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## On some spaces which are covered by a product space

# ON SOME SPACES WHICH ARE COVERED
# BY A PRODUCT SPACE

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The subject which we want to discuss here starts from the well known reducibility theorem proved by G. de Rham [16] for complete reducible Riemannian manifolds, which states that the universal covering space of such a manifold is a product manifold and for which several proofs as well as notable generalizations and applications are now available [8, 24, 25].

Particularly, S. Kashiwabara [5] proved a reducibility theorem for complete affinely connected manifolds and next he was able to give a similar proof for a class of locally product differentiable manifolds which satisfy some topological conditions [6] and to make a complete study of an important particular case [7].

On another side, Ia. L. Šapiro [17-20] and Šapiro-Žukova [21] gave a detailed description of the structure of the reducible complete Riemannian manifolds (see also Wang [24]) and of the locally product manifolds which can be given a reducible complete Riemann metric and which, also, have a product universal covering space.

In the theory of foliations there is also an important situation when a foliate manifold has a product universal covering space. This situation has been investigated in codimension one by G. Reeb [14] and S.P. Novikov [12] and in codimension $\geqslant 2$ by L. Conlon [1] and J. Meyer [11].

Finally, such a situation is encountered in an important theorem of Riemannian geometry due to Cheeger, Gromoll and Lichnerowicz [10].

All these clearly suggest that it would be interesting to analyse the geometric structure of the topological spaces which are covered by a topological product.

Actually, a careful reading of the above mentioned papers by Kashiwabara and Šapiro shows that the results of these authors, if considered in a proper frame are of a topological nature. Thus, one can get a significant information on the considered class of spaces by giving a topological version of the results of Kashiwabara and Šapiro.

It is just the aim of this Note to give such a topological version and this is accomplished in the first two sections. In the third section some complementary remarks are made including algebraic topology information and an application to topological groups. Finally, in the last section, the theory is applied to differentiable foliations which gives an alternative way for the deduction of the results of Reeb, Novikov and Conlon and of a part of the Cheeger-Gromoll-Lichnerowicz theorem cited above. Consequently, we can say that there are no essentially new geometrical facts in this paper, but that there is a new framework which gives the results their full generality.

## 1. Spaces covered by a product and the topological version of Kashiwabara's results.

Let us begin with the folIowing preliminary considerations. If $\varphi : S \to T$ and $\sigma : S \to S$ are two continuous maps of topological spaces, $\sigma$ is said to be $\varphi$-compatible if there is a continuous map $\sigma' : T \to T$ such that $\varphi \sigma = \sigma' \varphi$. Clearly, if $\varphi$ is epimorphic $\sigma'$ is unique. Let $\varphi$ be epimorphic and let $G$ be a group of $\varphi$-compatible homeomorphisms of $S$. Then it is obvious that

$$G' = \{g' : T \to T / g' \varphi = \varphi g , g \in G\}$$

is a group of homeomorphisms of $T$ and the map $g \mapsto g'$ defines an epimorphism $q : G \to G'$. Particularly, if $S = S_1 \times S_2$ with the projections $p_1, p_2$, a group $G$ of homeomorphisms of $S$ is said to be compatible with the product structure of $S$ if it is compatible with both $p_1$ and $p_2$. In this case, two corresponding groups $G_1, G_2$ of

homeomorphisms of $S_1$, $S_2$ respectively can be derived as above, together with two epimorphisms $q_i : G \to G_i$ ($i = 1, 2$) and it follows easily that $g \in G$ acts on $S$ by the formula

$$g (s_1 , s_2) = (q_1 (g) s_1 , q_2 (g) s_2)$$

This shows that $q_1 \times q_2 : G \to G_1 \times G_2$ is a monomorphism, which identifies $G$ with the subgroup $\overline{G}$ of $G_1 \times G_2$ given by the pairs $(g_1, g_2)$ such that $g_1 = q_1 (g)$, $g_2 = q_2 (g)$ for some $g \in G$.

Now, we can introduce the structures which we are interested in.

1. DEFINITION.– *Let X be a connected locally path-connected and semi-locally 1-connected metrizable topological space with a countable basis and at most countable fundamental group. A product covering (P.C.) structure on X is a regular covering map $p : \widetilde{X} \to X$ with $\widetilde{X}$ connected and such that $\widetilde{X} = Y \times Z$ and the covering transformation group G is compatible with the product structure of $\widetilde{X}$. Isomorphisms of P.C. structures on X can be considered in an obvious manner. A pair consisting of a space X as above and a P.C. structure on X is called, in this first section, a space covered by a product (C.P. space). If for the P.C. structure Y or Z reduces to one point the structure is called trivial.*

It is worth to remark that, though the C.P. spaces are characterized by a property of their covering spaces, this property does not depend on the fundamental group of the space only. Actually, $R^n$ and $S^n$ ($n \geqslant 2$) have the same (trivial) fundamental group, but while $R^n$ has a non-trivial decomposition as a product, i.e. a P.C. structure, $S^n$ has no such decomposition [2].

The following results will introduce the fundamental structural properties of the C.P. spaces.

2. THEOREM. – *If X is a C.P. space, it admits two disjoint partitions*

$$X = \bigcup_{\alpha \in A} L_\alpha = \bigcup_{\sigma \in \Sigma} \Lambda_\sigma \qquad (1)$$

*where $L_\alpha$ and $\Lambda_\sigma$ are connected subspaces of X and all the intersections $L_\alpha \cap \Lambda_\sigma$ are nonvoid totally disconnected subsets of X.*

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*Proof.* – Consider the P.C. structure of $X$ with the notation of definition 1. From the fact that the covering group $G$ is compatible with the product structure $\widetilde{X} = Y \times Z$ and that it acts transitively on the fibres of $p$, one deduces that $p\,(Y \times \{z\})\,(z \in Z)$ and $p\,(\{y\} \times Z)\,(y \in Y)$ are the two partitions looked for. We shall say that $L_\alpha$ are the leaves of type 1 and $\Lambda_\sigma$ are the leaves of type 2 of $X$.

Consider now $L_{\alpha_0} = p\,(Y \times \{z_0\})$, $\Lambda_{\sigma_0} = p\,(\{y_0\} \times Z)$. Then

$$p\,(y_0, z_0) \in L_{\alpha_0} \cap \Lambda_{\sigma_0},$$

hence the intersection is nonvoid. Moreover, one can deduce that

$$p^{-1}\,(L_{\alpha_0}) = Y \times G_2\,(z_0)\,,\, p^{-1}\,(\Lambda_{\sigma_0}) = G_1\,(y_0) \times Z,$$

$$p^{-1}\,(L_{\alpha_0} \cap \Lambda_{\sigma_0}) = G_1\,(y_0) \times G_2\,(z_0) \tag{2}$$

where $G_1\,(y_0)$ and $G_2\,(z_0)$ are the trajectories of $y_0$ and $z_0$ by the induced groups $G_1$ and $G_2$ in $Y$, $Z$ respectively.

Since $G$ is a factor group of the fundamental group of $X$, it is also at most countable and, hence, so is $L_{\alpha_0} \cap \Lambda_{\sigma_0}$. Then, by a well known theorem of the dimension theory, $L_{\alpha_0} \cap \Lambda_{\sigma_0}$ has the dimension zero and it is a totally disconnected space [9].

Next, let $X$ be an arbitrary topological space, which has two disjoint partitions of the form (1) with connected leaves. Then, for every $Y \subset X$ one gets the disjoint connected partitions given by

$$Y = \bigcup_{\alpha \in A} C\,(L_\alpha \cap Y) = \bigcup_{\sigma \in \Sigma} C\,(\Lambda_\sigma \cap Y) \tag{3}$$

where $C$ denotes connected components, and we denote by

$$\rho_i\,(Y)\,(i = 1\,,\,2)$$

the corresponding equivalence relations on $Y$. The equivalence classes of $\rho_i\,(Y)$ will be called *slices* of type 1 and 2 respectively and they will be denoted by brackets. Now, we can consider the map

$$\pi\,(Y) : Y \rightarrow (Y/\rho_1\,(Y)) \times (Y/\rho_2\,(Y)) \tag{4}$$

defined by $\pi\,(y) = ([y]_1, [y]_2)$ and we shall give

3. DEFINITION. – *The subset* $Y \subseteq X$ *is distinguished with respect to* (1) *if it is open and connected and if* $\pi\,(Y)$ *is a homeomorphism.*

4. THEOREM. – *Every point of a C.P. space X has a basis of neighbourhoods which are distinguished with respect to the partitions of theorem 2 and the corresponding slices are locally path-connected.*

*Proof.* – Clearly, if $x \in V \subset X$, where $V$ is open, $x$ has an open connected neighbourhood $\widetilde{U} \subseteq V$ for which $\widetilde{U} = A \times B \subseteq \widetilde{X}$ exists, with $A$ and $B$ connected open subsets of $Y$, $Z$ respectively, and such that $p/\widetilde{U}$ sends homeomorphically $\widetilde{U}$ onto $U$. Now, one has by (2)

$$U \cap L_{\alpha_0} = p\,(\widetilde{U} \cap p^{-1}\,(L_{\alpha_0})) = p\,[A \times (B \cap G_2\,(z_0))] \qquad (5)$$

and, since $G_2\,(z_0)$ is 0-dimensional hence totally disconnected, it follows that the slices of type 1 have the form $p\,(A \times \{b\})$. A similar result holds for the slices of type 2 and, since $p$ is a homeomorphism from $\widetilde{U}$ onto $U$ and $\widetilde{U}$ is distinguished in an obvious sense in $\widetilde{X}$, we get that $U$ is distinguished in $X$, which proves our first assertion. The second assertion is trivial.

We shall now establish an important connection with a generalization of the foliation theory due to Reeb [15].

5. PROPOSITION. – *Let $X$ be a $T_1$ topological space endowed with the two partitions (1) having connected leaves, and suppose that every point of $X$ has a basis of distinguished neighbourhoods with locally connected slices. Then, the distinguished neighbourhoods and the two corresponding equivalence relations $\rho_i\,(i = 1\,,2)$ define on $X$ two generalized dynamical systems in the sense of Reeb [15], whose leaves are just the leaves defined by (1).*

*Proof.* – From the existence of the homeomorphism (4), it follows easily that a distinguished neighbourhood $Y$ is homeomorphic with the product of any two of its slices of different types, whence the conditions of Reeb's definition (namely : (i) $\rho_i$ are open equivalence relations, (ii) the slices are connected and locally connected, (iii) the slices of $Y$ are closed in $Y$, (iiii) for every

$$a \in Y \cap Y'$$

there is a distinguished $Y''$ such that $\rho_i$, $\rho_i'$ induce the same equivalence relation in $Y''$) are trivially verified and the proposition follows.

Then, following Reeb [15], one obtains on $X$ of proposition 5
two new topologies which are finer than the initial topology of $X$
and whose open sets are unions of open sets of the slices of type 1
and 2 respectively. These will be called the *leaf topologies* and play
an important role in the sequel. From now on, we make the convention
that the leaves are always to be considered with the respective leaf
topology while $X$ has always the initial topology. We also remark
that the *leaf topologies* can be similarly defined without the assump-
tion that X be a $T_1$ space.

6. COROLLARY. — *The leaves of a C.P. space $X$ define generalized
dynamical systems. If* $L_{\alpha_0} = p$ $(Y \times \{z_0\})$ *is a leaf endowed with
the corresponding leaf topology,* $L_{\alpha_0}$ *is connected, locally path
connected and* $p/Y \times \{z_0\}$ *is a covering map, and similarly for*
$\Lambda_{\sigma_0}$.

In fact, only the second assertion has still to be verified and
this is straightforward.

We proceed now by considering a notion due to Kashiwabara
[6].

7. DEFINITION. — *Let $X$ be a topological space with the two
partitions* (1). *Then, let $x(\sigma)$ $(\sigma \in [0, 1])$ be a path with origin
$x_0 = x(0)$ in the leaf $L(x_0)$ and $x'(\tau)$ $(\tau \in [0, 1])$ a path with the
same origin $x_0 = x'(0)$ in $\Lambda(x_0)$. A continuous map $f : [0, 1]^2 \to X$
is called a latticed map with respect to the paths $x, x'$ if one has*

$$f(\sigma, 0) = x(\sigma), f(0, \tau) = x'(\tau), f(\sigma, \tau) \in L(x'(\tau)) \cap \Lambda(x(\sigma)). \quad (6)$$

8. PROPOSITION. – *If the space $X$ has the partition* (1) *and if
every point of $X$ has a distinguished neighbourhood, then, if the
latticed map of two paths exists, it is unique.*

*Proof.* — We already remarked that a distinguished neighbourhood
is homeomorphic with the product of any two of its slices of different
types and, hence, the maps with values in such a neighbourhood
can be projected on the two slices. Then the proof of the announced
proposition as given in [6] for manifolds holds without change in
the situation considered here.

We can derive now another important property of the C.P. spaces.

9. THEOREM. – *In a C.P. space X, latticed maps exist for every couple of paths satisfying the conditions of definition 7.*

*Proof.* – We use the notation of definitions 1 and 7 and consider $\tilde{x}_0 = (y_0, z_0) \in p^{-1}(x_0)$. Let $\tilde{x}(\sigma)$ be the lift of $x(\sigma)$ beginning at $\tilde{x}_0$. From connexity considerations, it follows that $\tilde{x}(\sigma)$ is contained in $Y \times \{z_0\}$. Similarly, the lift $\tilde{x}'(\tau)$ of $x'(\tau)$ beginning at $\tilde{x}_0$ is contained in $\{y_0\} \times Z$. Hence one has

$$\tilde{x}(\sigma) = (y(\sigma), z_0), \tilde{x}'(\tau) = (y_0, z(\tau)).$$

Now, it follows easily that $f(\sigma, \tau) = p(y(\sigma), z(\tau))$ is just the latticed map looked for.

Finally, we arrived at the main theorem of this section :

10. THEOREM. – *Suppose that the space X satisfies all the conditions of definition 1. Then X is given a P.C. structure (hence it becames a C.P. space) if and only if : 1° X has two partitions (1) with connected leaves, such that any two leaves of different types have a totally disconnected intersection, 2° every point of X has a basis of distinguished neighbourhoods and the leaf topologies are locally path connected, 3° for every couple of paths like in definition 7 a corresponding latticed map exists.*

*Proof.* – The announced conditions are necessary as shown by theorems 2, 4 and 9 above.

To prove the sufficiency, let $\omega : \tilde{X} \to X$ be the universal (1-connected) covering of $X$. Then, the connected components of $\omega^{-1}(L_\alpha)$ and $\omega^{-1}(\Lambda_\varrho)$ define two partitions of $\tilde{X}$ which are easily seen to satisfy on $\tilde{X}$ the conditions 1°, 2°, 3°. (Particularly, the latticed maps on $\tilde{X}$ are lifts of latticed maps of $X$). For the leaves of these partitions on $X$ and $\tilde{X}$, we shall always consider the leaf topologies which can be introduced in view of proposition 5.

Next, the proof of theorem 2 of [6] can be seen to work in our situation too. The various lemmas needed, which are proved in [6] for differentiable manifolds, remain true in $\tilde{X}$. The only detail to which attention must be paid is the fact that two leaves have a

totally disconnected intersection and, hence, any continuous map
with connected domain and with values in such an intersection is
constant. In [6] this detail has not been mentioned explicitly
because the mentioned property always holds there.

Hence, applying that theorem of Kashiwabara, we get

$$\widetilde{X} \approx \widetilde{L}_\alpha \times \widetilde{\Lambda}_\sigma,$$

where, obviously, $\widetilde{L}_\alpha$ and $\widetilde{\Lambda}_\sigma$ are the universal covering spaces of
some leaves $L_\alpha$, $\Lambda_\sigma$ of $X$, with the leaf topologies. Moreover, the
previous homeomorphism preserves the leaf structure or, equivalently,
the covering transformations of $\widetilde{X}$ are compatible with the product
structure $\widetilde{L}_\alpha \times \widetilde{\Lambda}_\sigma$.

We thus obtained a P.C. structure on $X$ which ends the proof
of theorem 10. It is important to remark that the sufficiency proof
made no use of the conditions that $X$ be metrizable and with countable
basis and fundamental group, which makes this result even more
general.

Theorem 10 suggests to give

11. DEFINITION. — *Two P.C. structures on $X$ are called equivalent
if the corresponding partitions of $X$ into leaves are the same for
the two structures.*

Then it obviously follows

12. COROLLARY. — *The universal covering space of a C.P. space $X$
is isomorphic to the product of the universal covering spaces of
two arbitrary leaves of different types of $X$. The equivalence classes
of P.C. structures of $X$ are in an one-to-one correspondence with
the isomorphism classes of decompositions of its universal covering
space $\widetilde{X}$ into products compatible with the covering transformations
of $\widetilde{X}$.*

13. CONVENTION. — *From now on, a C.P. space will be understood
as a pair consisting of a space $X$ as in definition 1 and an equivalence
class of P.C. structures on $X$.*

## 2. The topological version of Ia. L. Šapiro's results.

More results on the geometric structure of C.P. spaces can be obtained by generalizing the studies of Ia. L. Šapiro [17-21].

We consider again a C.P. space $X$ in the sense of convention 1.13 and we shall use all the notation of section 1 with the supplementary convention that $\widetilde{X}$ is always the universal covering space of $X$, whence it follows that the group $G$ is isomorphic with $\pi(X)$.

We also recall that $G$ has the associated groups $G_i$ $(i = 1, 2)$ acting on $Y$ and $Z$ respectively. In [17], the structure of $G$ is precised by introducing the groups $G_i' = q_i (\ker q_j)$ $(i \neq j = 1, 2)$. Then, the correspondence of section 1, which relates the pairs of $\overline{G} \approx G$ induces an isomorphism $\Theta : G_1/G_1' \to G_2/G_2'$ such that one has

$$G \approx \overline{G} = \{(g_1, g_2) \in G_1 \times G_2/\Theta [g_1] = [g_2]\}, \qquad (1)$$

where the brackets denote classes with respect to $G_i'$. In this manner, one sees that the group $G$ is equivalent to the system $(G_i, G_i', \Theta ; i = 1, 2)$.

In the sequel, we shall need the following fundamental lemma (which is folklore-type and replaces in our version the lemma V of [17]).

1. LEMMA. — *Let $S$ be a connected locally path-connected and semi-locally 1-connected topological space, $G$ be a group of homeomorphisms acting properly discontinuous on $S$ and $H$ an invariant subgroup of $G$. Then, the natural bijection $(S/H)/(G/H) \approx S/G$ is a homeomorphism.*

*Proof.* — The existence of this bijection requires only a technical verification, which is given in lemma IV of [17]. In order to see that one has a homeomorphism, we first remark that $H$ is acting properly discontinuous on $S$, which defines the regular covering map $p_1 : S \to S/H$ with covering group H. Next, $S/H$ is connected and $G/H$ acts properly discontinuous on $S/H$, which defines a regular covering map $p_2 : S/H \to (S/H)/(G/H)$ whose covering group in $G/H$. Since $S$ and hence $(S/H)/(G/H)$ are semilocally 1-connected, $p_2 \circ p_1$ is also a regular covering map and one sees that its covering group is

G. The investigated map is just the identification of the base space of $p_2 \circ p_1$ with S/G, hence it is a homeomorphism. (For details regarding covering spaces see for instance [22]).

*Remark.* – Lemma 1 remains valid if the hypothesis that $S$ is semi-locally 1-connected is replaced by the hypothesis that so is S/H or (S/H)/(G/H).

Now, in order to obtain the topological version of Šapiro's main theorem we give.

2. DEFINITION. – *The C.P. space X is called regular if on* $\widetilde{X} = Y \times Z$ *there is at least one point* $(y_0, z_0)$ *which is not a fixed point of any nonidentical transformation of* $G_1 \times G_2$.

One obtains now the following main theorem :

3. THEOREM. – *Let X be a regular C.P. space. Then there are point* $x_0 \in X$ *such that one has a leaf-preserving homeomorphism*

$$X \approx L(x_0) \times \Lambda(x_0)/\Psi,$$

*where* $L(x_0)$, $\Lambda(x_0)$ *denote the leaves through* $x_0$ *with the corresponding leaf topologies and* $\Psi$ *is a group of homeomorphisms acting properly discontinuous on the product of these leaves and which is isomorphic to*

$$\pi(X, x_0)/\pi(L(x_0), x_0) \times \pi(\Lambda(x_0), x_0).$$

*Also,* $\Psi$ *is compatible with the product structure and induces on* $L(x_0)$, $\Lambda(x_0)$ *groups which are isomorphic to* $G_i/G_i'$ $(i = 1, 2)$.

*Proof.* – The proof is exactly the same as that given by Šapiro [17] for the case of reducible complete Riemannian manifolds and it goes on the following lines. The universal covering space of $X$ being $\widetilde{X} = Y \times Z$, one has $X \approx Y \times Z/G$. Now, let $(y_0, z_0) \in Y \times Z$ be a points un-fixed by all the nonidentical elements of $G_1 \times G_2$; such points exist in view of the regularity of $X$ in the sense of defi - nition 2. Consider $x_0 = p(y_0, z_0)$ and

$$L(x_0) = p(Y \times \{z_0\}), \Lambda(x_0) = p(\{y_0\} \times Z).$$

Since $g_1 y_0 \neq y_0$ if id. $\neq g_1 \in G_1$ and $g_2 z_0 \neq z_0$ if id. $\neq g_2 \in G_2$, it

follows that the covering map $p/\{y_0\} \times Z$ has a covering group which can be identified with $G_2'$ and, similarly, $p/Y \times \{z_0\}$ has the covering group $G_1'$. Next, it is easy to see that $G_1' \times G_2'$ is an invariant subgroup of $G$. Hence, the announced homeomorphism follows from lemma 1 by taking there $S = Y \times Z$ (which is a 1-connected space), $G = G$ and $H = G_1' \times G_2'$. All the other assertions of theorem 3 follow by technical verifications.

The previous theorem suggests the consideration of

4. DEFINITION. – *Let* X *be a C.P. space. Then, any leaf-preserving homeomorphism*

$$\xi : X \approx L \times \Lambda/\Psi, \qquad (2)$$

*where* L *and* $\Lambda$ *are connected locally path-connected spaces and* $\Psi$ *is a group of homeomorphisms acting properly discontinuous on* $L \times \Lambda$, *compatible with the product structure and with associate groups* $\Psi_1' = \Psi_2' = $ id., *is called a presentation of* X.

In the case of such a presentation, one obviously has that the isomorphism $\Theta$ of (1) is an isomorphism of the induced groups $\Psi_1$, $\Psi_2$ and one has $\Psi_1 \approx \Psi_2 \approx \Psi$.

Let $X$ be a C.P. space and $\xi$ a leaf-preserving homeomorphism of the form (2), with a properly discontinuous group $\Psi$ of homeomorphisms. Denote again by $\tilde{X}$ the universal covering space of $X$ with the group $G$. Then, one has the following important result.

5. LEMMA. – *The homeomorphism* (2) *is a presentation if and only if the fundamental groups of* L *and* $\Lambda$ *are respectively isomorphic to* $G_1'$ *and* $G_2'$.

*Proof.* – Consider the homeomorphism (2) and denote by $\xi' : L \times \Lambda \to X$ the associated covering. Since this covering is regular with covering group $\Psi$, one has [22].

$$\Psi \approx \pi(X, \xi'(l_0, \lambda_0))/\xi_{\#}' \pi(L \times \Lambda, (l_0, \lambda_0)). \qquad (3)$$

Now, let $\tilde{L}$, $\tilde{\Lambda}$ be the universal covering spaces of $L$ and $\Lambda$. Since $X$ is semilocally 1-connected, one gets, by using the homotopy covering property of $\xi'$ and the discreteness of its fibres, that $L \times \Lambda$, hence $L$ and $\Lambda$ are also semilocally 1-connected. Hence, $\tilde{L}$ and $\tilde{\Lambda}$ are

1-connected, $\widetilde{L} \times \widetilde{\Lambda}$ is the universal covering space of $X$, and $G$ appears as the covering transformation group of this space.

Then, if we denote by $T_i \, (i = 1, 2)$ any groups which are respectively isomorphic with $\pi \, (L)$, $\pi \, (\Lambda)$, the relation (3) becomes

$$\Psi \approx G/T_1 \times T_2. \tag{4}$$

But, it follows by a technical verification that the isomorphism (4) induces isomorphisms

$$\Psi_1' \approx G_1'/T_1 , \, \Psi_2' \approx G_2'/T_2, \tag{5}$$

which proves the announced lemma.

The result given by lemma 5 has been proved for the reducible complete Riemannian manifolds in [17], by using the respective metrics, and it has been used to obtain a uniqueness theorem for presentations, which is valid in the topological version too. Namely :

6. THEOREM. — *Let $\xi$ and $\xi^*$ be two presentations of the C.P. space $X$ and denote with a star all the elements of $\xi^*$. Then,*

$$L \approx L^* , \, \Lambda \approx \Lambda^* , \, \Psi \approx \Psi^*$$

*and the induced groups act compatibly with these homeomorphisms.*

*Proof.* — Like in the proof of lemma 5, we get two universal covering spaces $\widetilde{L} \times \widetilde{\Lambda}$ and $\widetilde{L^*} \times \widetilde{\Lambda^*}$ of $X$. From the equivalence of these two coverings one gets $\widetilde{L} \approx \widetilde{L^*}$, $\widetilde{\Lambda} \approx \widetilde{\Lambda^*}$, $G \approx G^*$, whence, by the previous lemma, $T_i \approx T_i^*$. It follows $L \approx L^*$, $\Lambda \approx \Lambda^*$ and, by formula (4), $\Psi \approx \Psi^*$, which proves the theorem.

By theorem 3, every regular C.P. space has a presentation and by theorem 6 this presentation is essentially unique. Moreover, there are always points $x_0 \in X$ such that $L(x_0) \approx L$ and $\Lambda(x_0) \approx \Lambda$. In this manner, for a regular C.P. space one can use either universal coverings or presentations as representatives of the corresponding class of P.C. structures.

Let us also mention that the definition of the regularity has not been explicitly given by Šapiro, but in his case, of the reducible complete Riemannian manifolds, Šapiro proved that regularity always holds. (For the same case, see also the results of Wang [24]).

As for the possible non-regular C.P. spaces, one can again apply that part of the proof of theorem 3 which consists in factorizing by $G_1' \times G_2'$ and this provides again a presentation of the space, but the presentation covering space will be, generally, the product of some covering spaces of leaves and not of leaves as in theorem 3. Since we don't know very much about those covering spaces, we shall use presentations only in the case of regular C.P. spaces.

Let us also indicate the topological version of some of the other results of Šapiro [18].

7. DEFINITION. — *Let $X$ be a C.P. space and $x \in X$. Then $L(x)$ is a regular leaf if $L(x) = p(Y \times \{z_0\})$ where $g_2 z_0 \neq z_0$ for id. $\neq g_2 \in G_2$. In the contrary, $L(x)$ is nonregular. Similar definitions hold for $\Lambda(x)$. If both $L(x)$ and $\Lambda(x)$ are regular, $X$ is a regular point (and one can apply theorem 3 with $x_0 = x$).*

The notion is important since, if $L(x)$ is a regular leaf then, coming back to the reasoning in the proof of theorem 3, we have that $G_1'$ is the covering group of $p/Y \times \{z_0\}$ and if we factorize the universal covering space of $X$ by $G_1' \times$ id. we get a covering space of $X$ which is of the form $L(x) \times Z$.

As shown in [18], if $X$ has the presentation (2), we could use in definition 7 the covering map $\xi'$ instead of $p$. In this case, every leaf $L(x)$ ($\Lambda(x)$) is covered by $L(\Lambda)$, the regular leaves being those for which $\xi'$ induces a homeomorphism. Hence, all the regular leaves of the same type are homeomorphic and they cover the nonregular leaves.

The fact that all the leaves of a given type have homeomorphic covering spaces can be considered as a global stability property of the kind encountered in the theory of foliations [14]. Another connection with this theory has been considered in proposition VII of [18]. Namely, in the present version, we shall have

8. PROPOSITION. — *Suppose that the leaves $L_\alpha$ satisfy the definition of a regular foliation as given by Palais [13] and that all the non-identical elements of $G_2$ have, at most, a set of fixed points with empty interior. Then, $G_2$ acts properly discontinuous on $Z$ and the leaves $L_\alpha$ are regular in the sense of definition 7. Conversely, if $G_2$ acts properly discontinuous on $Z$, the "foliation" $L_\alpha$ is regular.*

*Proof.* – Let $x_0 = p(y_0, z_0) \in X$ and let U be a regular neigh-bourhood of $x_0$ in the sense of [13]. One may suppose that there is a neighbourhood $\widetilde{U} = A \times B$ of $(y_0, z_0)$ in $\widetilde{X}$ which is sent homeo-morphically by $p$ onto $U$. Now, if $B \cap g_2 B \neq \phi$ for some $g_2 \in G_2$, it is easy to see that every point of this intersection is fixed for $g_2$ and, since this intersection is open, it follows from the hypotheses that $g_2 = $ id. This proves the first part of the proposition. For the second part, we start with $B$ and construct $U$ such that the relation considered above between them holds ; then $U$ will be a regular neighbourhood.

Other interesting results are :

9. THEOREM. – *Let $X$ be a regular C.P. space. Then, the number of intersection points of two regular leaves of different types is equal to the order of the covering transformation group of a presentation of $X$.*

The proof is that of proposition IV of [18].

10. DEFINITION. – *If $X$ is a regular C.P. space, a leaf of $X$ is called completely nonregular when, considering a presentation of $X$ as in theorem 3, the covering leaves of that leaf of $X$ remain unchainged by all the transformations of $\Psi$.*

11. THEOREM. – *Let $X$ be a regular C.P. space and $L_\alpha$ be a completely nonregular leaf of $X$. Then $X$ is a locally trivial fibre bundle with base space $L_\alpha$, fibre $\Lambda$ and structure group $\Psi_2$.*

The proof is that of proposition VI of [18].

### 3. Miscellaneous complementary topological remarks.

A. It is obviously interesting to have a convenient notion of a morphism of C.P. spaces. Following Šapiro and Žukova [21], we shall consider in our version :

1. DEFINITION. – *Let $X$, $S$ be two C.P. spaces. By a morphism between them one understands a continuous map $f : X \to S$ which carries leaves of the same type of $X$ to leaves of the same type of $S$.*

*If, moreover, X and S are regular and f sends one regular point of X to a regular point of S, f is called a regular morphism.*

We already met such maps in the previous section. Here, we want to give one more characterization of morphisms by

2. PROPOSITION. – *Let $f : X \to S$ be a morphism of C.P. spaces. Then f has a lift to the universal  covering spaces of X and S and this lift is compatible with the corresponding product structures. If f is a regular morphism, it has a similar lift to the presentation covering spaces which are products of leaves.*

*Proof.* – Let $p : Y \times Z \to X$, $q : P \times Q \to S$ be the universal coverings of $X$ and $S$ and $G$, $\Gamma$ be the respective covering transformation groups. The existence of a lift $\tilde{f} : Y \times Z \to P \times Q$ such that $fp = q\tilde{f}$ is obvious from the lifting properties of coverings. To show that $\tilde{f}$ is a product map, let $(y_i, z) \in Y \times Z$ $(i = 1, 2)$. Then $p(y_i, z)$ belong to the same leaf $L_\alpha$ of $X$, hence $fp(y_i, z)$ belong to some precised leaf $F$ of $S$ which, for instance, may also be supposed of the first type. It follows $\tilde{f}(y_i, z) \in q^{-1}(F) = P \times \Gamma_2(q_0)$ for some $q_0 \in Q$ and, since $\tilde{f}(y_i, z)$ belong to the connected set $\tilde{f}(Y \times \{z\})$, one must have $\tilde{f}(y_i, z) = (p_i, q_1)$ $(i = 1, 2; q_1 \in Q)$. Hence, we obtain a continuous map $\tilde{f}_2 : Z \to Q$ defined by $\tilde{f}_2(z) = q_1$. Similarly, a continuous map $\tilde{f}_1 : Y \to P$ will be obtained and $\tilde{f} = \tilde{f}_1 \times \tilde{f}_2$, which proves the first assertion or the proposition.

We remark that the universal character of the coverings $p$, $q$ has been used only to show the existence of $\tilde{f}$. Hence, for every coverings for which $\tilde{f}$ exists, the second part of the proof applies and one gets that $\tilde{f}$ is a product map.

It follows that, in order to prove the second part of the proposition we must prove only the existence of the desired lift.

Let $\xi' = L \times \Lambda \to X$ and $\mu' : A \times B \to S$ be presentation coverings of the regular C.P. spaces $X$ and $S$, and let $p$, $q$ be the corresponding universal coverings as above. As shown in section 2, one can identify $L$ and $\Lambda$ with leaves of $X$ and $Y$, $Z$ with the universal covering spaces of these leaves, and similarly for $A$, $B$, $P$, $Q$. Moreover, suppose that $L$ and $\Lambda$ are the leaves through the regular point $x_0$ and $A$, $B$ through the regular point $f(x_0)$. By lemma 2.5, it follows that

$$\pi(L) \approx G_1', \pi(\Lambda) \approx G_2', \pi(A) \approx \Gamma_1', \pi(B) \approx \Gamma_2'$$

and, of course, $\pi(X) \approx G$, $\pi(S) \approx \Gamma$.

Now, one sees that, by these isomorphisms, the monomorphisms $\xi'_{\#}: \pi (L \times \Lambda) \to \pi (X)$, $\mu'_{\#}:\pi (A \times B) \to \pi (S)$ correspond to the inclusion maps $G'_1 \times G'_2 \subseteq G$, $\Gamma'_1 \times \Gamma'_2 \subseteq \Gamma$. Also, the induced homomorphism $f_{\#}: \pi (X) \to \pi (S)$ corresponds to some homomorphism $f_* : G \to \Gamma$. Since $f$ is leaf preserving, we get $f_* (G'_1 \times G'_2) \subseteq \Gamma'_1 \times \Gamma_2$ hence, equivalently $(f \xi')_{\#} \pi (L \times \Lambda) \subseteq \mu'_{\#} \pi (A \times B)$. Then, from the well known lifting property of the coverings [22], it follows that $f$ has a lift $\widetilde{f}: L \times \Lambda \to A \times B$, which ends the proof of the proposition.

Moreover, in just the same way, it follows that every morphism of C.P. spaces admits a product lift to general presentation covering spaces of the given spaces (i.e. which may not be products of leaves).

*Remarks* 1). Clearly, if $f : X \to S$ is a map of C.P. spaces which admits a "product-type" lift to any of the P.C. structures defining $X$ and $S$, $f$ is a morphism in the sense of definition 1.

2) Let $f : X \to S$ be a regular morphism and $\Psi$, $\chi$ the groups associated to presentations of $X$ and $S$ as in theorem 2.3. Then, in view of the structure of $\Psi$ and $\chi$ as given by the same theorem 2.3, one gets an induced homeomorphism $f_* : \Psi \to \chi$ which is compatible with the actions of $\Psi$ and $\chi$ on the spaces of the presentations. Now, one can deduce that the regular morphisms $f : X \to S$ may be identified with morphisms of triples $f_* : (L , \Lambda , \Psi) \to (A, B, \chi)$.

B. At the second place, let us briefly indicate the sources which could provide information about the homotopy and the homology of the C.P. spaces.

A first such source is the classical result that, for every covering map $p : \widetilde{X} \to X$ vhere $X$ is connected and locally path-connected and $\widetilde{X}$ is connected, the induced homomorphisms

$$p_{\#} : \pi_n (\widetilde{X} , \widetilde{x}) \to \pi_n (X , x) \; (p (\widetilde{x}) = x)$$

are isomorphisms for $n \geqslant 2$ and monomorphisms for $n = 1$. If we use again the notation of the previous section, we have

3. PROPOSITION. $-$ *If $X$ is a C.P. space and if $L_\alpha$, $\Lambda_\sigma$ are two arbitrary leaves of different types, then $\pi_n(X) \approx \pi_n (L_\alpha) \oplus \pi_n (\Lambda_\sigma)$ ($n \geqslant 2$).*

4. COROLLARY: – *If $X$ is a C.P. space and if at least one leaf of each family is an Eilenberg-Mac Lane space $K(\pi_i, 1)$ $(i = 1, 2)$, then $X$ is an Eilenberg-Mac Lane space $K(\pi_1, 1)$, where $\pi_1 \oplus \pi_2$ is some subgroup of $\pi$.*

5. COROLLARY. – *If the universal covering space $\widetilde{X}$ has finitely generated singular homology groups, then, for $n \geq 2$, $\pi_n(X)$ are finitely generated and the groups $\pi_n(X)$, $\pi_n(L_\alpha)$ determine the groups $\pi_n(\Lambda_\sigma)$.*

In fact, this follows from proposition 3 and from some results of Hilton [4].

Next, in order to get homology (cohomology) information for our spaces, one could use the known spectral sequences of a covering. For instance, for the homology of the C.P. space $X$, there will be a spectral sequence such that $E_{p,p}^2 \approx H_p(G, H_q(Y \times Z, .))$. These groups depend on the action of $G_i (i = 1, 2)$ on the homology of $Y$ and $Z$, since the homology of $Y \times Z$ can be calculated by the Künneth formula. E.g., in this way it follows that whenever $G_i$ acts trivially on the homology of $Y$ and $Z$ respectively, $G$ acts trivially on the homology of $Y \times Z$.

Some other information can be obtained by the use of the following commutative diagram

$$
\begin{array}{ccc}
 & \widetilde{X} & \\
{\scriptstyle p}\swarrow & & \searrow{\scriptstyle \sigma} \\
X & \xrightarrow{\quad\tau\quad} & B = (Y/G_1) \times (Z/G_2)
\end{array}
\qquad (1)
$$

where $\sigma$ is the canonical projection and $\tau$ is continuous since $p$ is open. The desired information can be obtained via the following proposition :

6. PROPOSITION. – *Consider the diagram (1) and suppose that the trajectories of $G_i (i = 1, 2)$ define on $Y$ and $Z$ equivalence relations having Godbillon's homotopy prolongation property [3]. Then, the fibres of $\tau$ define on $X$ an equivalence relation which also has the homotopy prolongation property. Particularly, if $\sigma$ of (1) is a Serre fibration, $\tau$ is a Serre fibration too.*

*Proof.* — Only a technical verification using the definition of [3] and the lifting properties of the covering map $p$ is needed.

Now, if the hypotheses of proposition 6 are satisfied, the homotopy and homology groups of $X$ enter in some exact sequences and spectral sequences as shown in section IV. 2 of [3] and which we do not write down here. E.g., in view of theorem III 3.3 of [3] this in the case if $G_i$ act without fixed points on $Y$ and $Z$.

Of course, a diagram similar to (1) can be constructed for a presentation covering of $X$ and similar results hold for that diagram too.


C. Finally, we shall consider an application of the developed theory for the topological groups.

Hence, let $X$ be a C.P. space which is a topological group. Let us suppose that the corresponding universal covering space $\tilde{X} = Y \times Z$ with its natural group structure is the product of two topological groups $Y$ and $Z$. Then, we call $X$ a C.P. topological group.

In this case, one derives easily that, if $e$ is the unit of the group $X$, the leaves $L(e)$, $\Lambda(e)$ are permutable subgroups of $X$ and the leaves $L_\alpha$ are obtained from $L(e)$ by translations with elements of $\Lambda(e)$ and $\Lambda_\sigma$ are obtained from $\Lambda(e)$ by translation with elements of $L(e)$.

Now, let $X$ be a topological group satisfying the topological conditions of definition 1.1 and let $e$ be its unit. Let us suppose that there are two topological subgroups $L(e)$, $\Lambda(e)$ of $X$ which are connected and satisfy the following conditions : a) the two subgroups are permutable and have a totally disconnected intersection, b) every $x \in X$ has at least one decomposition of the form $x = l\lambda$, $l \in L(e)$, $\lambda \in \Lambda(e)$, c) for some open connected neighbourhood $U$ of the unit $e$, there are the open and path-connected neighbourhoods $A$, $B$ of $e$ in $L(e)$, $\Lambda(e)$ respectively such that the map $(a, b) \to ab \, (a \in A, b \in B)$ is a homeomorphism, of $A \times B$ onto $U$.

Then, it is easy to see that, by translating $L(e)$ $(\Lambda(e))$ with elements of $\Lambda(e)$ $(L(e))$, one gets two partitions of $X$ which satisfy the three conditions of theorem 1.10. The distinguished neighbourhoods of $e$ are obviously defined by $U$ in view of the homeo-

morphism given by condition c) above, and the distinguished neigh-
bourhoods of an arbitrary $x \in X$ are defined by $x\, U$ since this
neighbourhood is homeomorphic to $l\, A \times \lambda\, B$ where $x = l\,\lambda$. As
for the latticed maps, if $x\,(\sigma)$ and $x'\,(\tau)$ are two paths beginning
at $x_0 = l_0\,\lambda_0$, in the leaves through this point, then

$$f\,(\sigma\,,\tau) = x\,(\sigma)\,\lambda_0^{-1}\,l_0^{-1}\,x'\,(\tau)$$

defines the corresponding latticed map.

It follows that, under the above conditions, $X$ is a C.P. topological
group and if, moreover, $e$ is a regular point for the corresponding
P.C. structure, one gets by theorem 2.3 a presentation of $X$ given
by the product $L\,(e) \times \Lambda\,(e)$.

## 4. Applications to differentiable foliations.

If the theory of the previous section is applied to differentiable
manifolds, one gets (in a slightly generalized version ) the results
of Kashiwabara and Šapiro.

Thus, suppose that $X$ is a finite or infinite dimensional connected
$C^r$-manifold ($r = 0,\ldots,\infty$ or real analytic) with a P.C. structure in
the sense of section 1 and with the universal covering $p : Y \times Z \to X$.
Then, $\widetilde{X} = Y \times Z$ has a canonical $C^r$-structure which makes $p$ a
$C^r$-covering map. Now, suppose that the following supplementary
conditions are satisfied : $Y$ and $Z$ are $C^r$-manifolds and the canonical
$C^r$-structure of $\widetilde{X}$ is $C^r$-equivalent to the $C^r$-product structure $Y \times Z$.
In this case, $X$ will be called a C.P. manifold of class $C^r$. Then, the
covering transformation group $G$ is a group of $C^r$-homeomorphisms
and it follows easily that the leaves of $X$ are submanifolds and
that they define two complementary $C^r$ foliations on $X$, i.e. a $C^r$ -local
product structure. Hence, $X$ is a locally decomposed $C^r$ -manifold
with latticed maps in the sense of Kashiwabara [6]. In this case,
by the proof of [6], theorem 1.10 is valid and gives necessary and
sufficient conditions for $X$ to be a C.P. manifold. Moreover, in
theorem 1.10 it suffices now to ask that latticed maps exists for
paths of the class $C^r$ only.

In [6], it is shown that these conditions hold, for instance,
in the case of reducible complete affinely connected, Finsler and
Riemann manifolds.

Next, if a C.P. manifold is regular in the sense of definition 2.2, we get, by theorem 2.3 a topological covering presentation which can easily be seen to be a $C^r$-covering presentation. The essential result of Šapiro [17] is just that a reducible complete Riemann manifold is regular, and in this case the $C^r$-covering presentation is a Riemannian covering. Also, in [19], it is shown that C.P. manifolds with regular foliations are regular in the sense of definition 2.2.

Finally, we remark that the previous theory applies also to complex analytic manifolds, in which case the theorem of Kashiwabara gives again that $\widetilde{X}$ is diffeomorphic to Y × Z. But, from the construction of this diffeomorphism as given in [6] and using a lemma of Kobayashi-Nomizu [8, Section IX. 8] it follows that, actually, $\widetilde{X}$ is holomorphic with Y × Z.

In the sequel, we want to use the theory of C.P. manifolds in order to obtain in a new manner the results of Reeb [13] and Conlon [1] on some particular class of differentiable foliations. For the notions and notation regarding foliations we send to [14], [1] and [23].

Let $X$ be a connected finite dimensional paracompact differentiable manifold and $\mathfrak{F}$ a foliation of codimension $q$ on $M$, where $q \leqslant m = \dim X$. For the sake of simplicity, differentiability will always be assumed $C^\infty$ although it would be sufficient to take it $C^r$ with $r \geqslant 2$. Let $T(X)$ be the tangent bundle of $X$, E the tangent bundle of $\mathfrak{F}$ and $Q = T(X)/E$ the normal (transverse) bundle. We agree to call *foliate elements* on $X$ all the elements (functions, vector fields, forms, bundles, etc.) which depend only on the leaves of $\mathfrak{F}$, which means that, if

$$\mathfrak{X} = \{U_\theta ; x^a , x^u\}(a = 1, \ldots , q , u = q + 1, \ldots , m)$$

is an *adapted atlas* of $\mathfrak{F}$ on $X$ in the sense that, in $\mathfrak{X}$, $\mathfrak{F}$ is given by the local equations $dx^a = 0$, then the respective elements depend locally on the $x^a$ only [23].

Following [1], we consider

1. DEFINITION. — $\mathfrak{F}$ *is called an e-foliation if Q admits q global foliate cross sections* $Z_1, \ldots , Z_q$ *which are independent at every point of X. If, moreover,* $\mathfrak{F}$ *admits a Haefliger cocycle whose derivative is the unit element of* $Gl (q , R), \mathfrak{F}$ *is called a strong e-foliation.*

In the sequel, it is useful to choose a complementary subbundle of $E$ in $T(X)$ and to identify this subbundle with $Q$. Then, with respect to the atlas $\mathcal{A}$ above, $E$ has the local bases $\{X_u = \partial/\partial x^u\}$ and $Q$ has local bases of the form $\{X_a = \partial/\partial x^a - t_a^u X_u\}$, where $t_a^u$ are some locally defined functions with a suitable transformation law [23].

Now, if $\mathcal{F}$ in an $e$-foliation, $Z_1, \ldots, Z_q$ of definition 1 are identified with vector fields on $X$ which can be locally expressed as

$$Z_a = \xi_a^b X_b \ (a, b = 1, \ldots, q), \tag{1}$$

and the fact that they are foliate means just that [23]

$$X_u \, \xi_a^b = 0. \tag{2}$$

If all the vector fields belonging to the real linear space generated by $\{Z_a\}$ are complete (i.e. define global 1-parameter groups of diffeomorphisms on $X$), $\mathcal{F}$ is called a *transversely complete e-foliation* [1].

As for strong $e$-foliations, the condition of definition 1 means just that there is an adapted atlas $\mathcal{A}$ such that the vector fields $Z_a$ above are given locally by

$$Z_a = X_a. \tag{3}$$

The following proposition provides a simple characterisation of the considered foliations in terms of differential forms.

2. PROPOSITION. — $\mathcal{F}$ *is an e-foliation if and only if it is defined by a system*

$$d\omega^a = 0 \ (a = 1, \ldots, q), \tag{4}$$

*where $\omega^a$ are globally defined independent Pfaff forms on $X$ such that*

$$d\omega^a = \frac{1}{2} \varphi_{bc}^a \, \omega^b \wedge \omega^c \ (a, b, c = 1, \ldots, q), \tag{5}$$

*for some globally defined functions $\varphi_{bc}^a$. $\mathcal{F}$ is a strong e-foliation if and only if it is defined by a system (4), where the forms $\omega^a$ are closed.*

*Proof.* – If $\mathcal{F}$ is an *e*-foliation with the structure defined by (1), we define $\omega^a$ by the conditions

$$\omega^a (Z_b) = \delta^a_b , \ \omega^a (X_u) = 0,$$

which, because $Z_a$ are globally defined, define globally the forms $\omega^a$. Namely, one gets $\omega^a = \mu^a_b \, dx^b$ where $\mu^a_b \, \mu^b_c = \delta^a_c$, which implies $X_u \, \mu^a_b = 0$, and, from these relations, (5) follows immediately. If, moreover, $\mathcal{F}$ is strong we have (3), whence locally $\omega^a = dx^a$ and this forms are closed.

Conversely, let $\mathcal{F}$ be given by (4) and (5). Consider the dual cobases of $(X_a , X_u)$ which are easily seen to be $(dx^a , \theta^u = dx^u + t^u_a \, dx^a)$ and define $Z_a$ by $\omega^b (Z_a) = \delta^b_a , \theta^u (Z_a) = 0$. It follows easily that $\mathcal{F}$ is an *e*-foliation with the structure defined by $Z_a$. If $\omega^a$ are closed, we have locally $\omega^a = dx^a$ whence (3) and $\mathcal{F}$ is a strong *e*-foliation.

Clearly, condition (5) means just that $\omega^a$ are foliate forms.

Let us now discuss the application which we have in mind. The main part of this application consist in the following result of Conlon [1].

3. THEOREM. – *Let $X$ be a compact manifold and $\mathcal{F}$ a strong e-foliation of codimension $q$ on $X$. Then, the universal covering manifold $\widetilde{X}$ of $X$ is diffeomorphic with $\widetilde{L} \times R^q$, where $L$ is an arbitrary leaf of $\mathcal{F}$.*

*Proof.* – The main step is to prove this theorem for $q = 1$. In this case it suffices to ask $\mathcal{F}$ to be an *e*-foliation and $\mathcal{F}$ is defined by a closed Pfaff form $\omega$. The result has then been proved by Reeb [14] and we prove it here again, by using, this time, the theorem 1.10 of Kashiwabara.

Namely, let $Z$ be the foliate vector field attached to $\mathcal{F}$ like in the proof of proposition 2. It clearly exists then an adapted atlas $\mathcal{A}'$ on $X$, with local coordinates $(y , x^u) \ (u = 2, \ldots , m)$ such that one has locally

$$\omega = d \, y , \ Z = \partial/\partial \, y \tag{6}$$

and it follows that $\exp (t \, Z)$ are diffeomorphisms of $X$ which send leaves of $\mathcal{F}$ to leaves of $\mathcal{F}$. Moreover, since $X$ is compact, $\mathcal{F}$ is transversaly complete and $\exp (t \, Z)$ make up a group.

Now, the atlas $\mathcal{Cl}'$ shows that $X$ is a locally decomposed manifold and we shall also see that it has latticed maps. In fact, let $p_0 \in X$ and let $L$ be the leaf of $\mathcal{F}$ and $T$ the trajectory of $Z$ through $p_0$. Then, let $a(\sigma)$, $b(\tau)$ ($\sigma, \tau \in [0, 1]$) be paths in $L$, $T$ respectively, beginning at $p_0$. We devide $[0, 1]$ by points $0 = \tau_0 < \tau_1 < \ldots < \tau_n = 1$ such that $b(\tau_{i-1} \tau_i)$ ($i = 1, \ldots, n$) belong to some coordinate neighbourhood of $\mathcal{Cl}'$ and $b$ is given there by $(y(\tau), x^u(p_0))$. Next, we introduce the functions

$$f(\sigma, \tau) = [\exp(y(\tau_i) - y(\tau_{i-1})) Z] a(\sigma) \quad (\sigma \in [0, 1], \tau \in [\tau_{i-1}, \tau_i])$$

and glue them up such that to get a continuous function on $[0, 1]^2$. It is simple to verify that the obtained function is just the latticed map looked for.

Hence, by the differentiable version of theorem 1.10, we get $\widetilde{X} \approx \widetilde{L} \times R$, where $\widetilde{L}$ is the universal covering space of $L$ and the real line $R$ appears as the universal covering space of the trajectory $T$ of $Z$.

We also make the following important remark : the previous proof and result are valid if the assumption that $X$ is compact is replaced by the assumption that $\mathcal{F}$ is transversaly complete.

Suppose now that theorem 3 is valid for codimension $q - 1$ foliations and let $\mathcal{F}$ be a codimension $q$ strong $e$-foliation on $X$ given by the equation $\omega^1 = 0, \ldots, \omega^q = 0$, where $\omega^1, \ldots, \omega^q$ are closed Pfaff forms. Denote by $\mathcal{F}'$ the strong $e$-foliation of codimension $q - 1$ on $X$ defined by $\omega^1 = 0, \ldots, \omega^{q-1} = 0$. By the induction hypothesis, we get $\widetilde{X} \approx \widetilde{L}' \times R^{q-1}$, where $\widetilde{L}'$ is the universal covering space of the leaf $L'$ of $\mathcal{F}'$. Obviously $\omega^q = 0$ induces on $L'$ a transversaly complete $e$-foliation of codimension $1$ whose leaves are leaves of $\mathcal{F}$ and, consequently, we have $\widetilde{L}' \approx \widetilde{L} \times R$, where $L$ is a leaf of $\mathcal{F}$. Hence, $\widetilde{X} \approx \widetilde{L} \times R^q$ and theorem 3 is proved.

A similar result does not hold for $e$-foliations of an arbitrary codimension, but, as it is shown in [1], such a result holds for $e$-foliations of codimension 2. We send also to [1] for the discussion of the relation between codimension 1 $e$-foliations and the foliations without limit cycles of Novikov [12], where the universal covering space is homeomorphic to $\widetilde{L} \times R$. Namely, a Novikov foliation is an $e$-foliation with respect to another differentiable structure of the manifold.

4. COROLLARY. — *If $X$ is a compact manifold and $\mathfrak{F}$ a strong e-foliation on $X$, $\mathfrak{F}$ admits a complementary transverse foliation. Also, $X$ has a covering space of the form $\overline{X} \approx L \times R^q$ or $\overline{X} \approx L \times T^q$, where $T^q$ is a q-dimensional torus and $L$ is a leaf of $\mathfrak{F}$. All the leaves of $\mathfrak{F}$ are diffeomorphic.*

*Proof.* — Indeed, following step by step the induction process of the proof of theorem 3, we see that the covering transformation group $G$ acts on $\widetilde{L} \times R^q$ compatible with the product structure, so that $G$ induces $G_1$ on $\widetilde{L}$ and $G_2$ on $R^q$. Hence, $X$ is a C.P. space and, if we consider the projection $p : \widetilde{X} \to X$, the transverse foliation is defined by $p(\{l\} \times R^q)$ $(l \in \widetilde{L})$.

Next, with the notation of theorem 3, the foliation $\widetilde{L} \times \{t\}$ $(t \in R^q)$ is defined on $\widetilde{X}$ by $p^* \omega^a = 0$ $(a = 1, \ldots, q)$ and, since this are closed forms and $\widetilde{X}$ is 1-connected, we can choose the coordinates $t^a$ in $R^q$ such that $p^* \omega^a = dt^a$. Since these forms are invariant by $G_2$, it follows that $G_2$ acts on $R^q$ by translations and, consequently without fixed points. In this case, it is known from section 2 that $G_1'$ (defined as in section 2) is the covering group of $p/\widetilde{L} \times \{t\}$ and we can factorize the universal covering by $G_1' \times$ id which provides the covering $\overline{X} \approx L \times R^q$. Then, if $G_2' \neq$ id. we can further factorize through id. $\times G_2'$ and get a covering $\overline{X} \approx L \times T^q$.

Finally, from the form of $\overline{X}$ it follows that the fixed leaf $L$ covers all the leaves of $\mathfrak{F}$ and since $L$ was arbitrary we immediately obtain that all the leaves are diffeomorphic.

Similar considerations can be made for *e*-foliations of codimension 2 [1]. Moreover, based on the above mentioned results, Conlon [1] and Meyer [11] derive many interesting consequences involving the fundamental groups of $X$ and $L$ and which generalize similar results of Novikov [12] for the case of the codimension 1.

As another corollary of theorem 3, we can obtain a part of the Cheeger-Gromoll-Lichnerowicz theorem [10] mentioned in the introduction. Namely :

5. COROLLARY. — *Let $X$ be a compact Riemannian manifold which has the Lichnerowicz tensor $C \geqslant 0$ [10] and let $k$ be the greatest first Betti number of the compact covering spaces of $X$. Then the universal covering space $\widetilde{X}$ of $X$ is diffeomorphic with a product space $L \times R^k$.*

*Proof.*—Let $\overline{X}$ be the compact Riemannian covering space of $X$ with $b_1(\overline{X}) = k$. Then $\overline{X}$ has also $C \geqslant 0$ and, from results of [10], every nonzero harmonic 1-form on $\overline{X}$ vanishes nowhere.

It follows that, if one takes $k$ independent harmonic 1-forms on $\overline{X}$, they define a strong $e$-foliation of codimension $k$ and theorem 3 applies. This proves the announced corollary. But the theorem of [10] is stronger : it gives also that $\widetilde{L}$ is compact and that $\widetilde{X}$ is isometric to $\widetilde{L} \times R^k$.

Finally, it would of course be interesting to have results like in theorem 3 for the complex analytic case. In this case, the difficulty consists in the fact that generally there is no complementary analytic subbundle of the foliation, since an exact sequence of the form

$$0 \to E \to T(X) \to Q \to 0 \tag{7}$$

where $T(X)$ is the complex analytic tangent bundle of the complex manifold $X$ and $E, Q$ are complex analytic vector bundles on $X$, does not split analytically.

But, let us define complex analytic strong $e$-foliations $\mathscr{F}$ on $X$ to be complex analytic foliations such that : *i*) they are defined by a system $\omega^a = 0$ ($a = 1, \dots, q$), where $\omega^a$ are closed holomorphic 1-forms everywhere independent on $X$, and *ii*) the tangent bundle $E$ of $\mathscr{F}$ admits an analytic complementary tangent bundle. Then, we get

6. THEOREM. — *Let $X$ be a compact complex analytic manifold and $\mathscr{F}$ a complex analytic strong e-foliation of complex codimension $q$ on $X$. Then, the universal covering manifold $\widetilde{X}$ of $X$ is holomorphic with $\widetilde{L} \times C^q$ where $L$ is an arbitrary leaf of $\mathscr{F}$.*

*Proof.* — Consider first $q = 1$. Then, choosing an analytic transverse vector field of $\mathscr{F}$ we get on $X$ a complex analytic reducible structure. Since $\mathscr{F}$ is a real strong $e$-foliation of codimension 2 on $X$ we can apply theorem 3 whence we deduce the existence of latticed maps on $X$. Finally, with the remark at the beginning of this section regarding the theorem 1.10 in the complex analytic case, we deduce the result given by theorem 6.

Now, the passage to an arbitrary codimension $q$ will be obtained just like in theorem 3, which ends the proof of theorem 6.

We shall also remark that one can remove the condition *ii*) of the definition of the strong *e*-foliations for complex analytic manifolds $X$ for which every exact sequence (7) splits analytically. In this case, we call $X$ a *tangentially splitting* complex analytic manifold. But the existence of compact tangentially splitting manifolds remains to be discussed. Thus, it is well known that (7) splits if and only if some obstruction which is a cohomology class in $H^1(X, \Omega(\text{Hom}(Q, E)))$ vanishes. By representing this obstruction by differential forms, one sees that it can be reinterpreted as a class in $H^1(X, \Omega(\text{Hom}(T(X), T(X))))$. Hence, the vanishing of this last cohomology space is a sufficient condition for $X$ to be tangentially splitting. Now, other sufficient conditions can be looked for, if $X$ is a compact Kähler manifold, using the theory of harmonic forms and a well known method of Bochner. These conditions will be expressed in terms of the curvature of the Kähler metric of $X$.

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