ANNALES DE L'INSTITUT FOURIER

BAHMAN SAFFARI
R. C. VAUGHAN

On the fractional parts of x/n and related sequences. II

Annales de l'institut Fourier, tome 27, nº 2 (1977), p. 1-30
<http://www.numdam.org/item?id=AIF_1977__27_2_1_0>

© Annales de l'institut Fourier, 1977, tous droits réservés.

L'accès aux archives de la revue « Annales de l'institut Fourier » (http://annalif.ujf-grenoble.fr/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

Ann. Inst. Fourier, Grenoble
27, 2 (1977), 1-30.

ON THE FRACTIONAL PARTS OF x/n AND RELATED SEQUENCES. II

by B. SAFFARI and R.-C. VAUGHAN

1. Introduction and statement of theorems.

1.1. In this paper we assume the notation of [9]. Throughout, the implicit constants in the O, $\ll$ and $\gg$ notations are absolute unless otherwise indicated. In addition, we use the symbol $\asymp$. By $U \asymp V$ one means that $U \ll V$ *and* $V \ll U$. The letter p always designates a prime number.

1.2. *The standard case.* In this section we study the case $h(n) = 1/n$. We are primarily interested in the behaviour of

$$\Theta_{x,y}(\alpha) = y^{-1} \sum_{n \leqslant y} c_{\alpha}(x/n) \tag{1.1}$$

where x and y tend to infinity together. We observe that this is essentially the same as taking the simple Riesz means (R, λ_n) with $\lambda_n = 1$ for $n \leqslant y$ and $\lambda_n = 0$ for $n > y$. In fact, we are considering the positive Toeplitz transformation

$$\mathscr{A} = (a_n(y) : y \in [1, \infty), n = 1, 2, \ldots)$$

with $a_n(y) = y^{-1}$ for $n \leqslant y$ and $a_n(y) = 0$ for $n > y$.
We recall the definition of $F(\alpha, \xi, \sigma)$ (cf. [9], (2.4), (2.5)).

$$F(\alpha, \xi, \sigma) = \begin{cases} 0 \ (\alpha \leqslant 0) \\ 1 \ (\alpha \geqslant 1) \\ \theta(\alpha, \xi)(1 - \xi^{\sigma}([\xi] + \alpha)^{-\sigma}) \\ \qquad\qquad + \xi^{\sigma} \sum_{k > \xi} (k^{-\sigma} - (k + \alpha)^{-\sigma}) \\ \quad (0 < \alpha < 1, \sigma > 0) \\ \alpha \ (0 < \alpha < 1, \sigma = 0) \end{cases} \tag{1.2}$$

where

$$(1.3) \qquad \theta(\alpha, \xi) = \begin{cases} 1 \text{ if } (\xi - \alpha, \xi] \cap \mathbf{N} \neq \emptyset \\ 0 \text{ otherwise,} \end{cases}$$

and write

$$(1.4) \qquad F(\alpha, \xi) = F(\alpha, \xi, 1).$$

The connection between $\Theta_{x,y}$ and the Dirichlet divisor problem can be seen, for example, *via*

$$(1.5) \qquad \Delta(x) = 2x^{\frac{1}{2}} \int_0^1 (\Theta_{x, x^{1/2}}(\alpha) - \alpha)\, d\alpha + O(1)$$

or

$$(1.6) \qquad \Delta(x) = x \int_0^1 (\Theta_{x,x}(\alpha) - F(\alpha, 1))\, d\alpha + O(1)$$

where

$$(1.7) \qquad \Delta(x) = \sum_{n \leqslant x} d(n) - x \log x - (2\gamma - 1)x$$

and as usual d is the divisor function and γ is Euler's constant.

Theorem 1. — *Suppose that* $1 \leqslant y \leqslant x$. *Then*

$$(1.8) \qquad \Theta_{x,y}(\alpha) = F(\alpha, x/y) + O\left(x^{\frac{1}{3}} y^{-1} \log x\right).$$

By adapting the Van der Corput method of trigonometric sums it would be possible to improve the error term here, much as in the Dirichlet divisor problem. However, we have carried out no detailed calculations in this direction, partly because we do not believe that the small improvements that could be obtained are anywhere near the truth. In fact, Theorem 2 below suggests that $\Theta_{x,y}(\alpha) \to \alpha$ even when $y = x^{\varepsilon}$ where ε is any fixed number with $0 < \varepsilon < 1$. There are three immediate consequences of Theorem 1.

Corollary 1.1. — *As* $x \to \infty$,

$$(1.9) \qquad \Theta_{x,x}(\alpha) = \sum_{k=1}^{\infty} \frac{\alpha}{k(k+\alpha)} + O\left(x^{-\frac{2}{3}} \log x\right).$$

Corollary 1.2. — *Let* t *be a fixed number with* $0 < t < 1$. *Then*

$$\Theta_{x,tx}(\alpha) = F\left(\alpha, \frac{1}{t}\right) + O\left(x^{-\frac{2}{3}} \log x\right). \tag{1.10}$$

Corollary 1.3. — *Suppose that* $y/x \to 0$ *as* $x \to \infty$. *Then*

$$\Theta_{x,y}(\alpha) = \alpha + O\left(yx^{-1} + x^{\frac{1}{3}} y^{-1} \log x\right). \tag{1.11}$$

If y is quite close to x, the error term in (1.11) is not very good, and at first sight one might hope to do better. However, on inspecting $F(\alpha, \xi)$ one finds that the error can indeed be this large, and is essentially due to the irregular behaviour of $F(\alpha, \xi)$ as a function of ξ at the points 2,3, ... (see Lemma 4 of [9]).

The next theorem suggests that Theorem 1 is some way from being best possible.

Theorem 2. — *Suppose that* $y = y(x)$ *is increasing,* $y = o(x)$ *and* $y \to \infty$ *as* $x \to \infty$. *Suppose further that* $0 < \alpha < 1$ *and* $\lim_{x \to \infty} \Theta_{x,y}(\alpha)$ *exists. Then*

$$\lim_{x \to \infty} \Theta_{x,y}(\alpha) = \alpha.$$

The next three theorems put some limitations on how good the error term can really be in (1.8) an don how small y can be for there to be an asymptotic distribution.

Theorem 3. — *Suppose that* $y(u)$ *is increasing and*

$$y(u) \to \infty$$

as $u \to \infty$. *Let* $\delta = \delta(\alpha)$ *be sufficiently small, and suppose that* x *and* X *satisfy the inequalities*

$$0 < X \leqslant x, \tag{1.12}$$

$$(y(x + X) - y(x))(\log(y(x + X)))^4 < \delta y(x) \tag{1.13}$$

and

$$2 \leqslant y(x) \leqslant \frac{1}{2} X^{\frac{1}{2}}. \tag{1.14}$$

Then, for $x > x_0(\alpha)$,

$$(1.15)\qquad (\sin \pi\alpha)^4 \frac{X}{y(x)} \ll \int_x^{x+X} |\Theta_{u,y(u)}(\alpha) - \alpha|^2\, du \ll \frac{X}{y(x)}.$$

As an immediate consequence we have

Corollary 3.1. — *Suppose that*

$$0 < \alpha < 1 \qquad \textit{and} \qquad 0 < \beta < \frac{1}{2}.$$

Then there are numbers $\delta_1(\alpha)$ *and* $x_1(\beta)$ *such that, whenever* $x > x_1(\beta)$,

$$(1.16)\qquad \int_x^{x+\delta_1(\alpha)x(\log x)^{-4}} |\Theta_{u,u^\beta}(\alpha) - \alpha|^2\, du \asymp_\alpha x^{1-\beta} (\log x)^{-4}.$$

Moreover

$$(1.17)\qquad \limsup_{x\to\infty} x^{\beta/2}|\Theta_{x,x^\beta}(\alpha) - \alpha| > 0.$$

Theorem 4. — *Suppose that the continuous function* $G(u)$ *satisfies the differential difference equation*

$$(1.18)\qquad uG(u) = -G(u-1) \quad (u > 1), \qquad G(u) = 1\ (0 \leqslant u \leqslant 1).$$

Then, for each $u > 0$,

$$(1.19)\qquad \limsup_{x\to\infty} \Theta_{x,y}(\alpha) \geqslant G(u) \quad (0 < \alpha < 1, y = (\log x)^u).$$

Theorem 5 is an immediate corollary of Theorems 2 and 4.

Theorem 5. — *Suppose that* $0 < \alpha < G(u)$. *Then*

$$\Theta_{x,y}(\alpha)\ \ (y = (\log x)^u)$$

does not have a limit as $x \to \infty$.

The function G, often called Dickman's function, has been studied by a number of people (see references in Norton [7]), who have shown that it is monotone decreasing and satisfies

$$(1.20)\qquad 0 < G(u) \leqslant \Gamma(u+1)^{-1}$$

and

$$(1.21)\qquad \int_0^\infty G(u)\, du = e^\gamma.$$

It is easily seen that

$$(1.22) \qquad G(u) = 1 - \log u \quad (1 < u \leqslant 2)$$

and

$$(1.23) \quad G(u) = 1 - \log u + \int_2^u \log (v - 1) \frac{dv}{v} \quad (2 < v \leqslant 3).$$

1.3. *The « logarithmic case ».* As one might expect, when one considers limit distributions of $\{x/n\}$ in the sense of the logarithmic density, things can be pushed a good deal further. Write

$$(1.24) \qquad \theta_{x,y}(\alpha) = (\log y)^{-1} \sum_{n \leqslant y} \frac{1}{n} c_\alpha(x/n).$$

There is a close connection between $\theta_{x,y}$ and the error term

$$E(x) = \sum_{n \leqslant x} \sigma(n) - \frac{\pi^2}{12} x^2$$

where σ is the sum of the divisors function. It is easily seen that

$$E(x) = x(\log z) \int_0^1 (\theta_{x,z}(\alpha) - \alpha)\, d\alpha + \frac{1}{2} x^2 z^{-2} \int_0^1 (\varphi_{x,x/z}(\alpha) - \alpha)\, d\alpha + O(x) + O(x^2 z^{-3})$$

where $z \leqslant x$ and

$$\varphi_{x,y}(\alpha) = 2y^{-2} \sum_{n \leqslant y} n c_\alpha(x/n).$$

When $x^{\frac{1}{2}} \leqslant z \leqslant x$ this reduces to

$$E(x) = x (\log z) \int_0^1 (\theta_{x,z}(\alpha) - \alpha)\, d\alpha + O(x).$$

There is also a simple relation connecting $\varphi_{x,y}$ with E, namely

$$E(x) = \frac{1}{2} x^2 \int_0^1 (\varphi_{x,x}(\alpha) - F(\alpha, 1, 2))\, d\alpha + O(x).$$

We do not study $\varphi_{x,y}$ in detail, since its general behaviour can be easily deduced from that of $\Theta_{x,y}$.

The next theorem shows that, not only does one obtain the uniform distribution for $\theta_{x,y}$ when $y = x^\varepsilon$, but even when $y = x$.

THEOREM 6. — *Suppose that* $y \leqslant x$. *Then*

$$(1.25)\qquad \theta_{x,y}(\alpha) = \alpha + O\left((\log x)^{\frac{2}{3}} (\log y)^{-1}\right).$$

We would conjecture that $\theta_{x,y}(\alpha) \to \alpha$ providing that

$$\log\log x = o\,(\log y).$$

THEOREM 7. — *Suppose that* $y = y(x)$ *is increasing to infinity and* $y \leqslant x$. *Suppose that* $0 < \alpha < 1$. *Then, whenever* $\theta_{x,y}(\alpha)$ *tends to a limit as* $x \to \infty$, *the limit must be* α.

In the opposite direction we can do somewhat better than the analogue of Theorem 4. (Note that by (1.20) and (1.21), for $u > 1$,

$$G(u) \leqslant \Gamma(1+u)^{-1} < 1/u$$

whereas $\int_0^u G(v)\,dv > 1$ and $\int_0^u G(v)\,dv \to e^{\gamma}$ as $u \to \infty$).

THEOREM 8. — *For each* $u > 0$,

$$(1.26)\qquad \limsup_{x\to\infty} \theta_{x,y}(\alpha) \geqslant \frac{1}{u}\int_0^u G(v)\,dv \qquad (0 < \alpha < 1,\ y = (\log x)^u)$$

where G *is given by* (1.18).

As an immediate consequence of Theorems 7 and 8 we have

THEOREM 9. — *Suppose that* $u > 0$ *and*

$$0 < \alpha < \frac{1}{u}\int_0^u G(v)\,dv.$$

Then

$$\theta_{x,y}(\alpha)\ (y = (\log x)^u)$$

does not have a limit as $x \to \infty$.

It is very likely that both Theorems 5 and 9 hold with the upper bounds 1 for α for every fixed u.

1.4. *The prime numbers.* The following theorem shows that the prime numbers, suitably normalized, behave in much the same way as the natural numbers. Let

$$(1.27) \qquad \vartheta_{x,y}(\alpha) = y^{-1} \sum_{p \leqslant y} (\log p) c_\alpha(x/p).$$

Theorem 10. — *Suppose that* $\varepsilon > 0$ *and* $x^{\frac{6}{11}+\varepsilon} < y \leqslant x$. *Then*

$$(1.28) \quad \vartheta_{x,y}(\alpha) = \mathrm{F}(\alpha, x/y) + \mathrm{O}\left(\exp\left(-\mathrm{C}(\varepsilon)\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right)\right)$$

where $\mathrm{C}(\varepsilon)$ *is a positive number depending at most on* ε.

We remark that on the density hypothesis concerning the distribution of the zeros of the Riemann zeta function the $\frac{6}{11}$ could be replaced by $\frac{1}{2}\cdot$ The $\frac{6}{11}$ arises as $\frac{c}{c+2}$ where c is such that

$$(1.29) \qquad \mathrm{N}(\sigma, \mathrm{T}) \ll \mathrm{T}^{c(1-\sigma)+\varepsilon}$$

and where $\mathrm{N}(\sigma, \mathrm{T})$ is the number of zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $\beta \geqslant \sigma$ and $|\gamma| \leqslant \mathrm{T}$. The ε in Theorem 10 could be made an explicit function of x, but there is little point in doing so.

As far as the un-normalized case is concerned, providing that the conditions of Theorem 10 are satisfied, partial summation gives

$$(1.30) \quad \sum_{p \leqslant y} c_\alpha(x/p) = \frac{y}{\log y}\mathrm{F}(\alpha, x/y) + \int_2^y \frac{\mathrm{F}(\alpha, x/v)}{(\log v)^2}\,dv + \mathrm{O}\left(y \exp\left(-\mathrm{C}(\varepsilon)\left(\frac{\log x}{\log\log x}\right)^{\frac{1}{3}}\right)\right).$$

The asymptotic distribution is the same, but there is a second order term which has no very simple closed form, although the main terms can be combined to give

$$\int_{x/y}^{\infty} \frac{\alpha x\, du}{(u-(1-\alpha)\{u\})^2 \log(x/(u-(1-\alpha)\{u\}))}.$$

It is trivial that $\vartheta_{x,y}(\alpha)$ does not have an asymptotic distribution when $y = (\log x)^u$ with $0 < u \leqslant 1$. (Indeed, this is so for all choices of λ_n. We hope to discuss this further in a later paper). However, we have not been able to extend this to the region $u > 1$.

It is a simple application of Theorem 5 of [9], that if

$$0 < \theta < 1, \quad y = x^{\theta} \quad \text{and} \quad \vartheta_{x,y}(\alpha)$$

has a limit as x tends to infinity, then the limit must be α. Moreover, this can be sharpened along the lines of Theorem 2.

1.5. *A « law of the iterated logarithm ».* In all the applications of Theorems 3 and 4 of [9] hitherto, the expressions

$$\sum_n \left(\sum_m \frac{1}{n} a_{mn}(y) \right)^2$$

and

$$\sum_n \left| \sum_m \frac{1}{m} a_{mn}(y)(1 - e(\alpha m)) \right|^2$$

have behaved very much like $\sum_n a_n^2(y)$. We now show that this is not always so, even under fairly reasonable conditions. In particular, the following theorem justifies our remark below Theorem 4 of [9] to the effect that taking $\sum_n a_n^2(y)$ in that theorem can lose a factor as large as $\log \log y$.

THEOREM 11. — *There is an infinite subset $\mathscr{D}$ of N^* with the following property. Let*

$$\mathscr{A} = (a_n(y) : y \in [1, \infty), n = 1, 2, \ldots)$$

be the Toeplitz transformation where the $a_n(y)$ are the simple Riesz means (R, λ_n) obtained by taking λ_n to be the characteristic function of $\mathscr{D}$. Then there are arbitrarily large y such that whenever $x_0 \geqslant 0$ and $x > 0$,

$$\max(0, x - y^2) \ll \frac{\sum\limits_{n<y} \lambda_n}{\log \log y} \sup_{\alpha} \int_{x_0}^{x_0+x} (\Phi_{u,y}(\alpha) - \alpha)^2 \, d\alpha \ll x + y^2$$

1.6. In conclusion we mention an example with $h(n) = 1/n$ in which the asymptotic distribution function differs from $\mathrm{F}(\alpha, \xi)$. Suppose that $k \in \mathbf{N}^*$ and let $\lambda_n = 1$ if n is a k th power and $\lambda_n = 0$ otherwise. Then trivially by the method of the hyperbola,

$$\Phi_{x,y}(\alpha) = F(\alpha, x/y), 1/k) + O\left(x^{\frac{1}{k+1}} y^{-\frac{1}{k}}\right),$$

and deeper methods doubtless enable one to improve a little further the range of validity for y.

2. Proof of theorem 1.

2.1. The following lemma is implied by Satz 566 of Landau [4].

LEMMA 1. — *Let*

$$b(z) = \begin{cases} \{z\} - \dfrac{1}{2} & (z \notin \mathbf{Z}) \\ 0 & (z \in \mathbf{Z}). \end{cases}$$

Suppose that $u < w$, $f(v)$ *is positive and twice differentiable for* $u \leqslant v \leqslant w$ *and* $f''(v)$ *is non-zero and always has the same sign. Suppose also that for* $u \leqslant v \leqslant w$ *we have*

$$0 < \lambda \leqslant f'(v) \leqslant \mu$$

and that ρ *is any real number with* $\rho > 1$, $\rho > \lambda^{-3}$ *and*

$$\rho \geqslant |f''(v)|^{-1}(1 + f'(v)^2)^{3/2} \quad (u \leqslant v \leqslant w).$$

Let N *be the number of pairs of integers* m, n *for which*

$$u \leqslant m \leqslant w \quad \text{and} \quad 0 \leqslant n \leqslant f(m)$$

where any pair m, n *for which either* $m = u$, $m = w$, $n = 0$ *or* $n = f(m)$ *is counted with weight* $\frac{1}{2}\cdot$ *Then*

$$N = \int_u^w f(u)\, du - b(w)f(w) + b(u)f(u) + O\left(\rho^{\frac{2}{3}} \mu\right).$$

2.2. To prove Theorem 1, consider six sets S_1, S_2, S_2', S_3, S_3' and S_4 of pairs of integers m, n defined as follows;

$$S_1: \quad \frac{x}{y} < m \leqslant x^{1/3}, \quad \frac{x}{m+\alpha} < n \leqslant \frac{x}{m}$$

$$S_2: \quad x^{1/2} < m \leqslant x^{2/3}, \quad \frac{x}{m+\alpha} < n \leqslant \frac{x}{m}$$

$$S_2': \quad \frac{x}{y} < m \leqslant x^{2/3}, \quad \frac{x}{m+\alpha} < n \leqslant \frac{x}{m}$$

$$S_3: \quad \frac{x}{x^{1/2}+\alpha} < m \leqslant x^{2/3}, \quad \frac{x}{m} - \alpha < n \leqslant \frac{x}{m}, \quad x^{1/3} < n \leqslant x^{1/2}$$

$$S_3': \quad \frac{x}{x^{1/2}+\alpha} < m \leqslant x^{2/3}, \quad \frac{x}{m} - \alpha < n \leqslant \frac{x}{m}, \quad \frac{x}{y} < n \leqslant x^{1/2}$$

$$S_4: \quad m \leqslant x^{1/3}, \quad \frac{x}{m} - \alpha < n \leqslant \frac{x}{m}, \quad x^{2/3} < n \leqslant x.$$

Let $|S|$ denote the number of elements of the set S. By (1.1),

$$\Theta_{x,y}(\alpha) = \left(\left[\frac{x}{y}\right] - \left[\frac{x}{y} - \alpha\right]\right)\left([y] - \frac{x}{\left[\frac{x}{y}\right] + \alpha}\right) + \sum_{j=1}^{4} M_j \tag{2.1}$$

where

$$M_1 = \begin{cases} |S_1| & \text{if } x^{2/3} < y \leqslant x \\ 0 & \text{if } y \leqslant x^{2/3} \end{cases} \tag{2.2}$$

$$M_2 = \begin{cases} |S_2| & \text{if } x^{1/2} < y \leqslant x \\ |S_2'| & \text{if } x^{1/3} < y \leqslant x^{1/2} \\ 0 & \text{if } y \leqslant x^{1/3} \end{cases} \tag{2.3}$$

$$M_3 = \begin{cases} |S_3| & \text{if } x^{2/3} < y \leqslant x \\ |S_3'| & \text{if } x^{1/2} < y \leqslant x^{2/3} \\ 0 & \text{if } y \leqslant x^{1/2} \end{cases} \tag{2.4}$$

and

$$M_4 = \begin{cases} |S_4| & \text{if } x^{1/3} < y \leqslant x \\ 0 & \text{if } y \leqslant x^{1/3}. \end{cases} \tag{2.5}$$

Suppose first of all that $x^{2/3} < y \leqslant x$. By (2.2) and (2.5),

$$M_1 = \sum_{\frac{x}{y} < m \leqslant x^{1/3}} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3}) \quad \text{and} \quad |M_4| \leqslant x^{1/3}. \tag{2.6}$$

If $x^{1/2} \leqslant m \leqslant x^{2/3}$, then there are $\leqslant 1$ integers n with

$$\frac{x}{m+\alpha} < n \leqslant \frac{x}{m},$$

and the number of pairs m, n with either $n(m+\alpha)=x$ or $mn=x$ is $\ll x^{\varepsilon}$. Hence, by (2.3),

$$(2.7)\quad M_2 = M_2' + O(x^{\varepsilon}) \quad \text{with} \quad M_2' = \sideset{}{'}\sum_{x^{1/2}\leqslant m\leqslant x^{2/3}} \sideset{}{'}\sum_{\frac{x}{m+\alpha}\leqslant n\leqslant\frac{x}{m}} 1$$

where the dashes are used to indicate that if the pair m, n is on the « boundary » of the region under consideration, then it is counted with weight $\frac{1}{2}\cdot$ The same argument is applied to M_3. Note that there is at most one integer n in

$$\left[\frac{x}{x^{2/3}+\alpha}, x^{1/3}\right]$$

and likewise in $[x/(x^{1/2}+\alpha), x^{1/2}]$. Hence, by (2.4),

$$(2.8)\quad M_3 = M_3' + O(x^{\varepsilon}) \quad \text{where} \quad M_3' = \sideset{}{'}\sum_{x^{1/2}\leqslant m\leqslant x^{2/3}} \sideset{}{'}\sum_{\frac{x}{m}\alpha\leqslant n\leqslant\frac{x}{m}} 1.$$

Now write

$$(2.9)\quad M_2' = N_2(0) - N_2(\alpha) \quad \text{and} \quad M_3' = N_3(0) - N_3(\alpha)$$

where, for β with $0 \leqslant \beta \leqslant 1$,

$$(2.10)\qquad N_2(\beta) = \sideset{}{'}\sum_{-x^{2/3}\leqslant m\leqslant -x^{1/2}} \sideset{}{'}\sum_{0\leqslant n\leqslant\frac{x}{\beta-m}} 1$$

and

$$(2.11)\qquad N_3(\beta) = \sideset{}{'}\sum_{-x^{2/3}\leqslant m\leqslant -x^{1/2}} \sideset{}{'}\sum_{0\leqslant n\leqslant\frac{x}{m}\beta} 1.$$

It is now a straightforward application of Lemma 1 to intervals of the kind $-2^{h+1}x^{1/2} \leqslant m \leqslant -2^h x^{1/2}$ to obtain

$$N_2(\beta) = \int_{x^{1/2}}^{x^{2/3}} \frac{x}{u+\beta}\,du + b\left(x^{\frac{1}{2}}\right)\frac{x}{x^{1/2}+\beta} - b\left(x^{\frac{2}{3}}\right)\frac{x}{x^{2/3}+\beta} + O(x^{1/3}\log x)$$

and

$$N_3(\beta) = \int_{x^{1/2}}^{x^{2/3}} \left(\frac{x}{u}-\beta\right)du + b\left(x^{\frac{1}{2}}\right)\left(x^{\frac{1}{2}}-\beta\right) - b\left(x^{\frac{2}{3}}-\beta\right) + O(x^{1/3}\log x).$$

Therefore, by (2.7), (2.8), (2.9), (2.10) and (2.11),

$$M_2 = \sum_{m > x^{1/2}} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3} \log x)$$

and

$$\begin{aligned} M_3 &= \alpha(x^{2/3} - x^{1/2}) + O\left(x^{\frac{1}{3}} \log x\right) \\ &= \sum_{x^{1/3} < m \leqslant x^{1/2}} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3} \log x). \end{aligned}$$

Hence, by (2.6),

$$\sum_{j=1}^{4} M_j = \sum_{m > x/y} \frac{\alpha x}{m(m+\alpha)} + O(x^{1/3} \log x)$$

and Theorem 1 in the case $x^{2/3} < y \leqslant x$ now follows from (2.1).

The cases $x^{1/2} < y \leqslant x^{2/3}$ and $x^{1/3} < y \leqslant x^{1/2}$ are treated similarly.

3. Proofs of theorems 2, 3 and 7.

3.1. First of all we state a lemma which is a consequence of Theorems 3 and 4 of [9].

LEMMA 2. — *Suppose that x and X are non-negative real numbers, $y \geqslant 1$ and $0 < \alpha < 1$. Then*

$$(\sin \pi\alpha)^4 (X - y^2) y \ll \int_x^{x+X} \Big| \sum_{n \leqslant y} (c_\alpha(u/n) - \alpha) \Big|^2 du \ll (X + y^2) y.$$

3.2. We require a result in which in the integrand y can be made a function of u. In order to obtain this we first of all require some information concerning short intervals.

LEMMA 3. — *Suppose that x, z and X are non-negative real numbers, $y \geqslant 1$, $Y = \max(z, y)$ and $0 < \alpha < 1$. Then*

$$\int_x^{x+X} \Big| \sum_{z < n \leqslant z+y} \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \Big|^2 du \ll (X + Y^2) y (\log 2Y)^2.$$

Proof. — By Theorem 3 of [9], the left hand side is

$$\begin{aligned} &\ll (x+Y^2)\sum_n \Bigl(\sum_{\substack{m\\ z<nm\leqslant z+y}} \frac{1}{m}\Bigr)^2 \\ &\ll (x+Y^2)\sum_n \Bigl(\sum_{z<nm\leqslant z+y} \frac{1}{m}\Bigr)\log 2Y \\ &\ll (x+Y^2)\sum_{z<q\leqslant z+y}\ \sum_{m\mid q} \frac{1}{m}\log 2Y \\ &\ll (x+Y^2)y\,(\log 2Y)^2, \end{aligned}$$

as required.

Lemma 4. — *On the hypothesis of Lemma* 3,

$$\int_x^{x+X} \sup_{v\leqslant y}\Bigl|\sum_{z<n\leqslant z+v}\Bigl(c_\alpha\Bigl(\frac{u}{n}\Bigr)-\alpha\Bigr)\Bigr|^2 du \ll (X+Y^2)y\,(\log 2Y)^4,$$

where the supremum is taken over all non-negative real numbers with $v \leqslant y$.

Proof. — This uses a technique which goes back to Menchov [5] and Rademacher [8]. It may certainly be supposed that the supremum is taken only over those numbers of the form

$$v = y\sum_{r=0}^{k} \varepsilon_r 2^{-r}$$

where $\varepsilon_r = 0$ or 1 and $k = [\log y/\log 2]$. For such a v let

$$m_r = m_r(v) = \sum_{j=0}^{r-1} \varepsilon_j 2^{r-j}, \quad m_0 = 0.$$

Then $m_r < 2^r v/y \leqslant 2^r \leqslant y$, $m_{r+1}2^{-r-1} = m_r 2^{-r} + \varepsilon_r 2^{-r}$ and

$$y m_{k+1} 2^{-k-1} = v.$$

Now for given u choose some $v = v(u)$ for which the supremum occurs. Then

$$\sup_{v\leqslant y}\Bigl|\sum_{z<n\leqslant z+v}\Bigl(c_\alpha\Bigl(\frac{u}{n}\Bigr)-\alpha\Bigr)\Bigr| \leqslant \sum_{r=0}^{k}\Bigl|\Sigma\Bigl(c_\alpha\Bigl(\frac{u}{n}\Bigr)-\alpha\Bigr)\Bigr|$$

where the inner summation is over those integers n such that

$$z + y m_r 2^{-r} < n \leqslant z + (m_r+\varepsilon_r)y2^{-r}.$$

Hence,

$$(3.1)\quad \int_x^{x+X} \sup_{v\leqslant Y}\left|\sum_{z<n\leqslant z+v}\left(c_\alpha\left(\frac{u}{n}\right)-\alpha\right)\right|^2 du \ll (\log 2Y)\int_x^{x+X}\sum_{r=0}^{k}\left|\Sigma\left(c_\alpha\left(\frac{u}{n}\right)-\alpha\right)\right|^2 du$$

where the inner sum is over those n such that

$$z+ym_r2^{-r} < n \leqslant z+y(m_r+1)2^{-r}.$$

The right hand side of (3.1) is

$$(3.2)\quad \ll (\log 2Y)\sum_{r=0}^{k}\sum_{m=0}^{2^r-1}\int_x^{x+X}\left|\Sigma\left(c_\alpha\left(\frac{u}{n}\right)-\alpha\right)\right|^2 du$$

where the inner summation is over those n such that

$$z+ym2^{-r} < n \leqslant z+y(m+1)2^{-r},$$

and, by Lemma 3, (3.2) is

$$\ll (\log 2Y)\sum_{r=0}^{k} 2^r(X+Y^2)y2^{-r}(\log 2Y)^2 \ll (X+Y^2)y(\log 2Y)^4,$$

as required.

3.3. First of all we prove Theorem 2. Observe that

$$\liminf_{x\to\infty}\frac{y(2x)-y(x)}{y(x)} \leqslant 1,$$

for otherwise $y(x) \gg x$. Therefore the set

$$(3.3)\qquad S=\left\{x>1:\frac{y(2x)-y(x)}{y(x)}\leqslant 2\right\}$$

is unbounded. By Lemma 4,

$$(3.4)\quad \inf_{x\leqslant u\leqslant 2x}\left|\sum_{n\leqslant y(u)}\frac{c_\alpha(u/n)-\alpha}{y(u)}\right|^2 \leqslant \frac{1}{xy(x)^2}\int_x^{2x}\left|\sum_{n\leqslant y(u)}\left(c_\alpha\left(\frac{u}{n}\right)-\alpha\right)\right|^2 du \ll \frac{x+y(2x)^2}{xy(x)^2}\,y(2x)(\log 2y(2x))^4.$$

If S contains an unbounded subset S^* such that

$$(3.5) \qquad y(2\xi) \leqslant \xi\,(\log \xi)^{-5} \text{ whenever } \xi \in S^*,$$

then by (3.3), (3.4) and (3.5)

$$\liminf_{x\to\infty} \left| \sum_{n \leqslant y(x)} \frac{c_\alpha(x/n) - \alpha}{y(x)} \right| = 0.$$

This gives the desired conclusion if such an S^* exists. Otherwise there is a constant $x_0 > 1$ such that

$$(3.6) \quad y(2x) > x\,(\log x)^{-5} \quad \text{whenever} \quad x \in S,\ x > x_0.$$

Then, by (3.6) and Corollary 1.3,

$$\lim_{\substack{x\to\infty \\ x \in S}} \Theta_{2x,\, y(2x)} = \alpha.$$

This completes the proof of Theorem 2.

3.4. To prove Theorem 3 we use both Lemma 2 and Lemma 4. By Lemma 2,

$$(\sin \pi\alpha)^4 X y(x) \ll \int_x^{x+X} \left| \sum_{n \leqslant y(x)} \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll X y(x),$$

and by Lemma 4,

$$\int_x^{x+X} \left| \sum_{y(x) < n \leqslant y(u)} \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll X y(x).$$

Thus, if y is sufficiently small in terms of x, then

$$(\sin \pi\alpha)^4 \frac{X}{y(x)} \ll \int_x^{x+X} \left| \Theta_{u,\, y(u)}(\alpha) - \alpha \frac{[y(u)]}{y(u)} \right|^2 du \ll \frac{X}{y(x)}.$$

This gives (1.15), provided that $x > x_0(\alpha)$.

3.5. The proof of Theorem 7 follows the same pattern as that of Theorem 2. We observe that Theorem 3 of [9] gives

$$\int_x^{2x} \left| \sum_{n \leqslant y(u)} \frac{1}{n} \left(c_\alpha\left(\frac{u}{n}\right) - \alpha \right) \right|^2 du \ll x.$$

Thus

$$\text{(3.7)} \quad \inf_{x \leqslant u \leqslant 2x} \left| \sum_{n \leqslant y(u)} \frac{c_\alpha(u/n) - \alpha}{n \log y(u)} \right|^2$$
$$\leqslant \frac{1}{x (\log y(x))^2} \int_x^{2x} \left| \sum_{n \leqslant y(u)} \frac{1}{n} \left(c_\alpha \left(\frac{u}{n} \right) - \alpha \right) \right|^2 du$$
$$\ll (\log y(x))^{-2} \left(1 + \log \frac{y(2x)}{y(x)} \right)^2.$$

If there exists an unbounded set of real numbers $x > 1$ on which $y(2x)/y(x)$ is bounded, then Theorem 7 follows at once from (3.7). Otherwise

$$\text{(3.8)} \qquad y(x) \gg x,$$

and Theorem 7 follows from (3.8) and Theorem 6, which we shall prove in Section 5.

4. Proofs of theorems 4 and 8.

Let

$$\text{(4.1)} \qquad x_n = \exp\left(\sum_{r=1}^{n} \Lambda(r) \right)$$

where Λ is Von Mangoldt's function, and

$$\text{(4.2)} \qquad y_n = (\log x_n)^u.$$

Then

$$\text{(4.3)} \quad \sum_{m \leqslant y_n} c_\alpha(x_n/m) = \sum_{m \leqslant n^u} c_\alpha(x_n/m) + O\left(\frac{n^u}{\log n} \right)$$
$$\geqslant \sum_{m \leqslant n^u}{}' c_\alpha(x_n/m) + O\left(\frac{n^u}{\log n} \right)$$

where Σ' means that the sum is restricted to those m which have no prime divisor exceeding n. (Very probably the part of the sum thrown away contributes an amount infinitely often as large as $(\alpha - \varepsilon)(1 - G(u))$, and if this is so, then Theorem 5 also holds when $G(u) \leqslant \alpha < 1$). By (4.1), the number of these m not exceeding n^u and not dividing x_n is a most

$$\sum_{\substack{p,\, k \\ k \geqslant 2,\, p > n^{1/k}}} n^u p^{-k} \ll n^{u - \frac{1}{2}}.$$

Thus we have

$$(4.4) \qquad \sum_{m \leqslant y_n} c_\alpha(x_n/m) \geqslant \sum_{m \leqslant n^u}{}' 1 + O\left(\frac{n^u}{\log n}\right).$$

de Bruijn [1] has shown that if $\psi(X, Y)$ is the number of natural numbers not exceeding X which have no prime factor exceeding Y, then

$$(4.5) \quad \psi(Y^u, Y) = G(u)Y^u + O(Y^{u-1}(u+1)^2 \max_{2 \leqslant x \leqslant Y} |R(x)|)$$

uniformly for $Y \geqslant 2$, $u \geqslant 0$, where $R(x) = \pi(x) - \mathrm{li}\, x$ is the error term in the prime number theorem. This with (4.4) and (4.2) gives Theorem 4.

The proof of Theorem 8 proceeds in the same manner. Thus

$$\sum_{m \leqslant y_n} \frac{1}{m} c_\alpha(x_n/m) \geqslant \sum_{m \leqslant n^u}{}' \frac{1}{m} c_\alpha(x_n/m) + O\left(\frac{1}{\log n}\right)$$

and

$$\sum_{\substack{m \leqslant n^u \\ m \nmid x_n}}{}' \frac{1}{m} \ll \sum_{\substack{p,\, k \\ k \geqslant 2,\, p > n^{1/k}}} \sum_{m \leqslant n^u/p^k} \frac{1}{mp^k} \ll n^{-1/2} \log n.$$

Hence

$$(4.6) \qquad \sum_{m \leqslant y_n} \frac{1}{m} c_\alpha(x_n/m) \geqslant \sum_{m \leqslant n^u}{}' \frac{1}{m} + O\left(\frac{1}{\log n}\right).$$

By partial integration,

$$(4.7) \quad \sum_{m < n^u}{}' \frac{1}{m} = n^{-u}\Psi(n^u, n) + (\log n) \int_0^u n^{-w}\Psi(n^w, n)\, dw.$$

Combining (4.5), (4.6) and (4.7) now establishes Theorem 8.

5. Proof of theorem 6.

Suppose that

$$(5.1) \qquad 0 < \beta < 1.$$

Let

$$(5.2) \qquad M_\beta = \left[\frac{x}{y} - \beta\right]$$

and

$$(5.3) \qquad S(\beta) = \sum_{M_\beta < m \leqslant x-\beta} \; \sum_{n \leqslant x/(m+\beta)} \frac{1}{n}.$$

Then

$$(5.4) \qquad \theta_{x,y}(\alpha) \log y = S(0) - S(\alpha) + (M_0 - M_\alpha) \sum_{n \leqslant y} \frac{1}{n}.$$

Let

$$(5.5) \qquad N = [x^{1/2}].$$

Then, by (5.3)

$$S(\beta) = \sum_{M_\beta < m \leqslant N} \left(\log \frac{x}{m+\beta} + \gamma + O\left(\frac{m}{x}\right)\right) + \sum_{n \leqslant \frac{n}{N+\beta}} \frac{1}{n} \left(\left[\frac{x}{n} - \beta \right] - N \right)$$

providing that $N \geqslant M_\beta$. This also holds when $N < M_\beta$, providing that the convention

$$\sum_{M_\beta < m \leqslant N} = - \sum_{N < m \leqslant M_\beta}$$

is adopted. Hence

$$\begin{aligned} S(0) - S(\alpha) &= \sum_{M_0 < m \leqslant N} \log\left(1 + \frac{\alpha}{m}\right) - (M_0 - M_\alpha) \log \frac{x}{M_0 + \alpha} + O(1) \\ &+ \sum_{n \leqslant \frac{x}{N}} \frac{x}{n^2} - \sum_{n \leqslant \frac{x}{N+\alpha}} \left(\frac{x}{n^2} - \frac{\alpha}{n}\right) - \frac{1}{2} \sum_{\frac{x}{N+\alpha} < n \leqslant \frac{x}{N}} \frac{1}{n} \\ &- \sum_{n \leqslant \frac{x}{N}} \frac{1}{n} B\left(\frac{x}{n}\right) + \sum_{n \leqslant \frac{x}{N+\alpha}} \frac{1}{n} B\left(\frac{x}{n} - \alpha\right) \end{aligned}$$

where $B(u) = \{u\} - \frac{1}{2}$. Therefore, by (5.4),

$$\begin{aligned} (5.6) \qquad \theta_{x,y}(\alpha) \log y = (M_0 - M_\alpha) \log \frac{y(M_0+\alpha)}{x} \\ + \alpha \log \frac{N}{M_0} + O(1) \\ + \alpha \log \frac{x}{N+\alpha} - T(0) + T(\alpha) \end{aligned}$$

where

$$(5.7) \qquad T(\beta) = \sum_{n \leqslant x^{1/2}} \frac{1}{n} B\left(\frac{x}{n} - \beta\right).$$

By (5.2)

$$\alpha \leqslant 1 + (\alpha - 1)\frac{y}{x} \leqslant \frac{y}{x}(M_0 + \alpha) < 2$$

and

$$\frac{y}{2} < \frac{xN}{M_0(N + \alpha)} \leqslant \frac{x}{\left[\frac{x}{y}\right]} \leqslant 2y.$$

Hence, by (5.6),

$$(5.8) \qquad \theta_{x,y}(\alpha) \log y = \alpha \log y + 0(1) + T(\alpha) - T(0).$$

The proof is completed by observing that a trivial modification of the proof of Satz 3.2.2 of Walfisz [11, p. 98] gives

$$T(\beta) \ll (\log x)^{2/3}.$$

6. Proof of theorem 10.

6.1. We require a lemma which has some independant interest. Let

$$(6.1) \qquad \vartheta(X) = \sum_{p < x} \log p.$$

Lemma 5. — *Let* $N(\sigma, T)$ *denote the number of zeros*

$$\rho = \beta + i\gamma$$

of the Riemann zeta function with $\beta \geqslant \sigma$, $|\gamma| \leqslant T$. *Suppose that there are positive constants* B, C (with $C \geqslant 2$) *such that*

$$(6.2) \qquad N(\sigma, T) \ll T^{C(1-\sigma)} (\log T)^B \; (T \geqslant 2).$$

Then, whenever $x \geqslant 4$ *and* $x^{\varepsilon - 2/C} < \theta \leqslant 1$, *we have*

$$(6.3) \qquad \int_x^{2x} |\vartheta(u + u\theta) - \vartheta(u) - u\theta|^2 \, du \ll \theta^2 x^3 \exp\left(- C_1 \left(\frac{\log x}{\log \log x}\right)^{1/3}\right),$$

where C_1 *is a suitable positive number depending at most on* ε.

If the Riemann hypothesis is assumed instead, then whenever $x \geqslant 4$

$$(6.4)\qquad \int_x^{2x} |\vartheta(u + u\theta) - \vartheta(u) - u\theta|^2\, du \ll \theta x^2 \left(\log \frac{2}{\theta}\right)^2$$

uniformly in θ *with* $0 < \theta \leqslant 1$.

This is essentially due to Selberg [10]. It differs firstly in that in (6.4) the bound is uniform for θ close to 1 whereas Selberg apparently requires $\theta \ll x^{-\varepsilon}$, and secondly it is slightly weaker when $\theta \leqslant x^{-C_2}$ with $0 < C_1 < 1$ since Selberg obtains

$$(6.5)\qquad \int_0^{\theta^{-1/C_2}} |\vartheta(u + u\theta) - \vartheta(u) - u\theta|^2 u^{-2}\, du \ll \theta \left(\log \frac{2}{\theta}\right)^2.$$

Moreno [6] has observed (6.3) with $C = 5/2$ and ϑ replaced by Ψ (where

$$(6.6)\qquad \Psi(x) = \sum_{n \leqslant x} \Lambda(n)$$

and $\Lambda(n)$ is von Mangoldt's function), and given only a weaker result for ϑ. In fact, there are at least two obvious ways of deducing a corresponding result for ϑ.

Proof of Lemma 5. — Clearly

$$(6.7)\qquad \int_x^{2x} (\Psi(u + \theta u) - \Psi(u) - \theta u)^2\, du \leqslant \int_1^2 \left(\int_{xv/2}^{2xv} (\Psi(u + \theta u) - \Psi(u) - \theta u)^2\, du\right) dv.$$

Let $\sum_\rho$ denote summation over all the complex zero of ζ grouped in complex conjugate pairs, that is, $\lim_{T\to\infty} \sum_{|\gamma|\leqslant T}$. Then, by the explicit formula (Ingham [3], Theorem 29), whenever $y \geqslant 2$

$$\sum_{n\leqslant y}{}' \Lambda(n) = y - \sum_\rho \frac{y^\rho}{\rho} - \frac{\zeta'(0)}{\zeta(0)} - \frac{1}{2}\log(1 - y^{-2})$$

where the dash means that if $y \in \mathbf{Z}$, then $\Lambda(y)$ is to be replaced by $\frac{1}{2}\Lambda(y)$. The sum over the zeros is boundedly convergent (cf. Ingham [3], p. 80). Thus

$$(6.8)\quad \int_{xv/2}^{2xv} (\Psi(u+\theta u) - \Psi(u) - \theta u)^2\, du \ll \int_{xv/2}^{2xv} \left| \sum_{\rho} \frac{(1+\theta)^{\rho}-1}{\rho} u^{\rho} \right|^2 du + \int_{xv/2}^{2xv} \left(\log\left(1 - \frac{(1+\theta)^2-1}{u^2(1+\theta)^2-1}\right)\right)^2 du$$

and

$$(6.9)\quad \int_{xv/2}^{2xv} \left| \sum \frac{(1+\theta)^{\rho}-1}{\rho} u^{\rho} \right|^2 du = \sum_{\rho_1} \sum_{\rho_2} \frac{(1+\theta)^{\rho_1}-1}{\rho_1} \cdot \frac{(1+\theta)^{\bar\rho_2}-1}{\bar\rho_2} \cdot \frac{2^{1+\rho_1+\bar\rho_2} - 2^{-1-\rho_1-\bar\rho_2}}{1+\rho_1+\bar\rho_2} (xv)^{1+\rho_1+\bar\rho_2}.$$

Trivially

$$(6.10)\quad \int_{xv/2}^{2xv} \left(\log\left(1 - \frac{(1+\theta)^2-1}{u^2(1+\theta)^2-1}\right)\right)^2 du \ll \theta^2 x^{-3}.$$

By Theorem 25*a* of Ingham [3],

$$(6.11)\quad N(0, T+1) - N(0, T) \ll \log T \quad (T \geqslant 1).$$

Thus, the double sum on the right of (6.9) converges absolutely, and uniformly in v on [1, 2]. Thus, by (6.6), (6.8), (6, 9) and (6.10)

$$(6.12)\quad \int_{x}^{2x} (\Psi(u+\theta u) - \Psi(u) - \theta u)^2\, du \ll \theta^2 x^{-3} + \Sigma_1$$

where

$$\Sigma_1 = \sum_{\rho_1} \sum_{\rho_2} \frac{(1+\theta)^{\rho_1}-1}{\rho_1} \cdot \frac{(1+\theta)^{\bar\rho_2}-1}{\bar\rho_2} \cdot \frac{2^{1+\rho_1+\bar\rho_2} - 2^{-1-\rho_1-\bar\rho_2}}{1+\rho_1+\bar\rho_2} \cdot \frac{2^{2+\rho_1+\bar\rho_2}-1}{2+\rho_1+\bar\rho_2} \cdot x^{1+\rho_1+\bar\rho_2}.$$

By the trivial inequality $|z_1 z_2| \ll |z_1|^2 + |z_2|^2$,

$$\Sigma_1 \ll \sum_{\rho_1} \sum_{\rho_2} x^{1+2\beta_1} \min(\theta^2, \gamma_1^{-2})(1 + |\gamma_1 - \gamma_2|)^{-2}.$$

Thus, by (6.11),

$$(6.13)\quad \Sigma_1 \ll \sum_{\substack{\rho \\ \gamma>0,\ \beta \geqslant 1/2}} x^{1+2\beta} \min(\theta^2, \gamma^{-2}) \log \gamma.$$

If the Riemann hypothesis is assumed, then at once from (6.11) and (6.13),

$$\Sigma_1 \ll x^2 \sum_{0<\gamma\leqslant\theta^{-1}} \theta^2 \log\gamma + x^2 \sum_{\gamma>\theta^{-1}} \frac{\log\gamma}{\gamma^2} \ll \theta x^2 \left(\log\frac{2}{\theta}\right)^2.$$

This with (6.12) establishes (6.4) with ϑ replaced by Ψ. To deduce the corresponding inequality involving ϑ, observe that for $y \geqslant 1, y \notin \mathbf{Z}$,

$$\Psi(y) - \vartheta(y) - y^{1/2} + 1 = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\frac{1}{2}-iT}^{\frac{1}{2}+iT} \left(- \frac{\zeta'(2s)}{\zeta(2s)} - \frac{1}{2s-1} + \sum_p \frac{\log p}{p^s(p^{2s}-1)} \right) \frac{y^s}{s}\, ds.$$

Let

$$\Delta(v, \theta) = (\Psi(e^v(1+\theta)) - \Psi(e^v) - \vartheta(e^v(1+\theta)) - \vartheta(e^v) - e^{v/2}(1+\theta)^{1/2} + e^{v/2})e^{-v/2}$$

and

$$F(t) = - \frac{\zeta'(1+2it)}{\zeta(1+2it)} - \frac{1}{2iT} + \sum_p \frac{\log p}{p^{\frac{1}{2}+it}(p^{1+2it}-1)}$$

Then, by Plancherel's theorem,

$$\int_0^\infty |\Delta(v, \theta)|^2\, dv \ll \int_{-\infty}^\infty \frac{|F(t)|^2}{(1+|t|)^2} \left|(1+\theta)^{\frac{1}{2}+it} - 1\right|^2 dt$$
$$\ll \int_{-\infty}^\infty (\log(1+|t|))^2 \min(\theta^2, (1+|t|)^{-2})\, dt$$
$$\ll \theta \left(\log\frac{2}{\theta}\right)^2.$$

This combined with the observation

$$\int_x^{2x} (u^{1/2}(1+\theta)^{1/2} - u^{1/2})\, du \ll \theta^2 x^2$$

enables one to deduce (6.4) from the corresponding result with replaced Ψ. Another line of approach is to use the relation

$$\vartheta(x) = \sum_k \mu(k) \Psi(x^{1/k})$$

where μ is the Mobius function, but in the proof of (6.3) this gives rise to complications of detail.

To prove (6.3) note that by (6.11),

$$\sum_{\substack{\rho \\ \gamma > x^4}} x^{1+2\beta} \frac{\log \gamma}{\gamma^2} \ll 1.$$

Thus, by (6.13),

$$\Sigma_1 \ll 1 + x\left(\theta^2 \left(\log \frac{2}{\theta}\right) \Sigma_2 + \Sigma_3\right) \tag{6.14}$$

where

$$\Sigma_2 = \sum_{\substack{\rho \\ 0 < \gamma \leqslant \theta^{-1} \\ \beta \geqslant 1/2}} x^{2\beta}$$

and

$$\Sigma_3 = \sum_{\substack{\rho \\ \theta^{-1} < \gamma \leqslant x^4 \\ \beta \geqslant 1/2}} x^{2\beta} \gamma^{-2} \log \gamma. \tag{6.15}$$

Hence

$$\Sigma_2 = x\mathrm{N}\left(\frac{1}{2}, \theta^{-1}\right) + 2 \int_{1/2}^{1} x^{2u} (\log x) \mathrm{N}(u, \theta^{-1})\, du. \tag{6.16}$$

By (2), page 226, of Walfisz [11], we have

$$\mathrm{N}(\sigma, x^4) = 0 \text{ whenever } \sigma \geqslant 1 - \mathrm{C}_3 (\log x)^{-2/3} (\log \log x)^{-1/3}. \tag{6.17}$$

This with (6.2) and (6.11) gives

$$\Sigma_2 \ll x\theta^{-1} \left(\log \frac{2}{\theta}\right) + 2 \int_{\frac{1}{2}}^{1-\frac{1}{\mathrm{C}}} x^{2u} (\log x) \theta^{-1} \left(\log \frac{2}{\theta}\right) du$$
$$+ 2 \int_{1-\frac{1}{\mathrm{C}}}^{1-\mathrm{C}_2 (\log x)^{-2/3} (\log\log x)^{-1/3}} x^{2u} (\log x) \theta^{-\mathrm{C}(1-u)} \left(\log \frac{2}{\theta}\right)^{\mathrm{B}} du.$$

It is assumed that $x^2 \theta^{\mathrm{C}} \geqslant x^{\mathrm{C}\varepsilon} > 1$. Thus

$$\Sigma_2 \ll x^{2-\frac{2}{\mathrm{C}}} \theta^{-1} \log \frac{2}{\theta} \tag{6.18}$$
$$+ x^2 (\log x)^{\mathrm{B}+1} \exp\left(\frac{\mathrm{C}_2 \left(\mathrm{C} \log \frac{1}{\theta} - 2 \log x\right)}{(\log x)^{2/3} (\log \log x)^{1/3}}\right).$$

The sum Σ_3 is estimated in the same way. By (6.16), (6.11), (6.17) and (6.2),

$$\Sigma_3 \ll \sum_{\substack{\rho \\ \theta^{-1}<\gamma<x^4 \\ \beta>1/2}} (x^{2\beta}-x)\left(\frac{\log \gamma}{\gamma^2} - \frac{4\log x}{x^8}\right) + \frac{(\log x)^2}{x^2} + \theta x\left(\log\frac{2}{\theta}\right)^2$$

$$\ll \theta x\left(\log\frac{2}{\theta}\right)^2 + \int_{\frac{1}{2}}^{1}\left(\int_{\theta^{-1}}^{x^4} x^{2u}(2\log x)\mathrm{N}(u, t)\frac{\log t}{t^3}\,dt\right)du$$

$$\ll \theta x^{2-\frac{2}{\mathrm{C}}}\left(\log\frac{2}{\theta}\right)^2 + x^2(\log x)^{\mathrm{B}+2}\exp\left(\frac{\mathrm{C}_2\left(\mathrm{C}\log\frac{1}{\theta} - 2\log x\right)}{(\log x)^{2/3}(\log\log x)^{1/3}}\right).$$

This, with (6.12), (6.14) and (6.18), gives (6.3) with ϑ replaced by Ψ. The deduction for ϑ is the same as in the proof of (6.4).

6.2. It is possible to deduce Theorem 10 directly from Lemma 5, or even from the corresponding result with ϑ replaced by Ψ. However, the argument is then somewhat more complicated than with the method we are going to use. Moreover, the following two lemmas also have some interest of their own.

LEMMA 6. — *Let h be any real number with $0 \leqslant h \leqslant x$. If* (6.2) *holds, then*

$$(6.19) \quad \int_x^{2x} (\vartheta(u+h) - \vartheta(u) - h)^2\,du \ll h^2 x \exp\left(-\mathrm{C}_1\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)$$

whenever $x^{\varepsilon-\frac{2}{\mathrm{C}}+1} < h \leqslant x$ *and* $x \geqslant 3$. *On the Riemnan hypothesis,*

$$(6.20) \quad \int_x^{2x} (\vartheta(u+h) - \vartheta(u) - h)^2\,du \ll h^2 x\left(\log\frac{2x}{h}\right)^2$$

uniformly in h.

Proof. — It suffices to prove the lemma with $h \leqslant x/6$. Suppose that $2h \leqslant v \leqslant 3h$ and $x \leqslant u \leqslant 2x$, so that

$$h \leqslant v - h \leqslant 2h \quad \text{and} \quad x \leqslant u + h \leqslant 3x.$$

Then, since

$$(\vartheta(u+h)-\vartheta(u)-h)^2 \leqslant (\vartheta(u+v)-\vartheta(u)-v)^2 + (\vartheta(u+v)-\vartheta(u+h))-(v-h))^2,$$

on making the substitution $w = \theta u (h \leqslant w \leqslant 3h)$ and on observing that

$$\frac{h}{3x} \leqslant \frac{h}{u} \leqslant \theta \leqslant \frac{3h}{u} \leqslant \frac{3h}{x} \leqslant \frac{1}{2},$$

one has

$$h\int_x^{2x} (\vartheta(u+h)-\vartheta(u)-h)^2\, du \leqslant \int_x^{3x}\left(\int_h^{3h} (\vartheta(u+w)-\vartheta(u)-w)^2\, dw\right) du \leqslant x\int_x^{3x}\left(\int_{h/3x}^{3h/x} (\vartheta(u+u\theta)-\vartheta(u)-\theta u)^2\, d\theta\right) du.$$

The integrand in the last double integral is continuous on $[x, 3x] \times [h/3x, 3h/x]$ except on a subset having zero content. Thus the order of integration can be inverted. Hence

$$\int_x^{2x} (\vartheta(u+h)-\vartheta(u)-h)^2\, du \leqslant \frac{x}{h}\int_{h/3x}^{3h/x}\left(\int_x^{3x} (\vartheta(u+\theta u)-\vartheta(u)-\theta u)^2\, du\right) d\theta. \tag{6.21}$$

Using this, (6.20) follows from (6.4). If $x \leqslant 3^{2/\varepsilon}$, then (6.19) is trivial. Thus it can be assumed that

$$h \geqslant 3x^{\frac{\varepsilon}{2}+1-\frac{2}{C}}.$$

Combining (6.3) and (6.21) then gives (6.19).

Lemma 7. — *Suppose that* (6.2) *holds. Then*

$$\int_x^{2x} \max_{0\leqslant v\leqslant h} |\vartheta(u+v)-\vartheta(u)-v|^2\, du \ll_\varepsilon h^2 x \exp\left(-C_4\left(\frac{\log x}{\log\log x}\right)^{1/3}\right) \tag{6.22}$$

whenever $x^{\varepsilon-\frac{2}{C}+1} < h \leqslant x$ *and* $x \geqslant 3$. *Moreover, on the*

Riemann hypothesis,

$$(6.23)\qquad \int_x^{2x} \max_{0\leqslant v\leqslant h} |\vartheta(u+v)-\vartheta(u)-v|^2\,du \ll hx(\log x)^4$$

whenever $0 \leqslant h \leqslant x$.

This follows from Lemma 6 by a similar argument to that used to deduce Lemma 4 from Lemma 3.

6.3. We now proceed with the proof of Theorem 10. By Σ'_m it is meant that possible terms with $m < [x/y]$ are omitted, $[x/y]$ is only counted when $x < y([x/y]+\alpha)$, and if $[x/y]$ is counted, then $x/[x/y]$ is replaced by y in all the appropriate places. Observe now that, by (1.27),

$$(6.24)\qquad y\vartheta_{x,y}(\alpha) = \sum'_{m\leqslant x^{1/2}} \sum_{\frac{x}{m+\alpha}<p\leqslant\frac{x}{m}} \log p + \sum_{p\leqslant x^{1/2}} \sum_{\substack{m\\ x-\alpha p<mp\leqslant x}} \log p - \sum_{m\leqslant x^{1/2}} \sum_{\frac{x}{m+\alpha}<p\leqslant x^{1/2}} \log p.$$

Clearly the contribution from the second double sum is $O(x^{1/2})$, and from the third is $O(\log x)$. Thus, by (6.24)

$$y\vartheta_{x,y}(\alpha) = \sum'_{m\leqslant x^{1/2}}\left(\vartheta\left(\frac{x}{m}\right)-\vartheta\left(\frac{x}{m+\alpha}\right)\right)+O(x^{1/2})$$

so that, by (1.2),

$$(6.25)\qquad y(\vartheta_{x,y}(\alpha)-F(\alpha,x/y)) = \sum'_{m\leqslant x^{1/2}}\left(\vartheta\left(\frac{x}{m}\right)-\vartheta\left(\frac{x}{m+\alpha}\right)-\frac{\alpha x}{m(m+\alpha)}\right)+O(x^{1/2}).$$

Suppose that

$$(6.26)\qquad 0 < \delta < 1$$

and

$$0 \leqslant u - \frac{x}{m+\alpha} \leqslant \frac{\delta x\alpha}{m(m+\alpha)}.$$

then

$$(6.27)\qquad \vartheta\left(\frac{x}{m}\right)-\vartheta\left(\frac{x}{m+\alpha}\right)-\frac{\alpha x}{m(m+\alpha)} = \vartheta\left(\frac{x}{m}\right)-\vartheta(u)-\left(\frac{x}{m}-u\right)+O\left(\left(\frac{\delta x}{m^2}+1\right)\log x\right).$$

Let X be of the form

$$\text{(6.28)} \qquad X = (1+\delta)^k,$$

where k is a non-negative integer, and suppose that

$$\text{(6.29)} \qquad X^2 \leqslant x.$$

Then

$$\text{(6.30)} \qquad \sum_{X<m\leqslant X+\delta X}{}' \left| \vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right) - \frac{\alpha x}{m(m+\alpha)} \right|$$
$$\ll \frac{X^2}{\delta x} \int_{x/(X+\delta X+1)}^{x/X} \sup_{v\leqslant xX^{-2}} |\vartheta(u+v) - \vartheta(u) - v|\, du$$
$$+ \sum_{X<m\leqslant X+\delta X}{}' \left(\frac{\delta x}{m^2} + 1\right) \log x.$$

Before proceeding further with the proof consider the consequence of assuming the Riemann hypothesis. By Lemma 7 and (6.30),

$$\sum_{X<m\leqslant X+\delta X}{}' \left| \vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right) - \frac{\alpha x}{m(m+\alpha)} \right|$$
$$\ll \frac{X^2}{\delta X}\left(\frac{\delta x}{X}\cdot\frac{x}{X^2}\cdot\frac{x}{X}(\log x)^4\right)^{1/2} + \sum_{X<m<X+\delta X}{}' \left(\frac{\delta x}{m^2} + 1\right) \log x.$$

Thus, summing over those X, given by (6.28), for which (6.29), holds, one finds that

$$\sum_{m\leqslant x^{1/2}}{}' \left| \vartheta\left(\frac{x}{m}\right) - \vartheta\left(\frac{x}{m+\alpha}\right) - \frac{\alpha x}{m(m+\alpha)} \right|$$
$$\ll x^{1/2}\delta^{-3/2}(\log x)^3 + \delta y \log x + x^{1/2}\log x,$$

which with (6.25) and the choice $\delta = y^{-2/5}x^{1/5}(\log x)^{4/5}$, which is consistent with (6.26), gives

$$\vartheta_{x,y}(\alpha) = F(\alpha, x/y) + O(y^{3/5}x^{1/5}(\log x)^{9/5})$$

whenever $y > x^{1/2}(\log x)^2$.

To return to the proof, suppose that (6.2) holds. Then, providing that

$$X \leqslant x^{\frac{2}{C+2}-\varepsilon},$$

one has, by (6.30), the Schwarz inequality and Lemma 7,

$$\sum_{X<m\leqslant X+\delta X}' \left|\vartheta\left(\frac{x}{m}\right)-\vartheta\left(\frac{x}{m+\alpha}\right)-\frac{\alpha x}{m(m+\alpha)}\right|$$
$$\ll \frac{X^2}{\delta x}\left(\frac{\delta x}{X}\cdot\frac{x^2}{X^4}\cdot\frac{x}{X}\exp\left(-C_5\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)\right)^{1/2}$$
$$+\sum_{X<m\leqslant X+\delta X}\left(\frac{\delta x}{m^2}+1\right)\log x.$$

Thus summing over all the numbers X of the form (6.28) for which (6.29) holds gives

$$\sum_{m<x^{1/2}}' \left|\vartheta\left(\frac{x}{m}\right)-\vartheta\left(\frac{x}{m+\alpha}\right)-\frac{\alpha x}{m(m+\alpha)}\right|$$
$$\ll \sum_{x^{\frac{2}{C+2}-\varepsilon}<m\leqslant<x^{1/2}}\left(\frac{x}{m^2}+1\right)\log x$$
$$+y\delta^{-3/2}\exp\left(-C_6\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)+(\delta y+x^{1/2})\log x.$$

This with the choice $\delta=\exp\left(-\frac{1}{2}C_6\left(\frac{\log x}{\log\log x}\right)^{1/3}\right)$ and Huxley's theorem [2] that (6.2) holds with $C=12/5$ establishes Theorem 10.

7. Proof of theorem 11.

Define N_j inductively by

(7.1) $$N_1=3 \quad\text{and}\quad N_{j+1}=\Pi p$$

where the product is over all those primes p such that

(7.2) $$p\leqslant e^{N_j}.$$

Let

(7.3) $$\mathscr{D}_j=\{n : n|N_j, \log N_j<n\leqslant N_j\}$$

and

(7.4) $$\mathscr{D}=\bigcup_{j=1}^{\infty}\mathscr{D}_j.$$

Further, let j be large and write

$$y = N_j. \tag{7.5}$$

Let λ_n be the characteristic function of $\mathscr{D}$ and

$$a_n(y) = \begin{cases} \lambda_n \Big/ \sum_{m \leqslant y} \lambda_m & (n \leqslant y) \\ 0 & (n > y). \end{cases} \tag{7.6}$$

By (7.1), (7.2), (7.3) and (7.4), all the elements of $\mathscr{D}$ are odd. Let $\alpha = 1/2$. Hence, by Theorems 3 and 4 of [9],

$$\begin{aligned} \text{(7.7)}\quad & \max(0, x - y^2) \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2 \\ & \leqslant \int_{x_0}^{x_0+x} (\Phi_{u,y}(\alpha) - \alpha)^2 \, d\alpha \leqslant (x + y^2) \sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2. \end{aligned}$$

By (7.1), (7.2), (7.3), (7.4) and (7.5), it is easily seen that y is squarefree and the elements of

$$\mathscr{D} \cap (\log N_j, N_j]$$

are precisely the divisors of y in the range $(\log N_j, N_j]$. Since λ_n is the characteristic function of $\mathscr{D}$,

$$\sum_{m \leqslant y} \lambda_m = 2^P + O(\log y) \tag{7.8}$$

where P is the number of prime divisors of y. Also,

$$\begin{aligned} \sum_{n \leqslant y} \left(\sum_{m \leqslant y/n} \frac{1}{m} \lambda_{mn} \right)^2 &= \sum_{\substack{n \\ \log y < n \leqslant y}} \left(\sum_{\substack{m \\ mn \mid y}} \frac{1}{m} \right)^2 + O((\log y)^3) \\ &= \sum_{\substack{n \\ n \mid y}} \left(\sum_{\substack{m \\ mn \mid y}} \frac{1}{m} \right)^2 + O((\log y)^3) \\ &= 2^P \prod_{p \mid y} \left(1 + \frac{1}{p} + \frac{1}{2p^2} \right) + O((\log y)^3). \end{aligned}$$

Hence, by (7.1), (7.2), (7.6) and (7.8),

$$\sum_{n=1}^{\infty} \left(\sum_{m=1}^{\infty} \frac{1}{m} a_{mn}(y) \right)^2 \asymp \frac{\prod_{p \mid y} \left(1 + p^{-1} + \frac{1}{2} p^{-2} \right)}{\sum_{n \leqslant y} \lambda_n} \tag{7.9}$$

Theorem 11 now follows in a straightforward manner from (7.1), (7.2), (7.5), (7.7) and (7.9).

BIBLIOGRAPHY

[1] N. G. DE BRUIJN, On the number of positive integers $\leqslant x$ and free of prime factors $> y$, *Ned. Akad. Wet. Proc.* Ser. A, 54 (1951), 50-60, *Indag. Math.*, 13 (1951), 50-60.

[2] M. N. HUXLEY, On the difference between consecutive primes, *Inventiones Math.*, 15 (1972), 164-170.

[3] A. E. INGHAM, The distribution of prime numbers, *Cambridge Tracts in Mathematics and Mathematical Physics*, 30, London, 1932.

[4] E. LANDAU, Vorlesungen uber Zahlentheorie, zweiter Band, Chelsea Pub. Co., New York, 1969.

[5] D. MENCHOV, Sur les séries de fonctions orthogonales. Première partie. La convergence, *Fundamenta Math.*, 4 (1923), 82-105.

[6] C. J. MORENO, The average size of gaps between primes, *Mathematika*, 21 (1974), 96-100.

[7] K. K. NORTON, Numbers with small prime factors, and the least k-th power non-residue, *Mem. Am. Math. Soc.*, 106 (1971).

[8] H. RADEMACHER, Einige Sätze über Reihen von allgemeinen Orthogonalfunktionen, *Math. Ann.*, 87 (1922), 112-138.

[9] B. SAFFARI and R.-C. VAUGHAN, On the fractional parts of x/n and related sequences. I, *Annales de l'Institut Fourier*, 26,4 (1976), 115-131.

[10] A. SELBERG, On the normal density of primes in small intervals, and the difference between consecutive primes, *Arch. Math. Naturvid.*, 47 (1943), 87-105.

[11] A. Z. WALFISZ, Weylsche Exponentialsummen in der neuren Zahlentheorie, Mathematische Forschungsberichte, XV, VEB Deutscher Verlag der Wissenschaften, Berlin, 1963.

Manuscrit reçu le 23 janvier 1976
Proposé par C. Chabauty.

B. SAFFARI,
1, place Corneille
92100 Boulogne-Billancourt.

R.-C. VAUGHAN,
Imperial College,
Dept. of Mathematics
Huxley Building
180 Queen's Gate
London SW7 2BZ (G. B.).