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Annales de l'institut Fourier, tome 27, n° 2 (1977), p. 145-159

http://www.numdam.org/item?id=AIF_1977__27_2_145_0

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SOME EXAMPLES OF NONSINGULAR MORSE-SMALE VECTOR FIELDS ON S^3

by F. Wesley WILSON, Jr.

A vector field is called Morse-Smale if it has a finite set of singular points $\alpha_1, \dots, \alpha_k$ and a finite set of periodic solutions β_1, \dots, β_l such that each of these elements is hyperbolic (in the generic sense), such that the various invariant manifolds of these elements intersect transversely, and such that the nonwandering set of the vector field consists entirely of these elements, cf. [11]. Besides their relatively simple qualitative structure, one reason for interest in such vector fields is that they are structurally stable [6]. Indeed, combining this result with Morse Theory provides the easiest proof that every compact manifold admits a structurally stable vector field.

If M has Euler characteristic zero, then M has nonsingular vector fields, and these vector fields can be partitioned into their (nonsingular) homotopy classes. It seems natural to ask whether each homotopy class contains a nonsingular Morse-Smale vector field (or any structurally stable vector field). D. Asimow [1] has shown that if $\dim > 3$, then each homotopy class contains a Morse-Smale vector field. The analogous result for the torus and the Klein Bottle follows from M. Peixoto's work on 2-dimensional structural stability [7]. We shall illustrate some techniques for producing examples of 3-dimensional Morse-Smale vector fields, and in particular our examples will include representatives of all vector field homotopy classes on S^3 except for two.

A related question is whether or not one can determine the homotopy class of a nonsingular vector field by examining some finite list of algebraic invariants which are associated with the flow which the vector field generates. B. L. Reinhart [10] has provided an affirmative answer for vector fields on the 2-torus. In this case, the periodic solutions determine a 1-dimensional homology class, up to sign. This homology class together with an integer $D = P - N$ determines the homotopy class of the vector field (P and N are determined by judiciously choosing one of the homology classes of the periodic solutions and determining that it contains P sources and N sinks). Thus we can determine the homotopy class from algebraic invariants which are associated with the periodic solutions. The author [13] has given examples in dimensions greater than two which indicate that algebraic invariants associated with the nonwandering set of the flow will in general not be sufficient to determine its homotopy class [8] or [13]. In these examples, the nonwandering set consists of a finite collection of periodic solutions, they all have index zero, are unknotted, are unlinked, and are the boundaries of embedded 2-disks. Moreover, these examples include representatives of all homotopy classes. By bifurcations of these periodic solutions into pairs of periodic solutions consisting of a saddle and a source or a saddle and a sink, these examples can be altered so that they are still nearly algebraically sterile and yet are almost Morse-Smale (these examples have periodic solutions with indices ± 1 , but equally many of each, and there will be some necessary degenerate intersections of the invariant manifolds). The following question remains open: Can one determine the homotopy class of a nonsingular Morse-Smale vector field from a list of algebraic invariants which are associated with the geometric structure of its flow? The examples which we construct in this paper tend to make us believe that the answer is no. Certainly they show that algebraic properties associated just with the set of periodic solutions is not sufficient. It remains possible that the set which is the closure of the 2-dimensional invariant manifolds may contain the information which we seek, but it will probably be very difficult to obtain such a result.

We should point out that in our examples there do occur

periodic solutions with various linking numbers and representing all toral knots. It is apparently unknown whether or not arbitrary knots can occur as a periodic solution of a Morse-Smale vector field.

1. Singular foliations and homotopy classes of vector fields.

A. Davis has introduced a notion of singular foliation and proved some results which are useful for studying the homotopy classes of vector fields [2]. We shall include a brief review of this approach. Let M be a compact manifold.

DEFINITION 1.1. — A *singular foliation* \mathcal{F} for M consists of a compact submanifold N of codimension two in M and a foliation $\tilde{\mathcal{F}}$ of $M-N$ of codimension one which satisfies the following condition: There is a tubular neighborhood U of N such that each fiber (2-disk) is transverse to $\tilde{\mathcal{F}}$, and such that along each component N_i of N , the induced regular curve families on the different fibers are topologically equivalent. The leaves of $\tilde{\mathcal{F}}$ are called the *regular leaves* of \mathcal{F} and the components of N are called the *singular leaves* of \mathcal{F} . The index $I(\mathcal{F}, N_i)$ of a singular leaf N_i is the index of the line field which is induced by \mathcal{F} on the fibers of a tubular neighborhood of that leaf.

THEOREM 1.2. (A. Davis). — *Every compact manifold M admits a singular foliation \mathcal{F} , and the following index formula is satisfied*

$$\chi(M) = \sum_{i=1}^m \chi(N_i) I(\mathcal{F}, N_i)$$

where χ denotes the Euler characteristic and $N = \bigcup_{i=1}^m N_i$.

The reason that singular foliations have been introduced here is that they are useful for describing certain vector fields on manifolds and for computing their homotopy classes. This approach is not necessary for this exposition, since the compu-

tations which we actually need have been carried out in [12]. However, these specific examples are more easily understood in this new setting.

Let \mathcal{F} be a singular foliation of M , and suppose that ζ is a nonsingular vector field on M which is tangent to the leaves of \mathcal{F} . Note that ζ does not exist for every \mathcal{F} . A necessary condition is that each component of N have zero Euler characteristic. Even in this case, there are obstructions to the existence of ζ . Since any $\zeta|N$ has an extension to $\mathcal{F}|U$, where U is a tubular neighborhood of N , then it follows that the obstructions to the existence of an extension of $\zeta|N$ to all of \mathcal{F} lie in $\{H^{k+1}[M, N; \pi_{k-1}(S^{n-1})]\}$, cf. [3]. Similarly, the obstructions to the uniqueness of extensions (up to homotopy as nonsingular vector fields tangent to \mathcal{F}) lie in

$$\{H^k[M, N; \pi_k(S^{n-2})]\}.$$

However, if we are only interested in the homotopy class as nonsingular vector fields on M , then we have the following interesting result.

THEOREM 1.3. (A. Davis). — *Let ζ, ξ be nonsingular vector fields on M which are tangent to the transversely orientable singular foliation \mathcal{F} . If $\zeta|N \cong \xi|N$ as vector fields on N , then $\zeta \cong \xi$ as nonsingular vector fields on M .*

Proof. — Both ζ and ξ are homotopic to a vector field η which coincides with ζ on N and which always has a non-zero component transverse to \mathcal{F} on $M - N$.

In [11], we gave the following description of the singular Hopf fiberings of S^3 over S^2 :

S^2 coordinates: $(z, \theta) \quad -1 \leq z \leq 1, 0 \leq \theta \leq 2\pi$

$$[\sqrt{1-z^2} \cos \theta, \sqrt{1-z^2} \sin \theta, z] \in S^2$$

S^3 coordinates: $(r, \mu, \nu) \quad 0 \leq r \leq 1, 0 \leq \mu, \nu < 2\pi$

$$[r \cos \mu, r \sin \mu, \sqrt{1-r^2} \cos \nu, \sqrt{1-r^2} \sin \nu] \in S^3$$

$h_{p,q}: S^3 \rightarrow S^2: (r, \mu, \nu) \rightsquigarrow (1-2r, p\mu - q\nu)$.

Note that $T_r = \{(r, \mu, \nu) | r \text{ fixed}\}$ is a circle if $r = 0$ or 1 and a torus otherwise. For $0 < r < 1$, $h_{p,q}^{-1}(1-2r, \theta)$

is a circle of type (p, q) in T_r . Thus the restriction of $h_{p,q}$ to a tubular neighborhood of T_0 can be viewed as the composition of the projection $D \times S^1 \rightarrow D$ and the branched p -fold covering of $D \rightarrow D$ given by $(r, \theta) \rightarrow (r, p, \theta)$, and the restriction to a tubular neighborhood of T_1 can be viewed as the analogous composition of a projection and the branched q -fold covering. Now let $\tilde{\zeta}$ be a vector field on S^2 which is singular precisely at the poles. A singular foliation \mathcal{F} is defined on S^3 by taking the $h_{p,q}$ preimage of the trajectories of $\tilde{\zeta}$. Note that T_0 and T_1 are singular leaves of \mathcal{F} and that each elliptic (or hyperbolic) region for the singular point at the south pole of S^2 corresponds to p elliptic (or hyperbolic, respectively) regions for the induced curve family on a disk transversal to T_0 . Using the formula

$$I_0(\tilde{\zeta}) = 1 + \frac{1}{2}(e - h)$$

to compute the index of the south pole, we see that the index of \mathcal{F} along T_0 is $I(\mathcal{F}, T_0) = 1 + \frac{p}{2}(e - h)$. A similar argument can be applied near T_1 .

LEMMA 1.4

$$I(\mathcal{F}, T_0) = 1 + p[I_0(\tilde{\zeta}) - 1]$$

and

$$I(\mathcal{F}, T_1) = 1 + q[I_1(\tilde{\zeta}) - 1],$$

where

$$I_0(\tilde{\zeta}) - 1 = - [I_1(\tilde{\zeta}) - 1].$$

Proof. — The first two relationships follow by the argument above. The last statement is just a special case of the Poincaré Index formula for S^2 .

Let ξ, ζ be unit vector fields on S^3 . From obstruction theory, we know that the homotopy difference between ξ and ζ is an element

$$d(\xi, \zeta) \in H^3(S^3; \pi_3(S^2)) \cong \pi_3(S^2) \cong \mathbf{Z}$$

where the last isomorphism is given by the Hopf invariant.

In [11], we developed a geometric interpretation of $d(\xi, \zeta)$ which we shall now review. On S^3 , and nonsingular vector field can be viewed as the first coordinate direction of a (unique to homotopy) parallelization Π . Expressing ζ in terms of the Π -coordinates, we obtain a mapping of S^3 into S^2 and the homotopy class of this mapping depends only on the homotopy classes of ξ and ζ . Applying the Hopf invariant to this mapping, we obtain the integer $d(\xi, \zeta)$ which measures the homotopy difference between ξ and ζ . In practice, we can evaluate $d(\xi, \zeta)$ only if we can compute the Hopf invariant of the induced mapping. The following lemma is useful in this matter. Part (4) is due to H. Hopf [5], and parts (1), (2), and (3) are proved in [12; § 2]. Let H_+ and H_- denote the respective homotopy classes of the unit vector fields which are tangent to the fibers of the Hopf fiberings with positive and negative fiber linkings.

LEMMA 1.5. — *Let ξ, ζ be unit vector fields on S^3 . Then $\xi \cong \zeta$ if and only if $d(\xi, \zeta) = 0$. Moreover,*

$$(1) \quad d(\xi, \zeta) = 0 \quad \text{if} \quad \xi(x) \neq -\zeta(x) \quad \text{for every} \quad x \in S^3,$$

$$(2) \quad d(\xi, -\xi) = 0,$$

$$(3) \quad d(H_+, H_-) = 1,$$

(4) *If the induced mapping by ξ and ζ carries each T_r onto a latitude circle of S^2 with degree m on meridian circles and degree l on longitude circles, then $d(\xi, \zeta) = -ml$.*

The third statement has an interesting interpretation. If $h: S^3 \rightarrow S^3$ is a diffeomorphism which reverses orientation, then $Dh(H_+) = H_-$. But by (3), there is no class between H_+ and H_- . Thus we can view the vector fields ξ on S^3 as being either above H_+ ($d(\xi, H_+) > 0$) or as being the Dh -image of a vector field which is above H_+ . For this reason, it suffices to restrict our attention to the vector fields which lie above H_+ .

Before completing our description of the examples of the vector fields in the various homotopy classes, we take this opportunity to point out that there is a sign error in [12]. The omission occurred in

PROPOSITION 4.6. — *The correct formulas are*

$$\begin{aligned} \lambda(\gamma_0; \gamma_1) &= \sigma \frac{p}{q'} = \frac{p'}{q} \\ r(\gamma_0; F)r(\gamma_1; F) &= \lambda(\gamma_0, \gamma_1)^2 \\ t_q(\gamma_0; F) &= \sigma \lambda(\gamma_0; \gamma_1)[i_q(\gamma_1; F) - 1] \\ t_{q'}(\gamma_1; F) &= \sigma \lambda(\gamma_0; \gamma_1)[i_{q'}(\gamma_0; F) - 1] \end{aligned}$$

where $\sigma = \pm 1$ is determined by the rule: $\sigma = +1$ if $F|N$ is homotopic rel ∂N to the vector field on N which has every trajectory periodic, and $\sigma = -1$ otherwise (only two cases are possible).

The proof which is given for Proposition 4.6 is correct, provided that one is careful about the orientation of γ . A consequence of this oversight is a sign change in the formulas of Theorem 6.4. The correct formulas are: If

$$p\lambda(\gamma_0; \gamma_1) < 0,$$

then

$$\begin{aligned} d(F, H_-) &= i_q(\gamma_0; F)i_p(\gamma_1; F) \quad \text{if } \lambda(\gamma_0; \gamma_1) > 0 \\ d(F, H_+) &= i_q(\gamma_0; F)i_p(\gamma_1; F) \quad \text{if } \lambda(\gamma_0; \gamma_1) < 0. \end{aligned}$$

These relatively minor changes culminate in a more substantial change in

THEOREM 6.7. — *There is a geometric vector field with two periodic solutions representing every homotopy class of nonsingular vector fields on S^3 except for the first class below H_- and the first class above H_+ .*

This theorem can be proved by applying the (correct) formulas from [12: Theorem 6.4] in a spirit similar to the previous proof. We omit this computation; instead we shall give another proof using Davis' approach.

By Theorem 1.3 and Lemma 1.5 (2), there are only two homotopy classes represented by vector fields tangent to the singular foliation \mathcal{F} which we constructed above. The vector field $H_{p,q}$ which is tangent to the $h_{p,q}$ -fibering of S^3 is tangent to \mathcal{F} , and so we see that one of these two classes is homotopic to H_+ (if $pq > 0$) or to H_- (if $pq < 0$). Thus

there is only one interesting homotopy class for each choice of $\tilde{\zeta}$, p , and q . Denote this class by $\tilde{\zeta}_{p,q}$. We note that $\tilde{\zeta}_{p,q}$ can be chosen to be of the form

$$\tilde{\zeta}_{p,q} = F + \alpha H_{p,q}$$

where $F \in Dh_{p,q}^{-1}(\tilde{\zeta})$ is normal to the $h_{p,q}$ fibers and $\alpha : S^3 \rightarrow [-1, 1]$ has values $+1$ on T_0 , -1 on T_1 and satisfies $\alpha|_{T_r}$ is constant for every r . Then the mapping $\tilde{\zeta}_{p,q}$ relative to a parallelization Π which satisfies $\Pi^{(3)} = H_{p,q}$ will satisfy the hypotheses of Lemma 1.5 (4). Thus it remains to determine the values of the integers l and m . Note that these values are the same for any choice of longitude and meridian on any T_r where $0 < r < 1$. Since Π is locally nearly constant near a point of T_0 , we can choose r to be very small and deduce that $m = I(\mathcal{F}, T_0)$. Similarly, we can choose r to be very near to one and compute l . In this case, the direction of a longitude is critical, since the longitude must be oriented as the boundary of a fiber in a tubular neighborhood of T_1 in order to be useful for computing the index. With the correct choice of orientation, $l = \text{sgn}(pq)I(\mathcal{F}, T_1)$. Thus we have

$$d(\tilde{\zeta}_{p,q}; H_{p,q}) = -\text{sgn}(pq)I_0\mathcal{F}, T_0)I(\mathcal{F}, T_1).$$

This formula is the analogue of [12; Theorem 6.4]. Using Lemma 1.4, and letting $k = I_0(\tilde{\zeta}) - 1$, we see that

$$d(\tilde{\zeta}_{p,q}; H_{p,q}) = -\text{sgn}(pq)(1 + pk)(1 - qk)$$

where $(p, q) = 1$ and k can be any integer (depending only on the choice of $\tilde{\zeta}$). Recall that $H_{p,q} \cong H_+$ if $pq > 0$ and $H_{p,q} \cong H_-$ if $pq < 0$.

THEOREM 1.6. — *There is a vector field of type $\tilde{\zeta}_{p,q}$ representing every homotopy class of nonsingular vector fields on S^3 except for the first class below H_- and the first class above H_+ .*

Proof. — If $pq > 0$, then $H_{p,q} \cong H_+$ and

$$d(\tilde{\zeta}_{p,q}, H_+) \cong -(1 + pk)(1 - qk).$$

Choosing $k = 1$, $p = 1$, $q > 0$, we obtain the positive even classes, and choosing $k = 2$, $q = 1$, $p > 0$ we obtain all positive odd classes except for $d(\tilde{\zeta}_{p,q}, H_+) = 1$. Indeed, this class is not represented. To achieve a representation would require the choice of $p > 0$, $q > 0$, k to satisfy

$$(1 + pk)(1 - qk) = -1.$$

Thus k must be an integer which satisfies

$$2pqk = q - p \pm \sqrt{p^2 + 6pq + q^2}$$

For $q \geq p \geq 2$,

$$2pq \geq 4q = q + \sqrt{9q^2} > q - p + \sqrt{p^2 + 6pq + q^2}$$

and so no positive k works. Similarly, no positive k works if $p \geq q \geq 2$. Choosing k negative merely reverses the roles of p and q in these arguments. The remaining cases

$$(p, q) = (1, 1), (1, 2), \text{ or } (2, 1)$$

have an irrational radical term.

2. Breaking Links.

The tangent vector fields to the Hopf fiberings of S^3 are tangent to the singular foliation of S^3 whose leaves are the sets $T_r (0 \leq r \leq 1)$, i.e. two linked circles and a continuum of nested toroidal shells. These vector fields belong to one of the homotopy classes H_+ or H_- . Also in these classes, there are vector fields which have two periodic solutions (T_0 and T_1), which are transverse to each other T_r , and which have the property that every other orbit has its α -limit at T_0 and its ω -limit at T_1 . We shall refer to these as the Max-Min vector fields. The homotopy class (H_+ or H_-) is determined by the linking number of the periodic solutions (H_+ is the class where the linking number is positive).

We shall now show that there are Morse-Smale vector fields in these classes for which there is no linking of the periodic solutions. Let K_0 and K_1 denote the solid tori with boundary $T_{1/2}$ which are tubular neighborhoods of T_0 and T_1 , respectively. Then K_0 (with the flow reversed) and K_1

contain flows similar to the flow which was the basis for the construction in [9]. In that paper, our goal was to modify this flow by a plug construction so that the resulting almost Morse-Smale vector field was not near to any Morse-Smale vector field. On the other hand we observed that by a different placement of the plugs we would have obtained an almost Morse-Smale vector field with the property that every nearby generic vector field is Morse-Smale (cf. the comments which conclude [8]). These Morse-Smale vector fields have the property that they have eight periodic solutions, two sources, four saddles, and two sinks, and that each of these periodic solutions bounds a small disk which is pierced by no periodic solution, i.e. these periodic solutions have linking number zero with every other periodic solution of the vector field. In separate oral communications, both Danial Asimov and John Franks have pointed out to me that examples of this kind can be constructed which have just two periodic solutions, one saddle and one source or one sink. Their constructions are very geometric. After some reflection on plug constructions [8] and [13] and their use in deriving Fuller's example [4], it became clear that the plug approach can also be used to sever links between certain periodic solutions.

LEMMA 2.1. — *Let K be a solid torus in M and let ξ be a Morse-Smale vector field on M which is nonsingular on K , which is transverse to the boundary of K , and which has a single periodic solution in K which attracts K in positive time if ξ is directed into K or attracts K in negative time if ξ is directed out of K . Then there is a Morse-Smale vector field ζ on M which has the following properties.*

1. ζ coincides with ξ on $M-K$ and on a neighborhood of ∂K ,
2. ζ has two periodic solutions, one saddle and either one sink (if ζ is directed into K) or one source (if ζ is directed out of K),
3. No periodic solutions of ζ links either of the periodic solutions of ζ which lie in K .

Proof. — In [8; Section 1], it is shown how Fuller's example with exactly one periodic solution can be obtained by using a

special plug construction where the plug is the three dimensional analogue of the flow indicated in Figure 1.a. Here, the mirror-image property is sacrificed but the strong contraction property of the attracting periodic solution is used to prevent unwanted recurrence from being introduced during the plug construction. For our present purposes, the same approach can be used, except that we use the Morse-Smale plug of Figure 1.b. Again, the strong contraction of the periodic solution of ξ is sufficient to compensate for the fact that the plug which we are using does not satisfy the mirror-image property. This situation is illustrated in Figure 1.c. In order to insure that ζ is a nonsingular vector field, we must be careful to insert the plug so that the invariant manifolds of the saddle orbit are transverse to the invariant manifold of the other critical elements of ζ . This can be arranged by an easy transversality argument along ∂K . Finally, to see that the two periodic solutions of ζ in K cannot link any other periodic solution of ζ , we observe that every trajectory which crosses ∂K has its ω -limit in K if ζ is directed into K or else has its α -limit in K if ζ is directed out of K . In either case, no periodic solution of ζ can cross ∂K . But by the plug

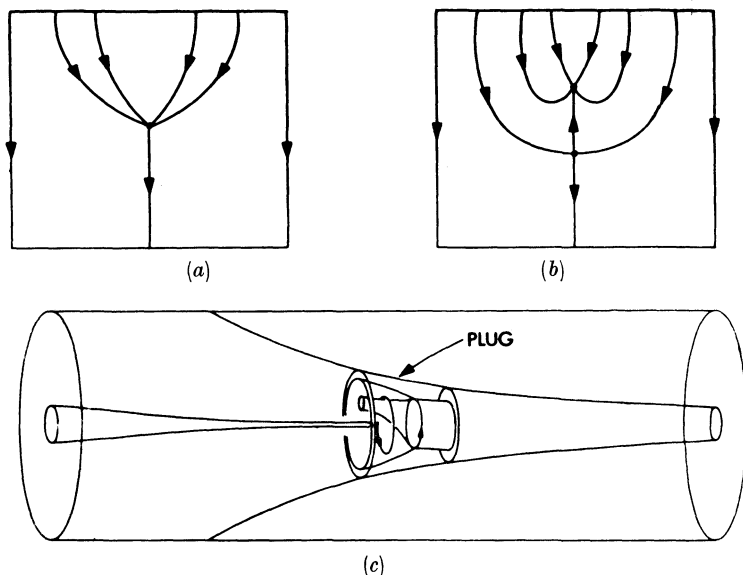


Fig. 1. — A special simple plug construction.

construction, the periodic solutions of ζ in K bound disjoint disks which lie in K ; hence no linking is possible.

Applying the constructions of this lemma to a Max-Min vector field, first in K_0 with $W = \emptyset$, and then in K_1 with W taken as the union of the intersections of the 2-dimensional invariant manifolds from K_0 with $\partial K_0 = \partial K_1$, we obtain a Morse-Smale vector field on S^3 which is homotopic to the Max-Min vector field (H_+ or H_-) and which has four periodic solutions which bound four mutually disjoint 2-disks, i.e. the periodic solutions are unknotted and unlinked. The sum of the indices of these periodic solutions is zero. In this case, there is no local property of the set of periodic solutions which reveals whether a particular example belongs to H_+ or H_- . However, in this case we could determine the homotopy class by combining our knowledge of the type of example which we have in hand with certain observations about the behavior of the 2-dimensional invariant manifolds. While this result seems to raise as many questions as it answers, we shall not pursue the matter at this time.

3. Examples from other homotopy classes.

We shall now show how the examples in section 1 can be used to obtain nonsingular Morse-Smale vector fields in each homotopy class of nonsingular vector field on S^3 which is represented by the vector fields $\tilde{\zeta}_{p,q}$. By lemma 1.4,

$$I(\mathcal{F}, T_0) = 1 + pk \quad \text{and} \quad I(\mathcal{F}, T_1) = 1 - qk$$

where the integer k is determined by $\tilde{\zeta}$. Observe that if $k = 0$, then these indices are both one. Indeed, this case would arise if we chose $\tilde{\zeta}$ to be the gradient flow on the standard 2-sphere, and then $\tilde{\zeta}_{p,q}$ is a Max-Min vector field on S^3 . This reflects the fact that the p -fold covering of an asymptotically stable singular point is still asymptotically stable. Since this is the only case in which index is preserved, we see that if we want to have the vector field $\tilde{\zeta}_{p,q}$ to be Morse-Smale, we had better choose $\tilde{\zeta}$ to be asymptotically stable

(\pm time) at the poles. We can achieve this with the following bifurcations.

Now we note that if $\tilde{\zeta}$ is a vector field whose singularities are generic and have the bifurcated form illustrated in Figure 2,

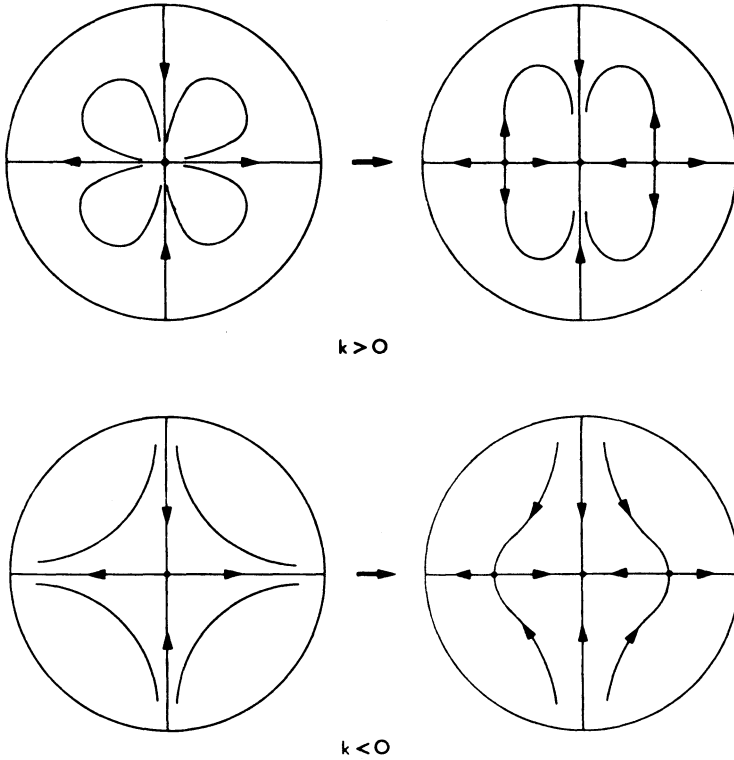


Fig. 2. — Bifurcations of the singularities of $\tilde{\zeta}$ at the poles.

then $\tilde{\zeta}_{p,q}$ will be Morse-Smale. For all of the saddles are near T_0 or T_1 , and there are no saddle connections for $\tilde{\zeta}$ for the induced singular foliation. Thus we have proved.

THEOREM 3.1. — *There is a nonsingular Morse-Smale vector field in each homotopy class of nonsingular vector fields on S^3 except for the class immediately above H_+ and the class immediately below H_- .*

Since we have hope that the understanding of these and other examples may lead to the understanding of a relations-

hip between the algebraic properties of the periodic solutions of a nonsingular Morse-Smale vector field and its homotopy class, we shall take the time to describe explicitly the parameters which describe the periodic solutions of these examples.

If a Morse-Smale vector field $\tilde{\zeta}_{p,q}$ has been constructed on S^3 by using a vector field ζ on S^2 which has index $1+k$ at the South Pole and $pq > 0$, then the homotopy class of $\tilde{\zeta}_{p,q}$ is described by

$$d(\tilde{\zeta}_{p,q}, H_+) = -(1 + pk)(1 - qk).$$

The vector field $\tilde{\zeta}_{p,q}$ has $2|k| + 2$ periodic solutions: T_0 and T_1 are sinks and are directed so that they have linking number -1 ; there are $|k|$ saddles and $|k|$ sources each of which lies on some T_r with type (p, q) ; if $k > 0$, then the saddles are near T_0 and the sources are near T_1 ; otherwise conversely. Since this example arises as a lifting from S^2 , it should not be surprising that the sum of the indices of the periodic solutions is $\chi(S^2) = 2$.

When we observe the vector field $\tilde{\zeta}_{p,q}$ on S^3 we can count the number $|k|$ of saddle orbits and since the saddles have linking number p with T_0 and q with T_1 , we can retrieve the numbers p and q (the other periodic solutions have linking numbers pq with each other). This is almost enough information to determine the homotopy class. On the other hand, we can alter $\tilde{\zeta}_{p,q}$ by applying Lemma 2.1 along each sink and source of $\tilde{\zeta}_{p,q}$. Let $\zeta_{p,q}^*$ denote this new vector field. It is Morse-Smale and has $|k| + 2$ sinks and sources, and $2|k| + 2$ saddles. All of the sinks and sources and $|k| + 2$ of the saddles bound disjoint disks, and the indices of all of this class of periodic solutions add to zero. The other $|k|$ saddles are the same as the saddles of $\tilde{\zeta}_{p,q}$. They link each other with linking number pq . Also, the sum of the indices of the periodic solutions of $\zeta_{p,q}^*$ is $-|k|$. It is not obvious how to view this data so that the homotopy class of $\tilde{\zeta}_{p,q}$ can be determined. It seems that if there is a solution to this problem, then some properties of the set which is the union of all of the 2-dimensional invariant manifolds will also have to be taken into consideration.

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Manuscrit reçu le 11 février 1976

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