a)\, d\lambda = \int_0^\infty B(\lambda; a)\, d\lambda,\end{aligned}$$

the last equality being true because the sequence $\{B_n(\lambda; a) : n = 1, 2, \ldots\}$ is monotonically increasing with limit $B(\lambda; a)$. As we already know the finiteness of the last integral, we have obtained the desired result.

Remark. — We shall give later an explicit evaluation of the integral $\int_0^\infty B(\lambda; a)\, d\lambda$.

2. Regularization of hyperbolic Riemann surfaces.

We first prove the following

LEMMA 2. — *If* R *is a hyperbolic Riemann surface for which* $B(\lambda; a) < \infty$ *for all* $\lambda > 0$ *and* $a \in R$, *then there exists a regular hyperbolic Riemann surface* R^+ *and a discrete subset* Σ *of* R^+ *such that* R *is conformally equivalent to* $R^+ \sim \Sigma$. *The regular surface* R^+ *is determined uniquely by* R *up to conformal equivalence.*

Proof. — (i) First we assume that the first Betti number of R is finite. Then, there exist a finite number of open subsets $U_1, \ldots, U_m$ of R satisfying the following: (a) the closures $\mathrm{Cl}\, U_i$ are mutually disjoint non-compact subsets of R; (b) for each i, the boundary ∂U_i of U_i is a simple closed analytic curve, say J_i, and there exists a conformal homeomorphism ψ_i of U_i onto the annulus $\{w \in \mathbf{C}: r_i < |w| < 1\}$ $(0 \leqslant r_i < 1)$ such that ψ_i is continuously extendable to a homeomorphism of $\mathrm{Cl}\, U_i$ onto $\{w \in \mathbf{C}: r_i < |w| \leqslant 1\}$ which maps J_i onto the unit circle; (c) $R \sim \bigcup_{i=1}^{m} U_i$ is a compact bordered Riemann surface.

If $r_i > 0$ for all i, then R itself is easily seen to be regular. So we may assume that $r_i = 0$ for some i. For the sake of simplicity, we assume that $r_i = 0$ for $i = 1, \ldots, s$ and $r_i > 0$ for $i = s + 1, \ldots, m$. Take a set of s elements, say $B = \{b(1), \ldots, b(s)\}$, and form a formal union

$$R^+ = R \cup B.$$

To each point p in R we assign a parametric disk (U_p, ψ_p), which is compatible with the given conformal structure of R. For each $i = 1, \ldots, s$, we put $U_{b(i)} = U_i \cup \{b(i)\}$ and

$$\psi_i^*(z) = \begin{cases} \psi_i(z) & \text{for } z \in U_i, \\ 0 & \text{for } z = b(i). \end{cases}$$

We regard $(U_{b(i)}, \psi_i^*)$ as a parametric disk about the point $b(i)$. It follows from our construction that these parametric disks define a conformal structure on R^+ which induces on R the original structure of R. As is well known, every bounded harmonic function on the punctured unit disk

$$\{w \in \mathbf{C}: 0 < |w| < 1\}$$

can be continuously extended to the full open unit disk so as to have a harmonic function. So the Green function $G(z', z)$ of R with pole at $z' \in R$ can be extended by continuity to the Green function of R^+. This function clearly vanishes on the ideal boundary of R^+. We may thus conclude that the surface R^+ is regular.

(ii) We now consider the general case. Let us fix a point $a \in R$. Take any $\alpha > 0$ and put $E_\alpha = R(\alpha; a)$. Since the first Betti number of the surface E_α is equal to $B(\alpha; a)$ and so is finite. As we have shown in (i), E_α can be completed to a regular surface E_α^+ by adding a finite number of points to E_α. It is not difficult to show that, if $\alpha > \beta > 0$, the surface E_α^+ may be canonically identified with the subregion of the surface E_β^+ defined by

$$\{z \in E_\beta^+ : G_\beta(a, z) > \alpha - \beta\},$$

where $G_\beta(a, z)$ denotes the Green function of E_β^+ with pole at a. With this identification for all α, β with $\alpha > \beta > 0$, we put $R^+ = \cup\{E_\alpha^+ : \alpha > 0\}$. If we give R^+ the conformal structure induced from those of E_α^+, then R^+ is a regular hyperbolic Riemann surface which satisfies the property of the lemma.

(iii) It is a routine matter to show the uniqueness of the pair (R^+, Σ). The proof is thus omitted. Q.E.D.

By means of the preceding lemma, we can easily find a simple relation between the Martin compactification of R and that of R^+. Before doing this, we shall fix some notations. Let E^* be the Martin compactification of a hyperbolic Riemann surface E and $\Delta(E) = E^* \sim E$ the Martin ideal boundary of E. Take a point $a_0 = a_0(E)$ in E, which is fixed throughout the discussion, and let $\alpha_0 = \alpha_0(E)$ be a fixed positive number so large that $\{z \in E : G_E(a_0, z) \geqslant \alpha_0\}$ is a parametric disk on E, where $G_E(a, z)$ denotes the Green function for E with pole at a. Let Φ be an infinitely differentiable real function on $[-\infty, +\infty]$ such that $\Phi(t) \leqslant t$, $\Phi(t) = t$ for $t \leqslant 0$, Φ is constant for $t \geqslant 1$, and $d^2\Phi(t)/dt^2 \leqslant 0$ and put $\Phi_0(t) = \Phi(t - \alpha_0) + \alpha_0$. Then, we define $k_E(b, z) = G_E(b, z)/\Phi_0(G_E(b, a_0))$ for b, $z \in E$. The function $b \to k_E(b, .)$, $b \in E$, is then extended by continuity to E^* and we get the Martin function $k_E(b, z)$ for $(b, z) \in E^* \times E$. In the following we shall refer to the point a_0 as the base point for the Martin function k_E. The point b is called the pole of the Martin function $z \to k_E(b, z)$. The Martin compactification E^* of E can be characterized as the smallest compactification of E to which every function $b \to k_E(b, z)$ $(z \in E)$ can be extended by continuity. Furthermore, let $\Delta_1(E)$ be the set of points $b \in \Delta(E)$ for which the function $z \to k_E(b, z)$ is a minimal harmonic function on E. We shall denote by $d\chi_E$ the harmonic measure, supported by $\Delta_1(E)$, for the point a_0. We call a function on $\Delta_1(E)$ integrable if it is so with respect to the measure $d\chi_E$. For any $b \in \Delta_1(E)$, let $\mathcal{G}_b(E)$ be the family of nonempty open sets D in E such that $k_E(b, .) \neq (k_E(b, .))_{E\sim D}$, where $(u)_{E\sim D}$, for a positive function u on E, is the greatest lower bound of the positive superharmonic functions which are not smaller than u quasi-everywhere on the set $E \sim D$. Let f be any function from E into a compact space X. For $b \in \Delta_1(E)$, put $f^\wedge(b) = \cap\{\mathrm{Cl}(f(D)) : D \in \mathcal{G}_b(E)\}$. Let $\mathcal{D}(f)$ be the set of $b \in \Delta_1(E)$ for which $f^\wedge(b)$ is a singleton and define $\hat{f}(b)$, for each $b \in \mathcal{D}(f)$, as the point in $f^\wedge(b)$. If $b \in \mathcal{D}(f)$, then we say that f has a fine limit $\hat{f}(b)$ at b. When f is a numerical function, we take as X the complex sphere. A detailed discussion on these concepts may be found in [2].

THEOREM 3. — *Let* R *be a hyperbolic Riemann surface for which* $B(\lambda; a)$ *is finite for any* $\lambda > 0$ *and* $a \in R$. *Then, the regular hyperbolic Riemann surface* R^+ *obtained in Lemma 2 can be constructed as follows. Let* Σ *be the subset of the Martin ideal boundary* $\Delta(R)$ *of* R *such that a point* $b \in \Delta(R)$ *belongs to* Σ *if and only if* $\limsup\{G(a, z) : z \in R, z \to b\} > 0$. *Then, the set* Σ *is at most countable and independent of the choice of* a, *and, for each* $b \in \Sigma$, *there exists a neighborhood* V *of* b *in the Martin compactification* R^* *of* R *with* $V \cap \Delta(R) = \{b\}$. *On the union* $R^+ = R \cup \Sigma$ *there exists a uniquely determined structure of hyperbolic Riemann surface, compatible with the relative topology of* R^+ *as a subspace of* R^*, *which satisfies the following*:

(i) R^+ *is a regular hyperbolic surface and the conformal structure of* R^+ *induces on* R *the original structure of* R;

(ii) *if* V *is a neighborhood of* $b \in \Sigma$ *in* R^* *such that* $V \cap \Delta(R) = \{b\}$, *then every bounded harmonic function* u *on* $V \cap R$ *can be extended by continuity to the point* b *so as to get a harmonic function on* V *with respect to the conformal structure of* R^+;

(iii) *the Green function* $G^+(z', z)$ *of* R^+ *(resp., the Martin function* $k^+(b, z)$ *of* R^+ *with pole at* b *and base point* $a_0(\in R)$*) is obtained by extending, by continuity in the topology of* R^*, *the Green function* $G(z', z)$ *(resp., the Martin function* $k(b, z)$ *of* R *with the same base point* a_0*) to the points in* Σ;

(iv) *the Martin compactification of* R^+ *can be identified with* R^* *and* $\Delta_1(R) = \Delta_1(R^+) \cup \Sigma$;

(v) *the harmonic measure* $d\chi^+$, *supported on* $\Delta_1(R^+)$, *for the point* a_0 *(in* R*) with respect to* R^+ *is nothing but the restriction, to the set* $\Delta_1(R^+)$, *of the harmonic measure* $d\chi$ *on* $\Delta_1(R)$ *for the point* a_0 *with respect to the surface* R;

(vi) *if* R *satisfies the hypothesis* (B) *(resp.,* (B′)*), then* R^+ *also satisfies* (B) *(resp.,* (B′)*).*

Proof. — Let (R^+, Σ) be a pair given by Lemma 2. Then, the properties (i) and (ii) are clearly satisfied. So we prove here that R^+ has the remaining properties.

(iii) The property (ii) implies that, for each fixed $z \in R$, the Green function $z' \to G(z', z)$ of R with pole at $z \in R$ is extended by continuity to the Green function $z' \to G^+(z', z)$ of R^+. So the definition of the Martin functions shows that, for each fixed $z \in R$, the function $b \to k^+(b, z)$ on R^+ is obtained by extending, by continuity, the function $b \to k(b, z)$ on R to the points in Σ.

(iv) For each fixed $z \in R$, the function $b \to k^+(b, z)$ on R^+ can be extended to a continuous function on the Martin compactification, say $R^\#$, of R^+ and therefore the function $b \to k(b, z)$ on R can also be extended to a continuous function on $R^\#$. Since the family of functions $b \to k^+(b, z^+)$ with $z^+ \in R^+$ separates points of $R^\#$ and since R is dense in R^+, the subfamily of functions

$$b \to k^+(b, z)$$

with $z \in R$ also separates points of $R^\#$. Hence, $R^\#$ can be identified with the Martin compactification of R, i.e., $R^* = R^\#$. In particular, we have $\Sigma = \Delta(R) \sim \Delta(R^+)$. Since R^+ is regular, we conclude that $b \in \Delta(R)$ belongs to Σ if and only if

$$\lim \{G(a, z) : z \in R,\ z \to b\} = G^+(a, b) > 0.$$

If $b \in \Sigma$, then the function $z \to k(b, z)$ is equal to a constant multiple of $G^+(b, z)$ and so is a minimal harmonic function on R. We thus have $\Sigma \subseteq \Delta_1(R)$ and therefore

$$\Delta_1(R) = \Delta_1(R^+) \cup \Sigma.$$

(v) This is clear from the above observation.

(vi) Let $B^+(\lambda; a)$ be the first Betti number of the region $R^+(\lambda; z) = \{z \in R^+ : G^+(a, z) > \lambda\}$, where we assume $a \in R$. Then, $B^+(\lambda; a) \leqslant B(\lambda; a)$ for all $\lambda > 0$ and therefore $\int_0^\infty B^+(\lambda; a)\, d\lambda \leqslant \int_0^\infty B(\lambda; a)\, d\lambda$. So R^+ satisfies (B′) whenever R does. Now, suppose that R satisfies (B). Since $R \subseteq R^+$, we have a natural homomorphism of $H_1(R; \mathbf{Z})$ onto $H_1(R^+; \mathbf{Z})$. So, we have a natural homomorphism of $\Pi(R^+)$ into $\Pi(R)$, which is injective. For any $\theta^+ \in \Pi(R^+)$, let θ be the element in $\Pi(R)$ corresponding to θ^+ under this homomorphism. Then, we have $\theta^+(\gamma) = 1$ for any cycle $\gamma \in H_1(R; \mathbf{Z})$ which is null-homologous in R^+. We put $u = -\log \delta(\theta)$. Since R satisfies (B), u is an outer harmonic function on R. Namely, u is harmonic and $\lim_{n\to\infty} (u \wedge n) = u$ on R. Since $u \wedge n$ is a bounded harmonic function on R, it can be extended, by continuity, to a harmonic function φ_n on R^+ in view of the property (ii). So, $\lim_{n\to\infty} \varphi_n$, say φ, exists and is a continuous extension of u to R^+. The function φ is clearly outer, so that $\exp(-\varphi)$ is an outer l.a.m. with character θ^+. Putting $\varphi = \delta(\theta^+)$, we see that R^+ satisfies the properties (a), (b) and (c) of the hypothesis (B). Let f be a function in $H^\infty(R^+)$ such that there exists a sequence $\{\theta^+(n) : n = 1, 2, \ldots\}$ in $\Pi(R^+)$ with $\lim_{n\to\infty} \delta(\theta^+(n)) = |f|$ pointwise on R^+. Then, the above construction says that $\delta(\theta(n))$ is the restriction of $\delta(\theta^+(n))$ to R. So, if we define h to be the restriction of f to R, then $\delta(\theta(n))$ converge to $|h|$ pointwise on R. By the property (d) of (B) for the surface R, $hH^\infty(R)$ is strictly dense in $H^\infty(R)$. Then, it is easy to see that $fH^\infty(R^+)$ is also strictly dense in $H^\infty(R^+)$. Hence, R^+ is shown to satisfy the hypothesis (B).

The theorem is now clear from these observations.

Remark. — In the author's original proof of Theorem 3, the existence of a regular surface R^+ was shown via the Martin compactification R^* of R. The present proof using Lemma 2 is due to Mikihiro Hayashi.

3. Applications of regularization.

In this section, we shall apply the regularization process given by Theorem 3 (or Lemma 2) to surfaces satisfying the hypothesis (B) or (B′) and obtain certain improved versions of our results in [4].

a) Invariant subspace theorems for R. — For $1 \leqslant p \leqslant \infty$, the mapping $h \to \hat{h}$ gives an isometric linear injection of the Hardy space $H^p(R)$ into $L^p(d\chi)$, where $H^p(R)$ for $1 \leqslant p < \infty$ (resp., for $p = \infty$) is normed by

$$\|f\|_p = (\mathrm{L.H.M.}(|f|^p)(a_0))^{1/p}$$

for $f \in H^p(R)$ (resp., $\|f\|_\infty = \sup \{|f(z)| : \quad z \in R\}$ for $f \in H^\infty(R)$). By use of this mapping we can identify, for each p, the space $H^p(R)$ with a subspace of $L^p(d\chi)$, which we denote by $H^p(d\chi)$. In particular, $H^\infty(d\chi)$ forms a subalgebra of the algebra $L^\infty(d\chi)$ and each $L^p(d\chi)$ can be seen as a topological module over $H^\infty(d\chi)$. We suppose that the surface R satisfies the hypothesis (B). By applying Theorem 3 to R, we construct a regular hyperbolic surface R^+ which also satisfies (B). The space $L^p(d\chi)$ can be identified with $L^p(d\chi^+)$ by Theorem 3, (v). On the other hand, every bounded harmonic function on R can be continued to the points in Σ so as to be harmonic on R^+. From this follows that every outer harmonic function can be continued to Σ so as to get an outer harmonic function on R^+. Since the least harmonic majorant u of $|f|^p$ for $f \in H^p(R)$ with $1 \leqslant p < \infty$ is outer by [4; Proposition 2.7, (*a*)], we see that every $f \in H^p(R)$ can be continued to Σ so as to get an analytic function $f^+ \in H^p(R^+)$ and that $\mathrm{L.H.M.}(|f^+|^p)$ is precisely the continuous extension of u to R^+. The space $H^p(R)$ is thus seen to be isometrically isomorphic to $H^p(R^+)$ and consequently $H^p(d\chi)$ is identified with $H^p(d\chi^+)$. Since the surface R^+ satisfies the hypotheses (A) and (B), all the results in [4] hold for R^+ in view of Theorem 1. Using the above correspondence between function spaces on R and those on R^+, we conclude that the same results hold for the surface R as for R^+. Hence we have the following

THEOREM 4. — *The invariant subspace theorems for* $L^p(d\chi)$ *and for* $H^p(R)$ (*namely, Theorem* 7.1 *and Corollary* 7.2 *of* [4]) *hold for any hyperbolic Riemann surface* R *satisfying the hypothesis* (B).

b) *The Cauchy-Read theorems.* — Our Cauchy-Read theorems [4; Theorems 5.3 and 5.12] should be amended to meet with the present situation. By considering the surface R^+ as in (*a*), we get the following

THEOREM 5. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B) *and let* $a \in R$. *Let* f *be a meromorphic function on* R *such that* $|f|\,g^{(a)}$ *has a harmonic majorant, where*

$$g^{(a)}(z) = \exp\left(-\Sigma\{G^+(w; z) : w \in Z(a; R^+)\}\right) \quad \text{for} \quad z \in R.$$

Then, the fine boundary function $\hat{f}$ *for* f *exists* a.e. *on* $\Delta_1(R)$, *is integrable and*

$$f(a) = \int_{\Delta_1(R)} \hat{f}(b) k(b, a)\, d\chi(b).$$

THEOREM 6. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B′) *and let* $a \in R$. *Let* $u^* \in L^1(d\chi)$ *and suppose that*

$$\int_{\Delta_1(R)} \hat{h}(b) u^*(b) k(b, a)\, d\chi(b) = 0$$

for each function h, *meromorphic on* R *such that* $|h|\,g^{(a)}$ *is bounded on* R *and* $h(a) = 0$. *Then, there exists an* f *in* $H^1(R)$ *such that* $\hat{f} = u^*$ a.e. *on* $\Delta_1(R)$.

That the latter theorem holds under the weaker hypothesis (B′) can be easily seen by looking into the proof of our earlier theorem in [4].

c) *Fine limits and limits along Green lines.* — An intimate connection between fine limits and limits along Green lines was found and played an important role in [4]. In view of Theorem 3, the same relation holds for surface satisfying (B′). We shall use the following notations. Let $\mathbf{G} = \mathbf{G}(R; a)$ be the totality of Green lines L issuing from a given point a.

Then, $\mathbf{G}$ is parametrized with $\omega \in [0, 2\pi)$ in a natural way as $L = L_\omega$, so that the usual topology of the circle can be transferred to $\mathbf{G}$. The Green measure dm on $\mathbf{G}$ is defined by $dm(L) = dm(\omega) = d\omega/2\pi$ with $L = L_\omega$. A Green line $L \in \mathbf{G}$ is called regular, if $\inf \{G(a, z) : z \in L\} = 0$. Let $\mathbf{L} = \mathbf{L}(R; a)$ be the totality of regular Green lines issuing from a. Then, $\mathbf{L}$ is a G_δ subset of $\mathbf{G}$ of Green measure one. For a Green line $L \in \mathbf{L}(R; a)$, let $z(L; \alpha)$ be the point z on L where $G(a, z) = \alpha$. We consider the following condition C: For a point $a \in R$, almost all Green lines L in $\mathbf{L}(R; a)$ are convergent to a point in Δ, i.e., $z(L; \alpha)$ converges to a point in Δ as $\alpha \to 0$ in the Martin topology.

We shall denote by $\Lambda(R; a)$ the totality of convergent Green lines L in $\mathbf{L}(R; a)$ and by b_L, for each $L \in \Lambda(R; a)$, the limit point of L in Δ.

THEOREM 7. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B′). *Then, the following hold:*

(i) *The pair* (R, R^*) *satisfies Brelot-Choquet's conditions* A, B *in* [1; p. 249] (*cf. also* [6]) *as well as the condition* C *mentioned above.*

(ii) *Let* $a \in R$ *be fixed. Then, the function* $L \to b_L$ *from* $\Lambda(R; a)$ *into* $\Delta(R)$ *is measurable and is measure preserving with respect to the Green measure* dm *on* $\Lambda(R; a)$ *and the harmonic measure* $k(b, a)\, d\chi(b)$ *on* $\Delta(R)$ *corresponding to the point* a.

(iii) *Let* f^* *be a bounded measurable function on* $\Delta_1(R)$ *and let* $f = h[f^*]$ *be the solution of the Dirichlet problem for* R *with the boundary data* f^*. *Then, the limit*

$$f(L) = \lim_{\alpha \to 0} f(z(L; \alpha))$$

exists for almost all $L \in \Lambda(R; a)$ *and* $f(L) = f^*(b_L)$ dm-a.e. *on* $\Lambda(R; a)$.

(iv) *Under the same assumption as in* (iii), *we have*

$$\hat{f}(b) = f^*(b) = f(L) \quad \text{a.e. } on \quad \Delta_1(R)$$

where $b = b_L$ *with* $L \in \Lambda(R; a)$.

d) Evaluation of the function $P(a, a'; z)$. — We put, as in [4], $P(a, a'; z) = \delta G(a', z)/\delta G(a, z)$. For this we have

PROPOSITION 8. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B'). *For a given pair of points* a, a' *in* R, *there exists a constant* $c > 0$, *depending only on* a, a' *and* R, *such that*

$$g^{(a)}(z) \exp(-G(a', z))|P(a, a'; z)| \leqslant c \quad \text{on} \quad R.$$

Moreover, $P(a, a'; z)$ *has the fine boundary values* a.e. *on* $\Delta_1(R)$ *and we have* $\hat{P}(a, a'; b) = k(b, a')/k(b, a)$ a.e. *on* $\Delta_1(R)$.

e) Evaluation of the integral $\int_0^\infty B(\lambda; a)\, d\lambda$. — Finally we wish to see the difference between the hypotheses (B') and (C). We have the following

PROPOSITION 9. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B') *and let* $a \in R$. *Then, the set* $Z(a; R)$ *of critical points of the function* $G(a, .)$ *consists of those elements of the set* $Z(a; R^+)$ *that lie in* R *and we have*

$$\begin{aligned}\int_0^\infty B(\lambda; a)\, d\lambda &= \int_0^\infty B^+(\lambda; a)\, d\lambda + \Sigma\{G^+(a, w) : w \in \Sigma\}\\ &= \Sigma\{G^+(a, z) : w \in Z(a; R^+)\} + \Sigma\{G^+(a, w) : w \in \Sigma\}.\end{aligned}$$

Proof (M. Hayashi). — Let $\Sigma = \{w_1, w_2, \ldots\}$ and put $R_n = R^+ \sim \{w_1, w_2, \ldots, w_n\}$ for $n = 1, 2, \ldots$. Denote by $B_n(\alpha; a)$ the first Betti number of the subregion

$$R_n(\alpha; a) = \{z \in R_n : G^+(a, z) > \alpha\}.$$

Since $R_n = R_{n-1} \sim \{w_n\}$, we have

$$B_n(\alpha; a) = \begin{cases} B_{n-1}(\alpha; a) & \text{for} \quad \alpha > G^+(a, w_n)\\ B_{n-1}(\alpha; a) + 1 & \text{for} \quad \alpha < G^+(a, w_n).\end{cases}$$

So we have

$$\begin{aligned}\int_0^\infty B_n(\alpha; a)\, d\alpha &= \int_0^\infty B_{n-1}(\alpha; a)\, d\alpha + G^+(a, w_n)\\ &= \int_0^\infty B^+(\alpha; a)\, d\alpha + \sum_{i=1}^n G^+(a, w_i).\end{aligned}$$

Since $B_n(\alpha; a)$ converges monotonically to $B(\alpha; a)$ as $n \to \infty$, we get the desired result.

4. Proof of [4; Lemma 5.5, (a)].

The suggestion which I made for the proof of Lemma 5.5 (a) in [4] looks quite misleading, for the theorems in Sario and Nakai [10] cited there do not deal with arbitrary bounded analytic functions. So we wish to describe an explicit proof of the lemma. For convenience's sake, we restate our assertion.

PROPOSITION 10 ([4; Lemma 5.5 (a)]). — *Let* R *be a hyperbolic Riemann surface which satisfies the hypotheses* (A), (C) *and let* $\mathbf{G}' = \mathbf{G}'(\mathrm{R}; a) = \cup\{\mathrm{L} : \mathrm{L} \in \mathbf{G}(\mathrm{R}; a)\} \cup \{a\}$ *be the Green star region with center at a point* $a \in \mathrm{R}$. *Then, every bounded analytic function* f *on* $\mathbf{G}'$ *possesses a radial limit a.e. on* $\mathbf{G}$ *and the limit function* $f(\mathrm{L})$ *is* m*-measurable on* $\mathbf{G}$. f *vanishes identically whenever* $f(\mathrm{L})$ *vanishes on a set of positive measure.*

Proof. — Let Ω_0 be the set of angles $\omega \in [0, 2\pi)$ for which the Green line L_ω is singular, i.e., L_ω ends in a critical point, say $z(\omega)$, of the Green function $\mathrm{G}(a, .)$. It is known that, if a critical point has order c, there exist exactly $c + 1$ singular Green lines, issuing from a, which end in that point. We know also that the function

$$z \to r(z) \exp (i\omega(z)),$$

defined in [4; Section 5], maps the Green star region $\mathbf{G}'$ conformally and univalently into the open unit disk. Its image, say D, is equal to $\mathrm{U} \setminus \cup\{\mathrm{S}_\omega : \omega \in \Omega_0\}$, where U is the open unit disk and S_ω, $\omega \in \Omega_0$, denotes the slit

$$\{re^{i\omega} : \exp (-\mathrm{G}(a, z(\omega))) \leqslant r \leqslant 1\}.$$

The usual topological boundary of D is clearly the union of the unit circumference and the slits S_ω, in view of the hypothesis (C). But we regard D somewhat differently.

For $\zeta_1, \zeta_2 \in \mathrm{D}$, let $d(\zeta_1, \zeta_2)$ be the infimum of the lengths of curves joining ζ_1 and ζ_2 in D. Then, d is a metric in D which induces there the usual Euclidean topology. Let D^* be the completion of D with respect to the metric d. We consider each slit S_ω, $\omega \in \Omega_0$, as the union of its two edges

S_ω^+ and S_ω^-, where $S_\omega^+ = \{z^+ = (z, +) : z \in S_\omega\}$ and $S_\omega^- = \{z^- = (z, -) : z \in S_\omega\}$. We regard z^+ (resp., z^-) with $z \in S_\omega$ as in the closure of $\{w \in D : \operatorname{Im}(wz^{-1}) > 0\}$ (resp., $\{w \in D : \operatorname{Im}(wz^{-1}) < 0\}$) with respect to the Euclidean topology of the complex plane. So, $z^+ = z^-$ when and only when $z = \exp(-G(a, z(\omega)) + i\omega)$. We put

$$L = \cup \{S_\omega^+ : \omega \in \Omega_0\} \cup \bigcup \{S_\omega^- : \omega \in \Omega_0\} \cup \{e^{i\omega} : \omega \notin \Omega_0\}.$$

Then, the completion D^* can be identified in a natural way with $D \cup L$, so that L can be seen as the boundary of D in the space D^*. It should also be noted that L can be identified with the set of prime ends for the domain D. The boundary L is rectifiable because the hypothesis (C) implies that $\Sigma\{1 - \exp(-G(a, z(\omega))) : \omega \in \Omega_0\} < \infty$. By a theorem on prime ends or by a slight modification of a well-known result on conformal mappings of Jordan domains, we have

LEMMA. — *Any one-to-one conformal mapping of the domain* D *onto the open unit disk* $\{|w| < 1\}$ *can be extended by continuity to* $D \cup L$ *so as to have a homeomorphism between* $D \cup L$ *and the closed unit disk* $\{|w| \leqslant 1\}$.

For the proof, see for instance Golusin [3; Chapter II, Section 3].

Let f be a univalent conformal mapping of the domain D onto the open unit disk and let $z = g(w)$ be the inverse of f. Then, the continuous extension of g to the closed unit disk, denoted by the same symbol g, maps $\{|w| \leqslant 1\}$ onto the disk $\{|z| \leqslant 1\}$ and $\{|w| = 1\}$ onto

$$\{|z| = 1\} \cup \bigcup \{S_\omega : \omega \in \Omega_0\}.$$

The extended function g is not necessarily one-to-one on $\{|w| = 1\}$ but is of bounded variation in view of the above lemma and the fact that L is rectifiable. By F. and M. Riesz's theorem (cf. Privalov [8; Chapter III, Section 1]), g is absolutely continuous on the unit circumference and therefore g' belongs to the Hardy class H^1 on the unit disk $\{|w| < 1\}$. If a subset e of $\{|w| = 1\}$ is mapped under $f^{-1} : \{|w| = 1\} \to L$ to a subset $E \cup L$, then the linear measure of E is equal to the integral $\int_e |g'(e^{i\theta})|\, d\theta$.

On the other hand, Lavrentiev's theorem (cf. [8; p. 125]) says that there exists a constant K such that

$$|e| \leqslant K/(|\log |E|| + 1),$$

where $|e|$ and $|E|$ denote the linear measures of e and E, respectively. Hence, the mapping f preserves the null sets on the boundaries.

Finally we wish to prove that, at almost every point $e^{i\theta}$ with $\theta \notin \Omega_0$, L has a tangent which coincides with that to the unit circumference. Since $g' \in H^1$, it admits nontangential boundary values $h(w) = (\partial g/\partial w)(w)$ almost everywhere on $\{|w| = 1\}$. The limit exists at every $w = e^{i\theta}$ for which

$$\lim_{t\to 0} t^{-1} \int_0^t |h(e^{i(\theta+t)}) - h(e^{i\theta})|\, dt = 0. \tag{3}$$

Let $w_0 = e^{i\alpha}$ be such a point with $h(w_0) \neq 0$. Draw a curve l which, except the initial point w_0, lies in $\{|w| < 1\}$ and has a tangent at w_0. We suppose that this tangent does not coincide with that to the circle. Let w, w_1 be two points on l distinct from w_0. We have

$$g(w) - g(w_1) = \int_{w_1}^{w} g'(t)\, dt.$$

Fix w and let w_1 tend to the point w_0 along l. So

$$g(w) - g(w_0) = \int_{w_0}^{w} g'(t)\, dt.$$

Since $g'(t)$ tends to $h(w_0)$ as $t \in l$ tends to w_0 by our hypothesis on w_0, we have shown that the derivative of g at w_0 along the curve l exists and is equal to $h(w_0)$. So the tangent to l at w_0 is rotated by $\arg h(w_0)$ under the mapping $w \to g(w)$, and therefore g preserves the angle between two curves l_1 and l_2, each being non-tangential at w_0 to the circumference $\{|w| = 1\}$. As

$$h(w_0) = (\partial g/\partial w)(w_0) \neq 0,$$

the boundary L has a tangent at $z_0 = g(w_0)$, which is obtained from the tangent to $\{|w| = 1\}$ at w_0 under a rotation by the angle $\arg h(w_0)$. Hence, the angle between the circle $\{|w| = 1\}$ and the curve l is equal to that between L and $g(l)$.

Now we take any $\omega \in [0, 2\pi)$ and any $\varepsilon > 0$. Since Ω_0 is countable, the set $\{e^{i\theta} : |\theta - \omega| < \varepsilon,\ \theta \notin \Omega_0\}$ contains a closed set E of positive measure. Let e be the subset of the circumference $\{|w| = 1\}$ such that $E = g(e)$. Since the mapping f preserves the sets of measure zero, we see that $|e| > 0$ and so there exists a null set $e_1 \subset e$ such that, for every $e^{i\theta} \in e \setminus e_1$, the relation (3) holds with $h(e^{i\theta}) \neq 0$. Let us take any $e^{i\theta} \in e \setminus e_1$ and put $e^{i\alpha} = g(e^{i\theta})$, where θ and α belong to $[0, 2\pi)$. Then

$$\lim_{\theta' \to \theta} \frac{g(e^{i\theta'}) - g(e^{i\theta})}{e^{i\theta'} - e^{i\theta}} \tag{4}$$

exists and is equal to $h(e^{i\theta})$. Since we have $g(e^{i\theta}) = e^{i\alpha}$, we can find a sequence $\{\theta(n) : n = 1, 2, \ldots\}$ in such a way that $\theta(n) \to \theta$ and $e^{i\alpha(n)} \to e^{i\alpha}$ with $e^{i\alpha(n)} = g(e^{i\theta(n)})$. It follows from this observation that the number defined by (4) has the same argument as $e^{i(\alpha-\theta)}$ and therefore that the tangent to L at the point $e^{i\alpha}$ is the same as that to the unit circumference. This is true of every point in the set $E \setminus g(e_1)$, where $g(e_1)$ is a null set. We thus conclude that, at almost every point in $\{e^{i\alpha} : \alpha \notin \Omega_0\}$, L has a tangent which is the same as that to the unit circumference at that point. In other words, the radius vector $\{re^{i\alpha} : 0 \leqslant r \leqslant 1\}$ is orthogonal to the boundary curve L for almost all $\alpha \notin \Omega_0$. Hence the Fatou theorem implies the following

PROPOSITION 11. — *Every bounded analytic function f on* D *has a limit along the radius $\{re^{i\alpha} : 0 \leqslant r < 1\}$ as $r \to 1$ for almost all angles $\alpha \notin \Omega_0$. The limit function is measurable with respect to α, and vanishes on a set of positive linear measure only when f vanishes identically on* D.

In view of the correspondance between the region D and the Green star region $\mathbf{G}'$, the last proposition is equivalent to what we wished to prove.

5. Miscellaneous remarks.

a) Correction to the paper [4]. — In Section 2 of [4], we called a meromorphic function f on R of bounded characteristic when it is a quotient of bounded analytic functions. But this gives a correct definition only for simply connected surfaces. For an arbitrary hyperbolic surface R, a (multiplicative) meromorphic function f on R is, by definition, of bounded characteristic if the function $\log^+|f|$ admits a superharmonic majorant on R (cf. [10; p. 270]). This is equivalent to saying that $\log|f| \in SP(R)$ in the notation of [4 : Section 2]. The paper [4] remains valid with this change of definition.

Secondly, the last statement of [4; Theorem 7.1, (*b*)] should read : « The i-function Q is determined uniquely by $\mathfrak{M}$ up to equivalence and a constant factor of modulus one ». There is no change in the proof.

b) *Wiener compactification.* — We shall reformulate a few of our results in [4] in terms of the Wiener compactification of our surface R. Let Y be a class of continuous functions on R with values in the extended real line $I = [-\infty, \infty]$ and let $C_0(R)$ be the set of all continuous real-valued functions on R with compact support. Then, the topological direct product $\Pi\{I_f : f \in Y \cup C_0(R)\}$, which we denote by I^Y, is compact, where $I_f = I$ for all $f \in Y \cup C_0(R)$. We define an injection i_Y of R into I^Y by

$$i_Y(z) = \{f(z) : f \in Y \cup C_0(R)\}.$$

i_Y is a homeomorphism of R into I^Y. Let R_Y^* be the closure of the image of R under i_Y. The compact space R_Y^* thus obtained is called the Y-compactification of the surface R. The Y-harmonic boundary, Γ_Y, of R is then defined as the set of all point $b \in R_Y^* \setminus R$ such that

$$\liminf \{p(z) : z \in R,\ z \to b\} = 0$$

holds for any potential p on R. If we have two classes Y and Y′ with $Y \subseteq Y'$, then we have a natural projection from $I^{Y'}$ onto I^Y, which induces a continuous mapping, say $i_{Y',Y}$, from $R_{Y'}^*$ into R_Y^* (resp., from $\Gamma_{Y'}$ into Γ_Y). When the mapping $i_{Y',Y}$ (resp., the restriction of $i_{Y',Y}$ to $\Gamma_{Y'}$) is a homeomorphism of $R_{Y'}^*$ with R_Y^* (resp., $\Gamma_{Y'}$ with Γ_Y), then we identify $R_{Y'}^*$ with R_Y^* (resp., $\Gamma_{Y'}$ with Γ_Y) and write $R_{Y'}^* = R_Y^*$ (resp., $\Gamma_{Y'} = \Gamma_Y$). We denote by W, BW, HB, and HP the classes of continuous Wiener functions on R, bounded continuous Wiener functions on R, bounded harmonic functions on R, and functions that are expressed as difference of two nonnegative harmonic functions on R, respectively. Then, $\mathrm{HB} \subseteq \mathrm{BW} \subseteq \mathrm{W}$, $\mathrm{HB} \subseteq \mathrm{HP} \subseteq \mathrm{W}$, and it is seen that $\mathrm{R}^*_{\mathrm{W}} = \mathrm{R}^*_{\mathrm{BW}}$ and $\Gamma_{\mathrm{HB}} = \Gamma_{\mathrm{BW}} = \Gamma_{\mathrm{W}} = \Gamma_{\mathrm{HP}}$ (cf., [2; Abschnitt 9]). We call $\mathrm{R}^*_{\mathrm{W}}$ (resp., Γ_{W}) the Wiener compactification (resp., the Wiener harmonic boundary) of the surface R. Every function $u \in \mathrm{HB}$ (resp., HP) can be extended uniquely to a bounded real-valued (resp., extended real-valued) continuous function on $\mathrm{R}^*_{\mathrm{W}}$, whose restriction to Γ_{W} is denoted by $\tilde{u}$. We know that the continuous extension of any harmonic function $u \in \mathrm{HB}$ attains its maximum on Γ_{W} and that the set of functions $\tilde{u}$ with $u \in \mathrm{HB}$ coincides with the space $\mathrm{C_R}(\Gamma_{\mathrm{W}})$ of all real-valued continuous functions on Γ_{W}. In fact, the correspondence $u \to \tilde{u}$ is a vector lattice isomorphism of HB onto $\mathrm{C_R}(\Gamma_{\mathrm{W}})$. Since HB is conditionally complete, $\mathrm{C_R}(\Gamma_{\mathrm{W}})$ is also conditionally complete with respect to the pointwise operations max and min, so that Γ_{W} is a Stonean space (cf., [2; Satz 9.6]). For each $f \in \mathrm{C_R}(\Gamma_{\mathrm{W}})$, let H_f be the function in HB such that $(\mathrm{H}_f)^{\sim} = f$. Then, for each fixed $a \in \mathrm{R}$, the mapping $f \to \mathrm{H}_f(a)$ is a positive linear functional on $\mathrm{C_R}(\Gamma_{\mathrm{W}})$, so that it determines a positive measure, say $d\omega_a$, on Γ_{W}. It is known that the closed support of the measure $d\omega_a$ is equal to Γ_{W} and also that, for any $a, a' \in \mathrm{R}$, the measures $d\omega_a$ and $d\omega_{a'}$ are mutually boundedly absolutely continuous. Fix a point $a_0 \in \mathrm{R}$ once for all and let $d\omega$ denote the harmonic measure corresponding to the point a_0. For each $f \in \mathrm{L}^1_{\mathrm{R}}(d\omega)$, $a \to \int_{\Gamma_{\mathrm{W}}} f\, d\omega_a$ defines a harmonic function, again denoted by H_f. The mapping $f \to \mathrm{H}_f$ is a vector lattice isomorphism of $\mathrm{L}^1_{\mathrm{R}}(d\omega)$ with the class HB' of quasibounded (or outer) harmonic functions on R and coincides on $\mathrm{C_R}(\Gamma_{\mathrm{W}})$ with the mapping $f \to \mathrm{H}_f$ defined before. We also see that $f = (\mathrm{H}_f)^{\sim}$ $d\omega$-a.e. for any $f \in \mathrm{L}^1_{\mathrm{R}}(d\omega)$.

Now we suppose that the surface R is hyperbolic and consider its Martin compactification R^*. Using the notations explained in Section 2 or in [4; Section 3], we see that the correspondence $u \to \hat{u}$ is a vector lattice isomorphism from HB (resp., HB') onto the space $\mathrm{L}^\infty_{\mathrm{R}}(d\chi)$ (resp.,

$L^1_R(d\chi)$) of all (equivalence classes of) real-valued bounded (resp., integrable) Borel functions on the Martin boundary $\Delta_1 = \Delta_1(R)$ equipped with the measure $d\chi$. The space $L^\infty_R(d\chi)$ is known to be isometrically isomorphic with the vector lattice $C_R(\Gamma)$ of all real-valued continuous functions on a compact Hausdorff space Γ, which turns out to be equal to the set of maximal order-ideals of the lattice $L^\infty_R(d\chi)$. Let ψ be the canonical vector lattice isomorphism of $L^\infty_R(d\chi)$ onto $C_R(\Gamma)$. Combining three vector lattice isomorphisms $C_R(\Gamma_W) \to HB$, $HB \to L^\infty_R(d\chi)$ and $L^\infty_R(d\chi) \to C_R(\Gamma)$ given respectively by $f \to H_f$, $u \to \hat{u}$ and ψ, we get a vector lattice isomorphism, say π^*, from $C_R(\Gamma_W)$ onto $C_R(\Gamma)$, which preserves the norm. This isomorphism then gives rise to a homeomorphism π from Γ_W onto Γ such that $\pi^*(f) = f \circ \pi^{-1}$ for every $f \in C_R(\Gamma_W)$ (cf., [5; Chapter 7, Section 45]). For each $a \in R$ and $h \in C_R(\Gamma)$, we have

$$H_f(a) = \int_{\Delta_1} \psi^{-1}(h) k(b, a)\, d\chi(b) \tag{5}$$

where $f = h \circ \pi$. If we define a measure $d\mu_a$ (resp., $d\mu$) on Γ by the right-hand side of (5) (with $a = a_0$), then the mapping π is an isomorphism of the measure space $(\Gamma_W, d\omega_a)$ (resp., $(\Gamma_W, d\omega)$) with the measure space $(\Gamma, d\mu_a)$ (resp., $(\Gamma, d\mu)$). Since the space Γ as well as the measure $d\mu_a$ with $a \in R$ are determined by the Martin functions $k(b, .)$ and the measure $d\chi$, we conclude that the Wiener harmonic boundary and the harmonic measures associated with it are determined by the Martin compactification up to an isomorphism. So, certain formulae related to the Martin compactification can be translated into the ones related to the Wiener compactification. By composing two canonical isometric vector lattice isomorphisms $L^1_R(d\omega) \to HB'$ and $HB' \to L^1_R(d\chi)$ given respectively by $f \to H_f$ and $u \to \hat{u}$, we get a vector lattice isomorphism, j_R, of $L^1_R(d\omega)$ onto $L^1_R(d\chi)$, which preserves the constants and the norm.

Proposition 12. — *(a) The mapping j_R is an isometric vector lattice isomorphism of $L^p_R(d\omega)$ onto $L^p_R(d\chi)$ for each p with $1 \leqslant p \leqslant \infty$.*

b) j_R is multiplicative in the following sense: for any

$f \in L^p_R(d\omega)$ *and* $g \in L^q_R(d\omega)$ *with* $1 \leqslant p,\ q \leqslant \infty$ *and*

$$p^{-1} + q^{-1} = 1,$$

we have $j_R(fg) = j_R(f)j_R(g)$.

c) For any $f \in L^1_R(d\omega)$ *we have*

$$\int_{\Gamma_W} f(b)\, d\omega_a(b) = \int_{\Delta_1} j_R(f)(b)k(b, a)\, d\chi(b). \tag{6}$$

Proof. — As it is easy to see the parts (*a*) and (*c*), we shall prove here only the part (*b*).

Let E be any measurable subset of Γ_W and let C_E be its characteristic function, which we regard as an element in $L^\infty_R(d\omega)$ or in $L^1_R(d\omega)$. Put $E' = \Gamma_W \setminus E$. Since

$$C_E + C_{E'} = C_E \vee C_{E'} = 1$$

and $C_E \wedge C_{E'} = 0$, we have

$$j_R(C_E) + j_R(C_{E'}) = j_R(C_E) \vee j_R(C_{E'}) = 1$$

and $j_R(C_E) \wedge j_R(C_{E'}) = 0$ as elements in $L^\infty_R(d\chi)$ or $L^1_R(d\chi)$. This shows that $j_R(C_E)$ corresponds to the characteristic function of some measurable subset of Δ_1. For any two measurable subsets E and F of Γ_W, we clearly have $C_E \wedge C_F = C_{E\cap F} = C_E . C_F$ and therefore

$$j_R(C_E . C_F) = j_R(C_E \wedge C_F) = j_R(C_E) \wedge j_R(C_F) = j_R(C_E) . j_R(C_F).$$

We thus see that $j_R(s)$ is a simple function whenever s is and that $j_R(s_1 . s_2) = j_R(s_1) . j_R(s_2)$ for any simple functions s_1 and s_2 in $L^\infty_R(d\omega)$. Since the simple functions are uniformly dense in $L^\infty_R(d\omega)$, we see that $j_R(fg) = j_R(f)j_R(g)$ for any $f,\ g \in L^\infty_R(d\omega)$. Since $L^\infty_R(d\omega)$ is norm-dense in each $L^p_R(d\omega)$ with $1 \leqslant p \leqslant \infty$, the desired result follows at once by making use of the Hölder inequality. Q.E.D.

Remark. — Clearly, the mapping j_R is then extended to a complex linear isomorphism of $L^1(d\omega)$ onto $L^1(d\chi)$, which is also denoted by the same symbol j_R. It is easy to see that the mapping j_R maps the space $L^p(d\omega)$ onto $L^p(d\chi)$ isometrically for all p with $1 \leqslant p \leqslant \infty$, is multiplicative in the sense explained in the preceding proposition, and satisfies (6) for any $f \in L^1(d\omega)$. If we set

$$H^p(d\omega) = \{\tilde{h} \in L^p(d\omega) : h \in H^p(R)\}$$

and $H^p(d\chi) = \{\hat{h} \in L^p(d\chi) : h \in H^p(R)\}$, then j_R maps $H^p(d\omega)$ isometrically onto $H^p(d\chi)$.

We now define the boundary values of certain meromorphic functions on R. Let h be a meromorphic function on R. We suppose that there exists a nonzero bounded analytic function F on R such that Fh is bounded on R. Then, both F and Fh can be extended as continuous functions to the whole R_W^* and define their boundary functions $\tilde{F}$ and $(Fh)^{\sim}$ on Γ_W. Since $\tilde{F}$ can vanish only on a set of measure zero by Riesz's theorem, $(Fh)^{\sim}/\tilde{F}$ is defined $d\omega$-a.e. If there exists another bounded analytic function, $G(\neq 0)$, on R such that Gh is bounded, then we have

$$\tilde{F}(Gh)^{\sim} = (FGh)^{\sim} = \tilde{G}(Fh)^{\sim}$$

on Γ_W and so $(Fh)^{\sim}/\tilde{F} = (Gh)^{\sim}/\tilde{G}$ $d\omega$-a.e. Thus, $(Fh)^{\sim}/\tilde{F}$ is determined by h uniquely up to equivalence. We define the boundary values $\tilde{h}$ of h on Γ_W as the (equivalence class of) function $(Fh)^{\sim}/\tilde{F}$. On the other hand, considering the Martin compactification, we have the boundary functions $\hat{F}$ and $(Fh)^{\wedge}$, and therefore $\hat{h} = (Fh)^{\wedge}/\hat{F}$ $d\chi$-a.e. As we have $\hat{F} = j_R(\tilde{F})$ and $(Fh)^{\wedge} = j_R((Fh)^{\sim})$, so

$$\hat{h} = (Fh)^{\wedge}/\hat{F} = j_R((Fh)^{\sim})/j_R(\tilde{F}).$$

If $\tilde{h}$ is bounded (or, more generally, integrable), then, by the remark after Proposition 12, $j_R((Fh)^{\sim}) = j_R(\tilde{F})j_R(\tilde{h})$ and consequently $\hat{h} = j_R(\tilde{h})$.

Now suppose that R satisfies the hypothesis (B′) and h is a meromorphic function on R such that $|h|g^{(a)}$ is bounded, where the function $g^{(a)}$ was defined in Theorem 5. Then, by use of the hypothesis (B′), there exists a bounded analytic function F_0 such that

$$|F_0| = g^{(a)}\,\delta(\theta(g^{(a)})^{-1})$$

where $\theta(g^{(a)})$ denotes the character of the l.a.m. $g^{(a)}$. F_0h

is bounded and analytic on R, and therefore $\tilde{h}$ and $\hat{h}$ are defined as above. Since

$$- \log g^{(a)}(z) = \Sigma\{G^{+}(w, z) : w \in Z(a; R^{+})\}$$

is a continuous Wiener potential on R, it vanishes everywhere on Γ_W and so $(g^{(a)})^{\sim}(b) = 1$ for every $b \in \Gamma_W$. By our assumption that $|h| g^{(a)}$ is bounded, $\tilde{h}$ should be bounded on Γ_W. Hence, we see that $\hat{h} = j_R(\tilde{h})$ and $\hat{h}$ is bounded, too. As a conterpart of Theorem 6 (or [4; Theorem 5.12]) we have the following

THEOREM 13. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B′) *and let* $a \in R$. *Let* $u^{\#} \in L^1(d\omega)$ *and suppose that*

$$\int_{\Gamma_W} \tilde{h}(b) u^{\#}(b)\, d\omega_a(b) = 0$$

for each function h, *meromorphic on* R *such that* $|h| g^{(a)}$ *is bounded on* R *and* $h(a) = 0$. *Then, there exists an* $f \in H^1(R)$ *such that* $\tilde{f} = u^{\#}$ $d\omega$-a.e. *on* Γ_W.

Proof. — As we have remarked above, $\tilde{h}$ exists and is bounded on Γ_W. If we define a function u^* on Δ_1 by setting $u^* = j_R(u^{\#})$, then $u^* \in L^1(d\chi)$ and

$$\hat{h}u^* = j_R(\tilde{h}) j_R(u^{\#}) = j_R(\tilde{h}u^{\#}).$$

By use of (6), we have

$$\begin{aligned}\int_{\Delta_1} \hat{h}(b) u^*(b) k(b, a)\, d\chi(b) &= \int_{\Delta_1} j_R(\tilde{h}u^{\#})(b) k(b, a)\, d\chi(b) \\ &= \int_{\Gamma_W} \tilde{h}(b) u^{\#}(b)\, d\omega_a(b) = 0.\end{aligned}$$

So, by Theorem 6, there exists a function $f \in H^1(R)$ such that $\hat{f} = u^*$ $d\chi$-a.e. on Δ_1. By applying the operation j_R^{-1}, we see that $\tilde{f} = u^{\#}$ $d\omega$-a.e. on Γ_W. Q.E.D.

Next we shall deal with invariant subspace theorems. In order to avoid any complication concerning multiple-valued functions, we shall proceed somewhat differently. Let R satisfy the hypothesis (B) and fix a family $\{\delta(\theta)\}$ satisfying the conditions in (B) throught our discussion. Since $\log \delta(\theta)$, for each $\theta \in \Pi(R)$, is a quasibounded (= outer) harmonic function, it admits well-defined boundary functions on the Martin boundary Δ_1 as well as on the Wiener boundary Γ_W of R. We denote by $\delta(\theta)^\wedge$ and $\delta(\theta)^\sim$ the boundary functions for $\delta(\theta)$ on Δ_1 and Γ_W, respectively. We call a function $w^* \in L^\infty(d\chi)$ (resp., $w^\# \in L^\infty(d\omega)$) a rigid function on Δ_1 (resp., Γ_W) if there is a character $\theta \in \Pi(R)$ such that $|w^*| = \delta(\theta)^\wedge$ (resp., $|w^\#| = \delta(\theta)^\sim$). For each p with $1 \leqslant p \leqslant \infty$, we set

$$H^p(d\omega)_0 = \{\tilde{h} \in H^p(d\omega) : \int \tilde{h}\, d\omega = h(a_0) = 0\},$$

and, as in [4], $H^p(d\chi)_0 = \{\hat{h} \in H^p(d\chi) : \int \hat{h}\, d\chi = h(a_0) = 0\}$. Then, we have the following

PROPOSITION 14. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B) *and let* $\mathfrak{M}$ *be a closed (weakly* closed, if* $p = \infty$) *subspace of* $L^p(d\chi)$. *Then,* $\mathfrak{M}$ *is simply invariant if and only if there exists a rigid function* w^* *on* Δ_1 *such that*

$$\mathfrak{M} = [w^* H^\infty(d\chi)]_p = [w^* H^p(d\chi)]_p,$$

where $[\ ]_p$ *denotes the norm closure (the weak* closure, if* $p = \infty$) *in* $L^p(d\chi)$. *The function* w^* *is determined by* $\mathfrak{M}$ *uniquely up to a constant factor of modulus one.*

Proof. — In this proof, we shall use the notations given in [4; Section 7] without further explanation. Suppose first that $\mathfrak{M}$ is simply invariant. Then, by [4; Theorem 7.1, (*b*)] which we know is valid under the hypothesis (B), there exists an i-function Q of some character θ such that

$$(7)\quad \mathfrak{M} = \{f^* \in L^p(d\chi) : f^*/Q \equiv \hat{h} \text{ for some } h \in MH^p(R)\}.$$

We take an $h_0 \in MH^\infty(R)$ with $|h_0| = \delta(\theta^{-1})$ and define w^* by putting $w^*(b) = Q(b;\, 0)\hat{h}_0(b;\, 0)$ for $b \in \Delta_1$. Then, w^* belongs to $L^\infty(d\chi) \cap \mathfrak{M}$ and therefore

$$w^* H^\infty(d\chi) \subseteq w^* H^p(d\chi) \subseteq \mathfrak{M}.$$

In order to prove the inclusion $\mathfrak{M} \subseteq [w^* H^\infty(d\chi)]_p$, take any $f^* \in \mathfrak{M}$, so that $f^*/Q \equiv \hat{h}$ for some $h \in MH^p(R)$.

Let $h_1 \in \mathrm{MH}^\infty(\mathrm{R})$ be such that $|h_1| = \delta(\theta)$; so $h_0 h_1 \in \mathrm{H}^\infty(\mathrm{R})$ and is outer. We have

$$f^* \cdot (\hat{h}_0 \hat{h}_1) = (\mathrm{Q}\hat{h})(\hat{h}_0\hat{h}_1) = \mathfrak{w}^* \cdot (\hat{h}_1 \hat{h}) \in \mathfrak{w}^* \mathrm{H}^p(d\chi).$$

It follows from this that

$$f^* \cdot (\hat{h}_0 \hat{h}_1 \mathrm{H}^\infty(d\chi)) \subseteq \mathfrak{w}^* \mathrm{H}^\infty(d\chi) \mathrm{H}^p(d\chi) \subseteq [\mathfrak{w}^* \mathrm{H}^\infty(d\chi)]_p.$$

Since $\hat{h}_0\hat{h}_1$ is outer, the invariant subspace theorem for $\mathrm{H}^\infty(d\chi)$ (cf., [4; Corollary 7.2], which is also valid under (B)) shows that the weak* closure of $\hat{h}_0\hat{h}_1\mathrm{H}^\infty(d\chi)$ is exactly $\mathrm{H}^\infty(d\chi)$. This implies that $f^* \in [\mathfrak{w}^*\mathrm{H}^\infty(d\chi)]_p$ and consequently that $\mathfrak{M}$ coincides with $[\mathfrak{w}^*\mathrm{H}^\infty(d\chi)]_p$.

Suppose conversely that $\mathfrak{M} = [\mathfrak{w}^*\mathrm{H}^\infty(d\chi)]_p$ for some rigid function $\mathfrak{w}^*$ on Δ_1. So there exists a character $\theta \in \Pi(\mathrm{R})$ such that $|\mathfrak{w}^*| = \delta(\theta)^\wedge$ on Δ_1. Take an $h_0 \in \mathrm{MH}^\infty(\mathrm{R})$ such that $|h_0| = \delta(\theta)$ on R and define an i-function Q by $\mathrm{Q}(b; \alpha) = \mathfrak{w}^*(b)/\hat{h}_0(b; a)$ with $b \in \Delta_1$ and $\alpha \in \mathrm{H}_1(\mathrm{R}; \mathbf{Z})$. Then, it is easy to see that $[\mathfrak{w}^*\mathrm{H}^\infty(d\chi)]_p$ is equal to the subspace of the form (7) and consequently that $\mathfrak{M}$ is simply invariant.

To see the uniqueness of the expression, suppose that we have $[\mathfrak{w}_1^*\mathrm{H}^\infty(d\chi)]_p = [\mathfrak{w}_2^*\mathrm{H}^\infty(d\chi)]_p$ for rigid functions $\mathfrak{w}_1^*$ and $\mathfrak{w}_2^*$ on Δ_1. We take characters θ_1 and θ_2 in such a way that $|\mathfrak{w}_1^*| = \delta(\theta_1)^\wedge$ and $|\mathfrak{w}^*| = \delta(\theta_2)^\wedge$. We also choose $h_1,\ h_2 \in \mathrm{MH}^\infty(\mathrm{R})$ such that $|h_1| = \delta(\theta_1)$ and $|h_2| = \delta(\theta_2)$ on R. If we put $\mathrm{Q}_i = \mathfrak{w}_i^*/\hat{h}_i$ for $i = 1, 2$, then we see that Q_1 and Q_2 are i-functions on Δ_1 and the invariant subspaces of the form (7) corresponding to Q_1 and to Q_2 are the same. By [4; Theorem 7.2, (*b*)] (cf., the subsection (*a*) of this section), Q_1 and Q_2 differ only by a constant factor of modulus one. This shows that $\theta_1 = \theta_2$, and so h_1 and h_2 differ only by a constant factor of modulus one. Hence, $\mathfrak{w}_1^*$ and $\mathfrak{w}_2^*$ differ only by a constant factor of modulus one. Q.E.D.

As a counterpart of Theorem 4 (or [4; Theorem 7.1)], we have the following.

THEOREM 15. — *Let* R *be a hyperbolic Riemann surface satisfying the hypothesis* (B) *and let* $1 \leqslant p \leqslant \infty$. *Let* $\mathfrak{N}$

be a closed (weakly closed, if* $p = \infty$) *subspace of* $L^p(d\omega)$ *such that* $H^\infty(d\omega)\mathfrak{N} \subseteq \mathfrak{N}$.

a) $\mathfrak{N}$ *is doubly invariant, i.e.,* $H^\infty(d\omega)_0\mathfrak{N}$ *is dense (weakly* dense, if* $p = \infty$) *in* $\mathfrak{N}$, *if and only if there exists a measurable subset* Ξ *of* Γ_W *such that* $\mathfrak{N} = C_\Xi L^p(d\omega)$, *where* C_Ξ *denotes the characteristic function of* Ξ. *The set* Ξ *is determined by* $\mathfrak{N}$ *uniquely up to a null set.*

b) $\mathfrak{N}$ *is simply invariant, i.e.,* $H^\infty(d\omega)_0\mathfrak{N}$ *is not dense (weakly* dense, if* $p = \infty$) *in* $\mathfrak{N}$, *if and only if there exists a rigid function* $w^\#$ *on* Γ_W *such that*

$$\mathfrak{N} = [w^\# H^\infty(d\omega)]_p = [w^\# H^p(d\omega)]_p.$$

The function $w^\#$ *is determined by* $\mathfrak{N}$ *uniquely up to a constant factor of modulus one.*

Proof. — We put $\mathfrak{M} = j_R(\mathfrak{N})$. In view of the remark after Proposition 12, $\mathfrak{M}$ is a closed (weakly* closed, if $p=\infty$) subspace of $L^p(d\chi)$ such that $H^\infty(d\chi)\mathfrak{M} \subseteq \mathfrak{M}$. Moreover, $\mathfrak{M}$ is doubly invariant (resp., simply invariant) if $\mathfrak{N}$ is. Thus, in order to get the desired result, we have only to transform [4; Theorem 7.1, (*a*)] and Proposition 14 by means of the mapping j_R^{-1}. Q.E.D.

Remark. — In Proposition 14 (resp., Theorem 15, (*b*)), it is not always possible to replace $[w^* H^p(d\chi)]_p$ (resp., $[w^\# H^p(d\omega)]_p$) by $w^* H^p(d\chi)$ (resp., $w^\# H^p(d\omega)$). For this, see [7; Chapter 7].

c) A comment on the hypothesis (B). — We are not satisfied with our hypothesis (B), because it is not quite intuitive. One may ask how restrictive this hypothesis is, compared with, for instance, the hypothesis (B′) of H. Widom. We do not have, at this moment, any satisfactory answer to this problem.

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