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LOCAL STRUCTURAL STABILITY OF C^2 INTEGRABLE 1-FORMS

by Alcides LINS Neto

In this paper we define a class of $C^r (r \geqslant 2)$ locally structurally stable integrable 1-forms with singularities. The main idea is to consider integrable 1-forms on $\mathbf{R}^n$ with singularities such that the 2-jet of the form in the singularities satisfy a « hyperbolicity condition » to be defined in § 1. With this condition we show in theorems A, B and E that the foliation induced by the form in a neighborhood of the singularities is topologically equivalent to a foliation induced by a hyperbolic linear action of $\mathbf{R}^{n-1}$ on $\mathbf{R}^n$. In theorem C we show that the set of singularities of the form is a cell complex which is stable if we impose transversality conditions. In theorem D we show that the foliation induced by the form in a neighborhood of the singularities is locally like a product of a singular codimension one foliation in $\mathbf{R}^3$ by codimension three planes in $\mathbf{R}^n$. In § 1 we give the definitions, state the results and give some examples. In § 2 we prove the results.

I wish to thank specially A. S. Medeiros and C. Camacho for helpful conversations and ideas.

1. Definitions and results.

1.1. *General definitions.*

Let M be a C^∞ manifold of dimension n. We shall denote the set of C^r k-forms on M by $\Lambda^{k,r}(M)$ and if $k = 1$, $\Lambda^{1,r}(M) = \Lambda^r(M)$. A 1-form on M will be integrable if $\omega \wedge d\omega = 0$. The set of integrable C^r 1-forms on M will be denoted by $\vartheta^r(M)$. In $\Lambda^r(M)$ we shall consider the

Whitney's C^r topology and in $\vartheta^r(M)$, the induced topology. If $\alpha \in \Lambda^{k,r}(M)$ we set $\operatorname{sing}(\alpha) = \{p \in M | \alpha_p = 0\}$. A point $p \in \operatorname{sing}(\alpha)$ is called a singularity of α. Frobenius' theorem implies that $\omega \in \vartheta^r(M)$ defines a codimension one foliation on M-sing (ω), which will be denoted by $\mathcal{F}(\omega)$.

1.1.1. DÉFINITION. — *Let* $\omega \in \vartheta^r(M)$, $\tilde{\omega} \in \vartheta^r(\tilde{M})$. *We say that* ω *and* $\tilde{\omega}$ *are topologically equivalent if there exists a homeomorphism* $h: M \to \tilde{M}$ *such that* $h(\operatorname{sing}(\omega)) = \operatorname{sing}(\tilde{\omega})$ *and* h *sends leaves of* $\mathcal{F}(\omega)$ *onto leaves of* $\mathcal{F}(\tilde{\omega})$. *If* $p \in M$, $q \in \tilde{M}$, *we say that* ω *and* $\tilde{\omega}$ *are locally equivalent at* p *and* q, *if there exist neighborhoods* U *of* p *and* V *of* q, *such that the restrictions* ω/U *and* $\tilde{\omega}/V$ *are topologically equivalent.*

1.1.2. DÉFINITION. — *We say that* $\omega \in \vartheta^r(M)$ *is structurally stable if there exists a neighborhood* υ *of* ω *in* $\vartheta^r(M)$ *such that for all* $\tilde{\omega} \in \upsilon$, ω *and* $\tilde{\omega}$ *are topologically equivalent. We say that* ω *is locally structurally stable at* $p \in M$ *if for each neighborhood* V *of* p, *there exists a neighborhood* υ *of* ω *in* $\vartheta^r(M)$, *such that if* $\tilde{\omega} \in \upsilon$ *there exists* $\tilde{p} \in V$, *such that* ω *and* $\tilde{\omega}$ *are locally equivalent at* p *and* $\tilde{p}$.

1.2. *Some known results.*

Singularities of integrable 1-forms were considered by Reeb in [1]. In his work Reeb showed that an integrable 1-form with non degenerate linear part of the type $\sum_{i=1}^{n} x_i dx_i$ is locally equivalent to the linear part. Furthermore he showed that in the case that the form is analytic it is sufficient that the linear part be non degenerate, for the local equivalence. Kupka in [2] considered this problem from the structural stability point of view. In this paper he gave some necessary conditions for C^1 structural stability. In [3] Medeiros extends the results of Reeb to the case in which the form is C^1 and the linear part of ω is of the type $\sum_{i=1}^{n} \varepsilon_i x_i \, dx_i$ $(\varepsilon_1 = \pm 1)$ and the number of $\varepsilon_{i's}$ with minus and plus sign is not two. Furthermore he considered the case $\omega_p = 0$ but $d\omega_p \neq 0$ and in this case he showed that singular foliation induced by ω

is locally equivalent to the product of a singular codimension 1 foliation on $\mathbf{R}^2$ by codimension 2 planes in $\mathbf{R}^n$ (see the picture below).

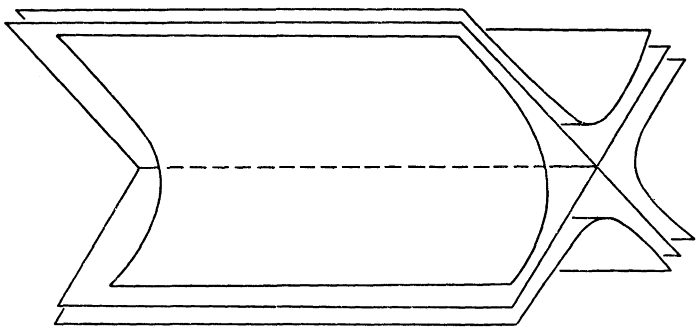

Fig. 1.

In this paper we analyze the case in which the linear part is zero but the two jet of the form in the singularity is not zero.

1.3. *The results.*

Let $\omega \in \vartheta^2(V)$, where V is an open set of $\mathbf{R}^n$. If

$$j^1(\omega)_p = 0 \quad \text{and} \quad j^2(\omega)_p = q,$$

then q is a 1-form with coefficients homogeneous of degree two and $q \wedge dq = 0$. Let $Q(n) = \{q \in \vartheta^2(\mathbf{R}^n) | q = \sum_{i=1}^{n} q_i\, dx_i$, where q_i is a homogeneous polynomial of degree two$\}$. If $\omega \in \Lambda^1(\mathbf{R}^3)$, we define rot (ω) to be the unique vector field X in $\mathbf{R}^3$ such that $d\omega = i_X\,(dx_1 \wedge dx_2 \wedge dx_3)$, where for a k-form η in $\mathbf{R}^n$, $i_X(\eta)$ is the $(k-1)$-form such that $i_X(\eta)(v_1, \ldots, v_{k-1}) = \eta(X, v_1, \ldots, v_{k-1})$. If $q \in Q(3)$ then rot (q) is a linear vector field in $\mathbf{R}^2$.

1.3.1. Définition. — *Let* $q \in Q(n)$, $n \geqslant 3$. *We say that* q *is simple if there exists a* 3*-plane* $\pi \subset \mathbf{R}^n$ *such that* rot (q/π) *is a hyperbolic vector field in* π, *where* q/π *is the restriction of* q *to* π. *If* $\omega \in \vartheta^2(M)$ *and* $p \in M$ *is such that*

$$j^2(\omega)_p = q + df,$$

where q *is simple, then we say that* p *is a simple point of* ω

We shall see below in 2.2.2 that a simple point of ω is a singularity of ω, therefore in this case we shall say that p is a simple singularity of ω. Observe that if $n = 3$, rot (ω) depends of a volume form on M, but the fact that p is a hyperbolic singularity does not depend. If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the linear part of rot (ω) in p, then

$$\lambda_1 + \lambda_2 + \lambda_3 = 0.$$

This is a consequence of the fact that rot (ω) preserves volume.

1.3.2. PROPOSITION. — *Let* $q \in Q(n)$, $n \geqslant 3$, *be simple. Then there exists a linear isomorphism* A *of* $\mathbf{R}^n$ *such that* $A^*(q)$ *has one of the following forms:*

i) $A^*(q) = ax_2x_3\,dx_1 + bx_1x_3\,dx_2 + cx_1x_2\,dx_3$.

ii) $A^*(q) = (ax_1 + bx_2)x_3\,dx_1 + (-\,bx_1 + ax_2)x_3\,dx_2 + c(x_1^2 + x_2^2)\,dx_3$

iii) $A^*(q) = (ax_1 + bx_2)x_3\,dx_1 - bx_1x_3\,dx_2 + cx_1^2\,dx_3$.

Case iii) *occurs only when the eigenvalues of* rot (q/π) *are of the form* $\lambda, \lambda, -2\lambda (\lambda \neq 0)$, *where* $\pi \subset \mathbf{R}^n$ *is like in* 1.3.1.

1.3.3. *Remark.* — Let $S = S(n) \subset Q(n)$ be the set of simple 1-forms in $Q(n)$. Then S is open (but not dense) in $Q(n)$. We remark that if $q \in S$, then it is not difficult to see that the three canonical types i), ii) and iii) can be obtained in the following way: Take two linear comutative vector fields in $\mathbf{R}^3$, say X and Y. Let $Z = X \times Y$, where $\times$ denotes the cross product in $\mathbf{R}^3$. If $Z = \sum_{i=1}^{3} Z_i \frac{\partial}{\partial x_i}$, we take

$$q(Z) = \sum_{i=1}^{3} Z_i\,dx_i.$$

It is not difficult to see that the map $(X, Y) \to q(X \times Y) \in S$ is surjective, therefore the singular foliation induced by $q \in S$ can be obtained as a foliation induced by an action of $\mathbf{R}^{n-1}$ in $\mathbf{R}^n$ such that two of the generators are linear and the others are constant.

1.3.4. *Remark.* — We shall prove in § 2 that cases i) with a, b, $c \neq 0$ and ii) with b, $c \neq 0$ are locally stable. We remark here that case iii) is not locally stable. In this case

$$q = (ax_1 + bx_2)x_3\, dx_1 - bx_1x_3\, dx_2 + cx_1^2\, dx_3$$

and

$$\operatorname{sing}(q) = \{x \in \mathbf{R}^n | x_1 = x_2 = 0 \text{ or } x_1 = x_3 = 0\}.$$

If $\varepsilon > 0$, let $\tilde{q} = q + \alpha x_2 x_3\, dx_1 + \beta x_2^2\, dx_3$ where $\alpha c = \beta a$ and $|\alpha|, |\beta| < \varepsilon$. Then $\tilde{q} \in S$ and

$$\operatorname{sing}(\tilde{q}) = \{x | x_1 = x_2 = 0\}$$

if $\beta c > 0$ or $\operatorname{sing}(\tilde{q})$ is the union of three codimension 2 subspaces if $\beta c < 0$. Then case iii) is not locally stable.

1.3.5. Définition. — *Let* $q \in S(n)$, $n \geqslant 3$. *We say that* q *is hyperbolic if there exists an isomorphism* A *of* $\mathbf{R}^n$ *such that* $A^*(q)$ *is of type* i) (*of* 1.3.2) *with* a, b, $c \neq 0$ or of type ii) with b, $c \neq 0$. Let $\omega \in \vartheta^r(M)$, $r \geqslant 2$, $n \geqslant 3$. We say that $p \in M$ is a hyperbolic singularity of ω if $j^2(\omega)_p = q$ is hyperbolic.

We have the following results :

1.3.6. Theorem A. — *Let* $\omega \in \vartheta^2(M)$, dim (M) = 3. *Suppose that* $p \in M$ *is a hyperbolic singularity of* ω *and that*

$$j_2(\omega)_p = q.$$

Then ω *and* q *are locally equivalent at* p *and* 0 *respectively.*

1.3.7. Corollary B. — *If* $p \in M$, dim (M) = 3, *is a hyperbolic singularity of* $\omega \in \vartheta^2(M)$, *then* ω *is locally structurally stable at* p.

1.3.8. *Remark.* — In [4] C. Camacho proves the local structural stability of hyperbolic actions of $\mathbf{R}^2$ on $\mathbf{R}^3$. We remark that this result can be obtained « generically » in the C^2 case as a corollary of theorem A and corollary B, by using the construction of 1.3.3. Observe that if we apply the construction of 1.3.3 to the vector fields $X = (x_1, x_2, x_3)$ and

$$Y = (x_1, 2x_2, 3x_3)$$

we obtain the 1-form $\omega = x_2x_3\,dx_1 - 2x_1x_3\,dx_2 + x_1x_2\,dx_3$ which is not simple and not stable.

If Σ and M are manifolds we denote by $\xi^k(\Sigma, M)$ the set of C^k embeddings of Σ in M with the C^k Whitney's topology.

1.3.9. THEOREM C. — *Let* $\omega \in \vartheta^r(M)$, $\dim(M) = n$, $r \geqslant 2$, $n \geqslant 3$. *Suppose that* ω *is simple in the points of* sing $(d\omega)$. *Then* sing $(d\omega) \subset$ sing (ω) *and in fact* $j^1(\omega)_p = 0$ *if*

$$p \in \text{sing}\,(d\omega).$$

Furthermore sing $(d\omega)$ *is a* C^{r-1} *codimension three embedded submanifold of* M. *Now suppose that* sing $(d\omega)$ *intersects* ∂M *transversally and* $r \geqslant 3$. *In this case there exists a neighborhood* υ *of* ω *in* $\vartheta^r(M)$ *such that if* $\tilde\omega \in \upsilon$ *then* sing $(d\tilde\omega)$ *is diffeomorphic to* sing $(d\omega)$ *and it is possible to define a continuous map* $\xi : \upsilon \to \xi^{r-2}(\text{sing}\,(d\omega), M)$ *such that the image of* $\xi(\tilde\omega)$ *is* sing $(d\tilde\omega)$.

We say that a 1-form on $\mathbf{R}^n$ depends of p variables in a open set $U \subset \mathbf{R}^n$, if there exists a decomposition

$$\mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^{n-p}$$

such that $\omega/U = \sum_{i=1}^{p} \omega_i\,dx_i$, where $\omega_i : U \to R$ depends only of the variables $x_1, \ldots, x_p \in \mathbf{R}^p$, for $i = 1, \ldots, p$.

1.3.10. THEOREM D. — *Let* $\omega \in \vartheta^r(M)$, $\dim(M) = n \geqslant 4$, $r \geqslant 4$. *Suppose that* p *is a simple singularity of* ω. *Then there exist open sets* $0 \in U \subset \mathbf{R}^n$, $p \in V \subset M$ *and a* C^{r-3} *diffeomorphism* $\varphi : (U, O) \to (V, p)$ *such that* $\varphi^*(\omega) \in \vartheta^r(U)$ *and depends of three variables. In particular the foliation induced by* ω *in* V *is equivalent to the product of a singular foliation in* $\mathbf{R}^3$ *by a regular foliation of codimension three.*

1.3.11. *Remark.* — Let $\omega \in \vartheta^r(M)$, $\dim(M) \geqslant 4$, $r \geqslant 4$. Suppose that ω is hyperbolic in the points of sing $(d\omega)$. It follows from theorems A, C and D that there exists a neighborhood V of sing $(d\omega)$ such that sing $(\omega) \cap V$ is a cell complex with codimension 2 and 3 cells.

1.3.12. Corollary E. — *If* p *is a hyperbolic singularity of* $\omega \in \vartheta^4(M)$, *then* ω *is locally structurally stable at* p.

A global structural stability theorem for forms with singularities of the type above can be found in [10].

1.4. *Some problems.*

There are some problems and questions which arise naturally :

1. In which situation is the hyperbolicity condition necessary for local structural stability?

2. Let ω be the integrable 1-form in $\mathbf{R}^n$ defined by

$$\omega = \sum_{i=1}^{n} a_i x_1 \ldots x_{i-1} x_{i+1} \ldots x_n \, dx_i.$$

Is it locally stable for a dense set of $a_{i's}$?

3. Generalize the definitions and theorems for systems of integrable 1-forms or for k-forms.

4. Study k-parameter families of integrable 1-forms. In 1.3.4 we have an example of a 2-parameter family of 1-forms. Is it stable?

5. Does the space of germs in 0 of hyperbolic 1-forms have a structure of a Banach manifold? Notice that Medeiros has a proof that in the case $d\omega_0 \neq 0$ the answer is yes.

1.5. *Pictures.*

Here we sketch the pictures of the foliations induced by the forms i) and ii) of 1.3.2 in $\mathbf{R}^3$.

Case i. $\omega = ax_2x_3\, dx_1 + bx_1x_3\, dx_2 + cx_1x_2\, dx_3$ with

$$a, b, c \neq 0.$$

We have two cases : i.1) a, b, c have the same sign and i.2) a, b, c do not have the same sign. In the pictures below we sketch the pictures of the intersection of the foliations with a sphere. Some of these pictures can be found in [4].

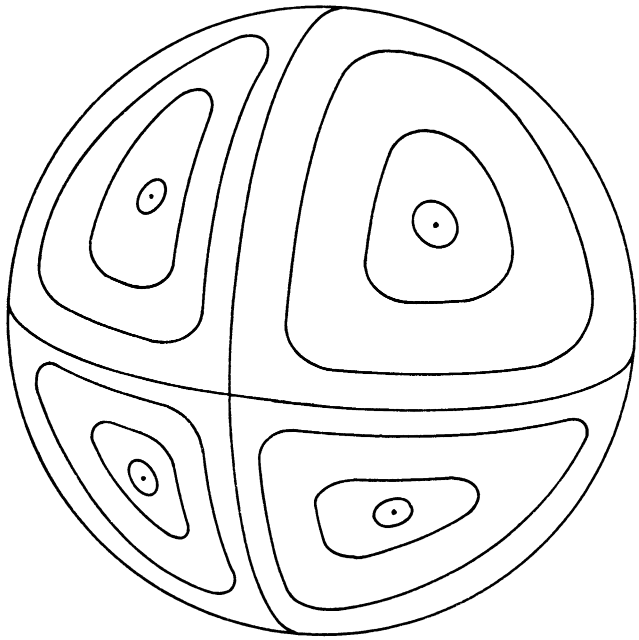

Fig. 2.1 (case i.1).

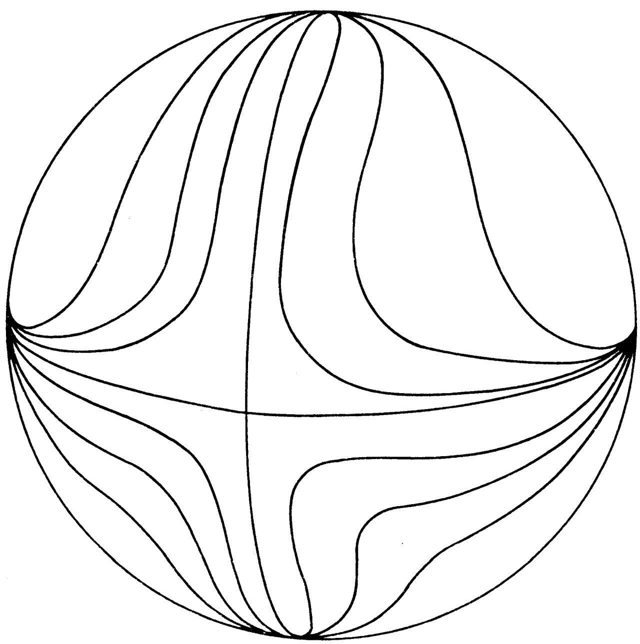

Fig. 2.2 (case i.2).

Case ii.

$\omega = (ax_1 + bx_2)x_3\, dx_1 + (-\, bx_1 + ax_2)x_3\, dx_2 + c(x_1^2 + x_2^2)\, dx_3$ with $b,\ c \neq 0$. We have three cases: ii.1) $a,\ c$ with the same sign, ii.2) $a > 0 > c$ or $c > 0 > a$, ii.3) $a = 0$. We

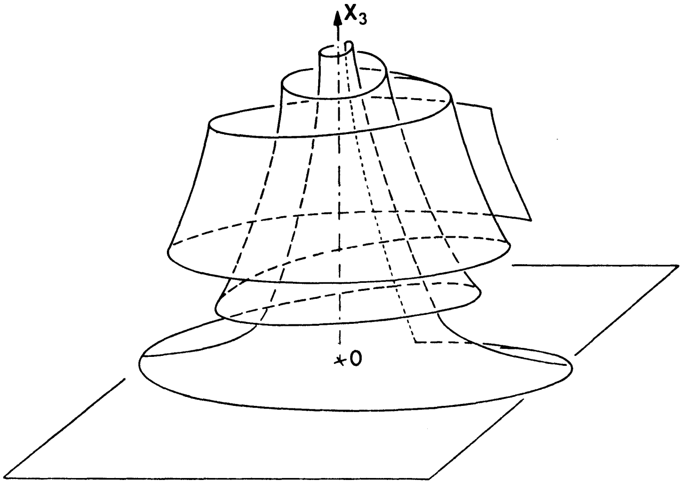

Fig. 3.1 (case ii.1).

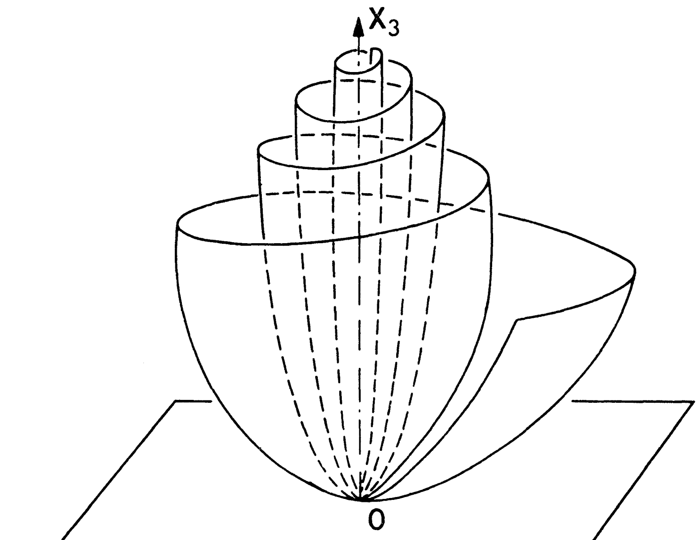

Fig. 3.2 (case ii.2).

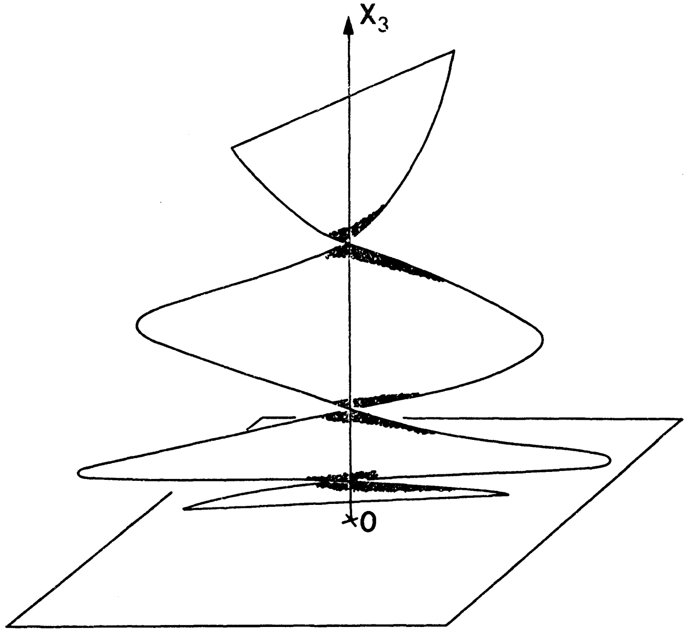

Fig. 3.3 (case ii.3).

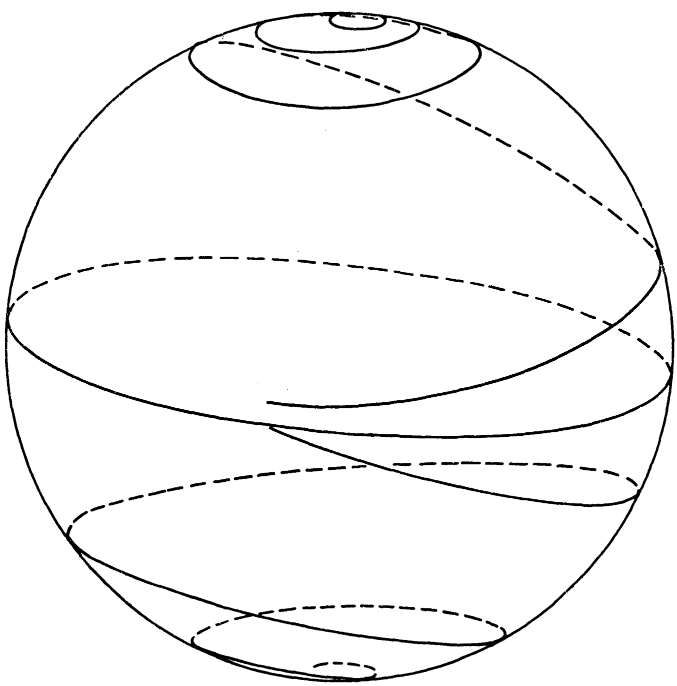

Fig. 3.4.

remark that all these cases are topologically equivalents as we shall see in the proof of corollary B. We sketch below some typical leaves.

2. Proof of the results.

2.1. *Proof of proposition* 1.3.2.

First case: $n = 3$.

By definition we have $dq = i_X(\Gamma)$ where

$$\Gamma = dx_1 \wedge dx_2 \wedge dx_3 \quad \text{and} \quad X = \text{rot}\,(q)$$

is linear and trace $(X) = 0$. Let A be a linear transformation of $\mathbf{R}^3$. Then

$$d(A^*q) = A^*\,(dq) = A^*(i_X(\Gamma)) = i_{A^*(X)}(A^*\Gamma) = \det\,(A) i_{A^*(X)}(\Gamma),$$

where $A^*(X) = A^{-1}(X.A)$. Let A be a linear isomorphism of $\mathbf{R}^3$ such that $\det\,(A) = 1$ and $A^*(X)$ is in Jordan's canonical form, with respect to the canonical base of $\mathbf{R}^3$. We have three possibilities :

a)

$$d(A^*q) = \lambda_1 x_1\, dx_1 \wedge dx_2 + \lambda_2 x_2\, dx_3 \wedge dx_1 + \lambda_3 x_3\, dx_1 \wedge dx_2$$

where $\lambda_1 + \lambda_2 + \lambda_3 = 0$.

b)

$$d(A^*q) = (\alpha x_1 + \beta x_2)\, dx_2 \wedge dx_3 + (-\,\beta x_1 + \alpha x_2)\, dx_3 \wedge dx_1 - 2\alpha x_3\, dx_1 \wedge dx_2$$

c)

$$d(A^*q) = \lambda x_1\, dx_2 \wedge dx_3 + (x_1 + \lambda x_2)\, dx_3 \wedge dx_1 - 2\lambda x_3\, dx_1 \wedge dx_2.$$

Now suppose we have *a*). The other cases are analogous. Let $\tilde{q} = \lambda_2 x_2 x_3\, dx_1 - \lambda_1 x_1 x_3\, dx_2$. We have $d\tilde{q} = d(A^*q)$, therefore $A^*q = \tilde{q} + df$, where $f\colon \mathbf{R}^3 \to \mathbf{R}$ is cubic. By the integrability condition we must have $d\tilde{q} \wedge df = 0$, or

$\sum_{i=1}^{3} \lambda_i x_i \frac{\partial f}{\partial x_i} = 0$. As f is homogeneous of degree three, it is not difficult to see that we have two possibilities.

a') $\lambda_i \neq \lambda_j$ if $i \neq j$. In this case $f = kx_1x_2x_3$ and

$$A^*(q) = ax_2x_3\, dx_1 + bx_1x_3\, dx_2 + cx_1x_3\, dx_3,$$

where $a = \lambda_2 + k$, $b = k - \lambda_1$, $c = k$.

a'') $\lambda_1 = \lambda_2 = \lambda$, $\lambda_3 = -2\lambda$. In this case

$$f = x_3(kx_1^2 + lx_1x_2 + mx_2^2)$$

and $\tilde{q} = \lambda x_3(x_2\, dx_1 - x_1\, dx_2)$. Observe that if B is any linear transformation such that $B(x_1, x_2, x_3) = (B_1(x_1, x_2), B_2(x_1, x_2), x_3)$, then $B^*(\tilde{q}) = \det(B)\tilde{q}$. Let B be such that $B^*(kx_1^2 + lx_1x_2 + mx_2^2) = k(x_1^2 + x_2^2)$ or kx_1x_2 or kx_1^2. Then we have $B^*A^*q = B^*\tilde{q} + d(B^*f)$ has one of the forms i), ii) or iii).

Second case: $n > 3$.

It is sufficient to show that q can be reduced to a 1-form depending of 3 variables. Let $q \in Q(n)$ and $\pi \subset \mathbf{R}^n$ be such that $\dim \pi = 3$ and ω/π is simple. Suppose that $(x_1, x_2, x_3, 0, \ldots, 0)$ is the parametrization of π. Let

$$q = \sum_{i=1}^{n} q_i\, dx_i, \quad dq = \sum_{i<j} \alpha_{ij}\, dx_i \wedge dx_j,$$

where $\alpha_{ij} = -\alpha_{ji} = \frac{\partial q_j}{\partial x_i} - \frac{\partial q_i}{\partial x_j}$. By the integrability condition we have $q \wedge dq = 0$, and then $dq \wedge dq = 0$ so that

$$(*) \qquad \alpha_{ij}\alpha_{kl} + \alpha_{il}\alpha_{jk} + \alpha_{ik}\alpha_{lj} = 0$$

Let $X: \mathbf{R}^n \to \mathbf{R}^3$ and $Y_k: \mathbf{R}^n \to \mathbf{R}^3$ $(k = 4, \ldots, n)$ be defined by $X = (\alpha_{23}, \alpha_{31}, \alpha_{12})$ and $Y_k = (\alpha_{1k}, \alpha_{2k}, \alpha_{3k})$. Observe that X and Y_k are linear and the condition « q/π is simple » means that the matrix $M = (\partial X_i/\partial x_j)_{1 \leq i, j \leq 3}$ is non singular. *Assertion:* sing $(dq) = \ker(X)$. Let $M_k = (\partial \alpha_{ik}/\partial x_j)_{1 \leq i, j \leq 3}$. By $(*)$ we have $X.Y_k = 0$ and by differentiation

$$M^t Y_k + M_k^t X = 0,$$

where M^t, M_k^t are the transposes of M and M_k. As M^t is non singular, $Y_k = -(M^t)^{-1}.M_k^t.X$ and then

$$\ker(Y_k) \supset \ker(X) \ (k = 4, \ldots, n).$$

Now it is sufficient to show that $\ker(\alpha_{jk}) \supset \ker(X)$ if $j, k \geqslant 4$. This is an immediate consequence of

$$\alpha_{12}\alpha_{jk} + \alpha_{1k}\alpha_{2j} + \alpha_{1j}\alpha_{k2} = 0$$

and $\ker(Y_k) \supset \ker(X)$ $(k = 4, \ldots, n)$. By the assertion $\operatorname{sing}(dq)$ is a codimension three sub-space of $\mathbf{R}^n$, transversal to π. Let $\tilde{\pi} = \{(x_1, \ldots, x_n) | x_1 = x_2 = x_3 = 0\}$ and A be a isomorphism of $\mathbf{R}^n$ such that

$$A(\pi) = \pi \quad \text{and} \quad A(\tilde{\pi}) = \operatorname{sing}(dq).$$

Assertion: $A^*(dq)$ *depends of three variables.* Let

$$A^*(dq) = \sum_{i<j} \beta_{ij}\, dx_i \wedge dx_j.$$

Then β_{ij} is linear and $\beta_{ij}(0, 0, 0, x_4, \ldots, x_n) = 0$. Let us show that $\beta_{ij} = 0$ if $j \geqslant 4$ and $1 \leqslant i \leqslant n$. As $A^*(dq)$ is exact we have $\dfrac{\partial \beta_{ij}}{\partial x_k} + \dfrac{\partial \beta_{ki}}{\partial x_j} + \dfrac{\partial \beta_{jk}}{\partial x_i} = 0$ and taking $k \geqslant 4$, we have $\dfrac{\partial \beta_{ij}}{\partial x_k} = 0$ therefore $\dfrac{\partial \beta_{ik}}{\partial x_j} = \dfrac{\partial \beta_{jk}}{\partial x_i}$. In particular if $j \geqslant 4$ and $i \leqslant 3$, $\dfrac{\partial \beta_{jk}}{\partial x_i} = 0$, therefore $\beta_{jk} = 0$ if $j, k \geqslant 4$. As $\dfrac{\partial \beta_{ik}}{\partial x_j} = \dfrac{\partial \beta_{jk}}{\partial x_i}$ $(k \geqslant 4)$, we have $\beta_j = df_j$, where

$$\beta_j = \sum_{i=1} \beta_{ij}\, dx_i.$$

By the condition $A^*(dq) \wedge A^*(dq) = 0$ we have

$$\beta \wedge df_j = 0,$$

where $\beta = \sum_{1 \leqslant i < j \leqslant 3} \beta_{ij}\, dx_i \wedge dx_j$, which implies that $\beta_j = 0$ if $j \geqslant 4$. Therefore $A^*(dq)$ depends of three variables (x_1, x_2, x_3).

We have to show now that $A^*(q)$ depends only of the variables x_1, x_2, x_3. As $A^*(dq)$ depends only of x_1, x_2, x_3, there exists an integrable 1-form $\bar{q} \in Q(n)$ depending only of the variables x_1, x_2, x_3, such that $d\bar{q} = A^*(dq)$ (take $\bar{q} = A^*q/\pi$). We have $A^*q = \bar{q} + df$, where f is homogeneous of degree three. By the integrability condition we have $df \wedge d\bar{q} = 0$, or

$$\beta_{jk} \frac{\partial f}{\partial x_i} + \beta_{ki} \frac{\partial f}{\partial x_j} + \beta_{ij} \frac{\partial f}{\partial x_k} = 0.$$

If $i, j \leqslant 3$ and $k \geqslant 4$, we have $\beta_{ij} \dfrac{\partial f}{\partial x_k} = 0$, where $\beta_{ij} \neq 0$, which implies that $\dfrac{\partial f}{\partial x_k} = 0$ for $k \geqslant 4$.

2.1.1. *Remark.* — Observe that in the above proof we use only that dq/π is hyperbolic and the relation $dq \wedge dq = 0$ to show that there exists a linear isomorphism A of $\mathbf{R}^n$ such that $A^*(dq)$ depends of three variables.

2.2. *Proof of Theorem* A.

As the theorem is local we shall consider $M = \mathbf{R}^3$ and $p = 0$. We need some lemmas.

2.2.1. Lemma. — *Let* $\omega \in \vartheta^1(M^n)$ *and suppose that the interior of* sing $(d\omega)$ *is empty. Let* X *be a vector field in* M *such that* $i_X(d\omega) = 0$. *Then* $i_X(\omega) = 0$ *and the Lie derivative* $L_X(\omega) = 0$. *In particular if* $p \in M$ *and* γ *is the orbit of* X *by* p *then* $\gamma \subset$ sing $(d\omega)$, $\gamma \subset$ sing (ω) *or* $\gamma \subset L$, *if*

$$p \in \text{sing}\,(d\omega), \quad \text{sing}\,(\omega)$$

or L *respectively, where* L *is the leaf of* $\mathcal{F}(\omega)$ *by* p, *if* $p \notin$ sing (ω). *If* $n = 3$ $M = \mathbf{R}^3$ *and* $X = \text{rot}\,(\omega)$ *then*

$$i_X(d\omega) = 0.$$

Proof. — Let $p \in M -$ sing $(d\omega)$. Then

$$0 = (i_X(\omega \wedge d\omega))_p = \omega_p(X(p)) \,.\, d\omega_p - \omega_p \wedge (i_X(d\omega))_p$$
$$= \omega_p(X(p))\, d\omega_p.$$

But $d\omega_p \neq 0$, then $(i_X(\omega))_p = 0$. As the interior of sing $(d\omega)$

is empty, $i_X(\omega) = 0$ in M. We have

$$L_X(\omega) = i_X(d\omega) + d(i_X(\omega)),$$

therefore $L_X(\omega) = 0$. Now suppose $p \notin \operatorname{sing}(\omega)$. By the condition $i_X(\omega) = 0$, the orbit γ of X by p is contained in the leaf by p. Now suppose $p \in \operatorname{sing}(\omega)$ and let X_t be the local flow of X. By the condition $L_X(\omega) = 0$, we have $\frac{d}{dt}(X_t^*\omega)_{t=0} = 0$, therefore $\omega_p(v) = \omega_{X_t(p)}(DX_t(p).v) = 0$ if $v \in TM_p$, so that the trajectory of X by p is contained in $\operatorname{sing}(\omega)$. The proof for $p \in \operatorname{sing}(d\omega)$ is analogous. If $n = 3$, $M = \mathbf{R}^3$, then $\operatorname{rot}(\omega)$ is the unique vector field X in $\mathbf{R}^3$ such that $i_X(dx_1 \wedge dx_2 \wedge dx_3) = d\omega$ and of course

$$i_X(d\omega) = 0.$$

2.2.2. Lemma. — *Let* $\omega \in \vartheta^2(\mathbf{R}^3)$ *and suppose that* $p \in \mathbf{R}^3$ *is a simple point of* ω. *Let* $X = \operatorname{rot}\omega$ *and* W_p *be the stable or unstable manifold of* X *in* p. *Then* $j^1(\omega)_p = 0$ *and* W_p *is the union of leaves of* $\mathcal{F}(\omega)$ *and singularities of* ω. *Furthermore if* $\dim(W_p) = 1$, *then* $W_p \subset \operatorname{sing}(\omega)$.

Proof. — Suppose $W_p = W_p^s$ the stable manifold of X (the other case is analogous). Let $q \in W_p^s$ and $v \in T_q(W_p^s)$. If X_t is the flow of X, we have

$$\omega_p(v) = \omega_{X_t(q)}(DX_t(q).v), \; t \in [0, \infty).$$

But $v \in T_q(W_p^s)$ therefore $\lim_{t\to\infty} DX_t(q).v = 0$, which implies that $\omega_q(v) = 0$. This proves that W_p^s and W_p^u are the union of leaves and singularities of ω. Since

$$T_p(W_p^s) \oplus T_p(W_p^u) = T_p(\mathbf{R}^3), \text{ then } \omega_p = 0.$$

Let us show that $j^1(\omega)_p = 0$. We can suppose $p = 0$ and $j^2(\omega)_0 = df + q$ where f and the coefficients of q are quadratic. The integrability condition implies that

$$0 = j^2(\omega \wedge d\omega)_0 = df \wedge dq,$$

or $\alpha_1 \frac{\partial f}{\partial x_1} + \alpha_2 \frac{\partial f}{\partial x_2} + \alpha_3 \frac{\partial f}{\partial x_3} = 0$, where $\alpha_1, \alpha_2, \alpha_3$ are the components of rot (q), which is linear hyperbolic. Let A be the jacobian matrix of rot (q) and B be the jacobian matrix of grad $(f) = (\partial f/\partial x_1, \partial f/\partial x_2, \partial f/\partial x_3)$. By the above relation we have $AB + BA = 0$, where B is symmetric and A is non-singular and has trace zero. This implies that $B = 0$, therefore $df = 0$ and $j^1(\omega)_0 = 0$.

It remains to show that if $\dim(W_p^s) = 1$ then

$$W_p^s \subset \text{sing}(\omega).$$

Let $q \in W_p^s$, $\nu \in T_q(\mathbf{R}^3) - T_q(W_p^s)$. We must show that $\omega_q(\nu) = 0$. We have for

$$t \geqslant 0, \qquad |\omega_q(\nu)| = |\omega_{X_t(q)}(DX_t(q).\nu)| \leqslant \|\omega_{X_t(q)}\| \, \|DX_t(q).\nu\|.$$

Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of rot (q) and $\beta_i = \text{Re}(\lambda_i)$. We have $\sum_{i=1}^{3} \beta_i = 0$ and $\beta_i \neq 0$, $i = 1, 2, 3$, therefore we can suppose that $\beta_3 < 0$, $\beta_1, \beta_2 > 0$ and

$$\max\{\beta_1, \beta_2\} = \rho < |\beta_3|.$$

Let $2|\beta_3| - \rho > 3\varepsilon$, $\varepsilon > 0$. Then using Gronwall's inequality (cf. [9], pg. 243, thms. 6.1 and 6.2) and $j^1(\omega)_0 = 0$ we have

$$\|X_t(q)\| \leqslant C_1 e^{-(|\beta_3|-\varepsilon)t}, \quad \|\omega_{X_t(q)}\| \leqslant C_3 e^{-(2|\beta_3|-\varepsilon)t}$$

and $\|DX_t(q).\nu\| \leqslant C_2 e^{(\rho+\varepsilon)t}$, therefore

$$|\omega_q(\nu)| \leqslant C e^{(-2|\beta_3|+\rho+3\varepsilon)t} = C e^{-\alpha t}, \quad \alpha > 0,$$

which implies that $\omega_q(\nu) = 0$.

2.2.3. *Remark.* — Let $p = 0$ be a simple singularity of ω and $S_\rho^2 = \{x \in \mathbf{R}^3 | \, \|x\|^2 = \rho^2\}$. In S_ρ^2 we take coordinates $x_i = \rho u_i$, where $u = (u_1, u_2, u_3) \in S_1^2$. Define the « intersection » of ω with S_ρ^2 as the cross product

$$Y_\rho(x) = u \times \text{grad}(\omega)_x,$$

where $u = \dfrac{x}{\|x\|}$ and grad $(\omega) = (\omega_1, \omega_2, \omega_3)$, $\omega = \sum_{i=1}^{3} \omega_i \, dx_i$. It is obvious that Y_ρ is tangent to both $\mathcal{F}(\omega)$ and S_ρ^2 and $Y_\rho(x) = 0$ iff $x \in \text{sing}(\omega) \cap S_\rho^2$, or S_ρ^2 is tangent to the leaf of $\mathcal{F}(\omega)$

through x. By projection, we can consider Y_ρ as a vector field in S_1^2. Now $j^1(\omega)_0 = 0$, then $\omega = q + R$, where $q = \sum_{i=1}^{3} q_i \, dx_i \in Q(3)$, $R = \sum_{i=1}^{3} R_i \, dx_i$ and $\lim_{t \to 0} \frac{R_i(x)}{\|x\|^2} = 0$. Therefore $Y_\rho = u \times \text{grad}\,(\omega) = \rho^2(Z_0 + R_\rho) = \rho^2 Z_\rho$ where

$$Z_0(u) = (u_2 q_3(u) - u_3 q_2(u), \quad u_3 q_1(u) - u_1 q_3(u), \\ u_1 q_2(u) - u_2 q_1(u))$$

and $R_\rho(u) = \frac{1}{\rho_2} u \times (R_1(\rho u), \; R_2(\rho u), \; R_3(\rho u))$, $u \in S_1^2$. We call Z_ρ the « blowing up » of the intersection. Now let $\mathfrak{r}^1(S_1^2)$ be the Banach space of C^1 vector fields in S_1^2 with the uniform C^1-topology.

Assertion. — $\lim_{\rho \to 0} Z_\rho = Z_0$ in the C^1 topology.

Proof. — It is sufficient to show that

$$\lim_{\rho \to 0} \frac{1}{\rho^2} R_i(\rho u) = 0$$

and

$$\lim_{\rho \to 0} \frac{\partial}{\partial u_j} \left(\frac{1}{\rho^2} R_i(\rho u) \right) = 0, \quad 1 \leqslant i, \; j \leqslant 3.$$

The first is only a consequence of the fact that $j^1(\omega)_0 = 0$. For the second we have $\frac{\partial}{\partial u_j} \left(\frac{1}{\rho^2} R_i(\rho u) \right) = \frac{1}{\rho} \frac{\partial R_i}{\partial x_j}(\rho u)$ and as ω is C^2 we have $\lim_{\rho \to 0} \frac{1}{\rho} \frac{\partial R_i}{\partial x_j}(\rho u) = 0$, uniformly in u.

2.2.4. *Remark.* — Suppose now that 0 is a hyperbolic singularity of ω. By 2.2.2 we can suppose that

$$j^2(\omega)_0 = q \in Q(3)$$

and by proposition 1.3.2 that q has one of the two forms i) or ii) of 1.3.2. Let us analyze Z_0 in these two cases.

Case i: $q = a x_2 x_3 \, dx_1 + b x_1 x_3 \, dx_2 + c x_1 x_2 \, dx_3$; $a, b, c \neq 0$. In this case $Z_0 = (u_1(b u_3^2 - c u_2^2), \; u_2(c u_1^2 - a x_3^2), \; u_3(a x_2^2 - b x_1^2))$

and we have two sub-cases:

i′) a, b, c have the same sign.

i″) a, b, c do not have the same sign.

Case i′): We can suppose $a, b, c > 0$. In this case Z_0 has 14 singularities: 8 centers, corresponding to tangencies of $\mathscr{F}(q)$ with S_1^2 and 6 sadles corresponding sing $(q) \cap S_1^2$. The phase space is like in picture 2.1. In this case Z_0 is not structurally stable.

Case i″): We can suppose $b > c > 0 > a$. In this case Z_0 has 6 hyperbolic singularities corresponding to sing $(q) \cap S_1^2$. These singularities are 2 sadles, 2 sinks and 2 sources. In fact it is not difficult to see that Z_0, in this case, is a Morse-Smale vector field in S_1^2 (cf. [5]) and its phase space is like in picture 2.2.

Case ii):

$q = (ax_1 + bx_2)x_3\, dx_1 + (-\, bx_1 + ax_2)x_3\, dx_2 + c(x_1^2 + x_2^2)\, dx_3$ where $b, c \neq 0$. In this case

$$Z_0 = (u_3^2(au_2 - bu_1) - cu_2(u_1^2 + u_2^2),$$
$$u_3^2(-\, au_1 - bu_2) + cu_2(u_1^2 + u_2^2), \quad bu_3(1 - u_3^2))$$

and it is not difficult to see that the non-wandering set of Z_0 is the union of two hyperbolic singularities (which are sinks or sources) and one hyperbolic closed trajectory. The phase portrait of Z_0 is like in pictures 3.4. Observe that Z_0 is Morse-Smale in all cases. By 2.2.3 and 2.2.4 we can conclude the following facts:

1) If ρ is small, then Z_0 and Z_ρ have the same number of singularities. This fact implies that the restrictions ω/S_ρ^2 and q/S_ρ^2 have the same number of singularities. Furthermore $\mathscr{F}(\omega)$ and $\mathscr{F}(q)$ have the same number of tangencies with S_ρ^2 and sing $(\omega) \cap S_\rho^2$, sing $(q) \cap S_\rho^2$ have the same number of points.

2) For cases i″) and ii) Z_0 is topologically equivalent to Z_ρ, if ρ is small, which means that the restrictions of ω and q to S_ρ^2 are topologically equivalent.

Now let $\omega \in \vartheta^2(\mathbf{R}^3)$ and 0 be a hyperbolic singularity of ω.

Let $j^2(\omega)_0 = q \in Q(3)$, $X = \operatorname{rot}(\omega)$, $L = \operatorname{rot}(q)$. Suppose that $\dim W_0^s(L) = 2$ and $W_0^s(L) = \{x \in \mathbf{R}^3 | x_3 = 0\}$. Let $U_{\rho,\varepsilon}$ be the cilinder $\{x \in S_\rho^2 | \, |x_3| \leqslant \varepsilon\}$ and $U_{\rho,\varepsilon}(X)$, $U_{\rho,\varepsilon}(L)$ be the saturated sets of $U_{\rho,\varepsilon}$ by X and L restricted to $B_\rho = \{x \in \mathbf{R}^3 | \, \|x\| \leqslant \rho\}$.

2.2.5. Lemma. — *There exist* $\rho_0, \varepsilon_0 > 0$ *such that:*

a) X *and* L *are transversal to* $U_0 = U_{\rho_0,\varepsilon_0}$ *and*

$$U_0(X) \cup W_0^u(X), \quad U_0(L) \cup W_0^u(L)$$

contain neighborhood V_X *and* V_L *of the origin, where* V_X, $V_L \supset U_0$.

b) *The restrictions of* ω *and* q *to* U_0 *are topologically equivalent.*

c) *If* l *is a leaf of the restriction of* ω *to* V_X, *then* $l \cap U_0$ *has only one connected component. The same is true for the leaves of* q.

Proof. — From the theory of invariant manifolds (cf. [8]), if ρ is small, the intersection of the stable manifold of X at 0 with $U_{\rho,\varepsilon}$ is a closed curve and X restricted to

$$W_0^s(X) \cap U_{\rho,\varepsilon}$$

is transversal to $U_{\rho,\varepsilon}$. If we take ε small enough then X is transversal to $U_{\rho,\varepsilon}$ and the same is true for L. By the λ-lemma (cf. [5]), if $U_0 = U_{\rho,\varepsilon}$ then $U_{\rho,\varepsilon}(X) \cup W_0^u(X)$ contains a neighborhood of the origin $V_X \supset U_0$. Let us prove *b*). As the restriction of ω and q to S_ρ^2 are topologically equivalent to Z_ρ and Z_0 respectively, it is enough to show that Z_0 and Z_ρ are topologically equivalent in a neighborhood $U_\delta = U_{1,\delta}$ of $\{x_3 = 0\}$ in S_1^2. In cases i″) and ii), it is obvious, since Z_0 and Z_ρ are Morse-Smale and are transversal to ∂U_δ if δ and ρ are small (cf. [6]). Let us consider case i′). In this case Z_ρ is not transversal to ∂U_δ, but Z_0 has eight tangencies with ∂U_δ, which are generic, therefore if ρ is small, Z_ρ has eight tangencies too. As $W_0^s(X) \cap U_{\rho,\varepsilon}$ is a closed curve C, and Z_ρ must have four sadle points in U_δ, this sadles must lie in C and C-{sadles} has four components which are sadle connections. The phase space of Z_ρ in U_δ is like in the picture below

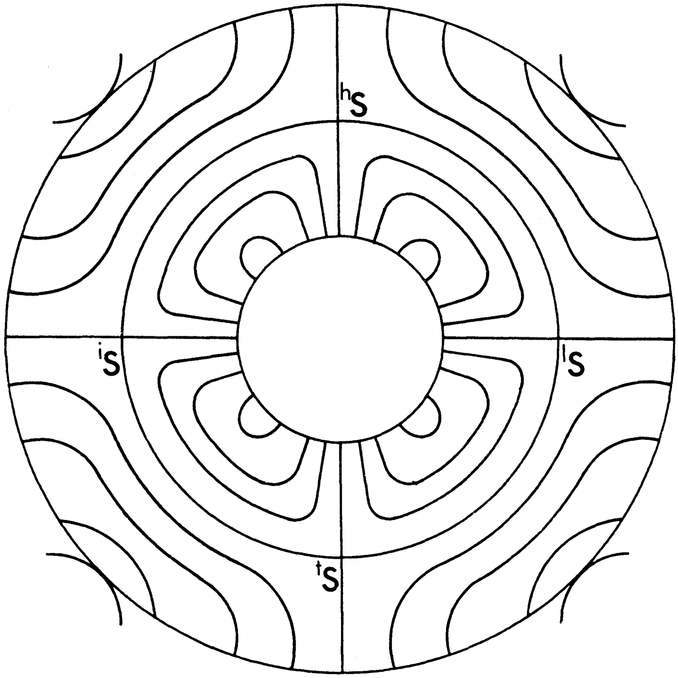

Fig. 4.

Using a known argument of arc length it is possible to construct a topological equivalence between Z_0 and Z_ρ in U_δ (cf. [6] and [7]).

Let us prove *c*). It is sufficient to show that the intersection of any leaf of $\mathscr{F}(\omega/V_X)$ with S^2_p has only one connected component, because Z_ρ is transversal to ∂U_δ in cases i″) and ii) and in case i′) the tangencies of Z_ρ with ∂U_δ are generic (of the type $y = x^2$). Consider $X = \text{rot}\,(\omega)$, $L = \text{rot}\,(q)$. It is not difficult to see that if ρ is small then the set of tangencies of X with S^2_ρ is the union of two closed disjoint curves $\gamma_1(\rho)$ and $\gamma_2(\rho)$. Let $\delta(\rho) = W^s_0(X) \cap S^2_\rho$ and $\{p_1(\rho), p_2(\rho)\} = W^u_0(X) \cap S^2_\rho$. Then it is not difficult to see that $S^2_\rho - [\{p_1(\rho), p_2(\rho)\} \cup \gamma_1(\rho) \cup \gamma_2(\rho) \cup \delta(\rho)]$ is the union of four cilindric regions A_i, B_i, $i = 1, 2$, as in the picture below.

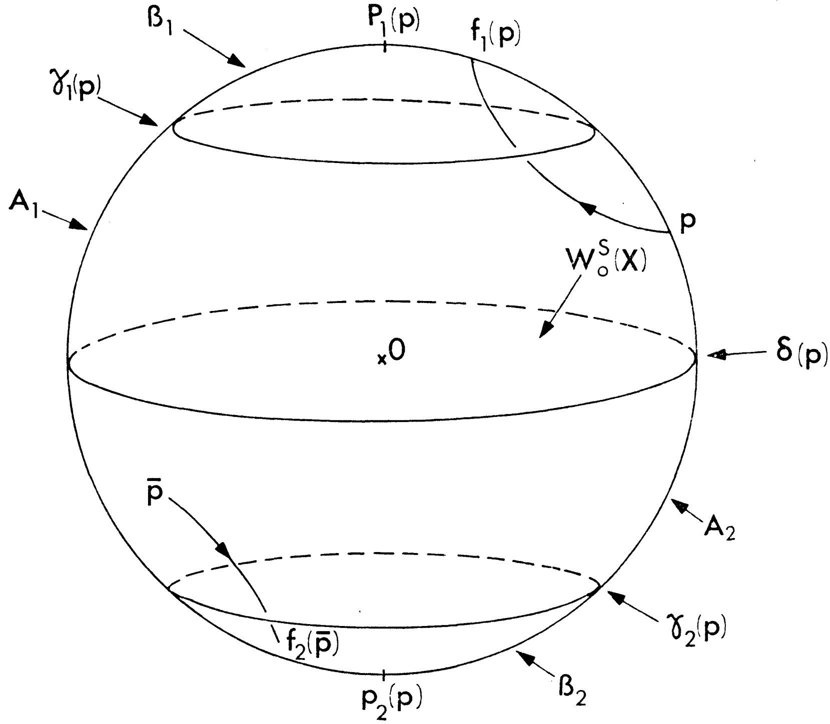

Fig. 5.

By a know construction we can define two Poincaré transformations $f_i : A_i \to B_i$, $i = 1, 2$, so that if $q \in A_i$, $f_i(q)$ is the first point of the positive trajectory of X by q in B_i. In fact f_i can be extended to $\gamma_i(\rho) \cup A_i = \tilde{A}_i$ by setting $f_i(q) = q$ if $q \in \gamma_i(\rho)$. Now, let l be a leaf of ω restricted to the interior region V bounded by S^2_ρ. Let C be a component of $l \cap S^2_\rho$ such that $C \cap A_1 \neq \varnothing$. Then the projection of C in S^2_1 is a trajectory of Z_ρ, which implies that

$$C \cap \gamma_1(\rho) \neq \varnothing,$$

if ρ is small. Let $C_X = \bigcup_{q \in \tilde{A}_1 \cap C} \widehat{qf_1(q)}$, where $\widehat{qf_1(q)}$ is the segment of the orbit of X between q and $f_1(q)$ (inside S^2_ρ). Then $C_X \subset l$ (because $i_X(\omega) = 0$) and it is open and closed in l (because X has no singularities in l), therefore $C_X = l$.

If $C \cap \gamma_1(\rho)$. is just one point (cases i′ and ii) then

$$l \cap S^2_\rho = C_X \cap S^2_\rho = C.$$

If $C \cap \gamma_1(\rho)$ contains two points, say q_1 and q_2, then the image of the segment of C between q_1 and q_2, $\widehat{q_1q_2}$, by f_1 is a curve with q_1 and q_2 as end points, therefore C is a closed curve and we have $l \cap S^2_\rho = C_X \cap S^2_\rho = C$.

Observe that the above argument shows that in case i′) the phase portrait of Z_ρ is like in the picture 2.1, if ρ is small.

2.2.6. *End of the proof.* — Let $0 \in \mathbf{R}^3$ be a hyperbolic singularity of ω, $j^2(\omega)_0 = q$, $X = \mathrm{rot}\,(\omega)$, $L = \mathrm{rot}\,(q)$. We can suppose that q has one of the forms i) or ii) of 1.3.2 and that $W^s_0(L) = \{x \in \mathbf{R}^3 | x_3 = 0\}$, $W^u_0(L) = \{x \in \mathbf{R}^3 | x_1 = x_2 = 0\}$. Let $f(x_1, x_2, x_3) = x_1^2 + x_2^2 - x_3^2$. Then it is not difficult to see that the non-singular trajectories of L are transversal to the surfaces $f^{-1}(c)$, $c \in \mathbf{R}$, and the same is true for X in a small neighborhood of the origin. We shall define a topological equivalence h between ω and q in a neighborhood V of the origin, such that $f \circ h(x) = f(x)$ for every $x \in V$. Observe that $W^u_0(X)$ and $W^u_0(L)$ intersect each $f^{-1}(c)(c \leqslant 0)$ in a unique point. Let $A_{\rho,\varepsilon} = \{x \in \mathbf{R}^3 | f(x) = \rho^2, |x_3| \leqslant \varepsilon\}$. By lemma 2.2.5 if ρ and ε are small we have:

a) X and L are transversal to $A = A_{\rho,\varepsilon}$ and the sets $A(X) \cup W^u_0(X)$, $A(L) \cup W^u_0(L)$ contain neighborhood V_X and V_L of the origin.

b) The restriction of ω and q to A are topologically equivalent.

c) If l is a leaf of the restriction of ω to V_X, then $l \cap A$ has only one connected component.

Let $\tilde{h}: A \to A$ be a topological equivalence between ω/A and q/A. We want to extend $\tilde{h}$ to $h: V_X \to V_L$. If $p \in V_X$ we have two possibilities: $p \in W^u_0(X)$ or the negative trajectory $0^-(X, p)$ of p in V_X intersects A in a unique point p'. If $p \in W^u_0(X)$ we define $h(p)$ to be the unique point of $W^u_0(L)$ such that $f(h(p)) = f(p)$. If $p \notin W^u_0(X)$, let

$$p' = 0^-(X, p) \cap A.$$

We define $h(p)$ to be the unique point of the positive trajectory $0^+(L, \tilde{h}(p'))$ (of $\tilde{h}(p')$ by L) such that

$$f(h(p)) = f(p).$$

It is not difficult to see that if h is continuous then it is a local equivalence between ω and q. The continuity of h in $V_X - W_0^u(X)$ is obvious. Let us show that h is continuous in $W_0^u(X) \cap V_X$. Let $p_n \to p \in W_0^u(X)$ as $n \to \infty$. Then $f(p_n) \to f(p)$ and the sequence $p'_n = 0^-(X, p_n) \cap A$ has its acumulation points in $W_0^s(X) \cap A$, therefore the sequence $\tilde{h}(p'_n)$ acumulates in $W_0^s(L) \cap A$ which implies that

$$0^+(L, \tilde{h}(p'_n)) \cap f^{-1}(f(p_n))$$

accumulates in $W_0^u(L) \cap f^{-1}(f(p)) = h(p)$, therefore

$$\lim_{n\to\infty} h(p_n) = h(p)$$

and h is continuous.

2.3. *Proof of Corollary B.*

Let p be a hyperbolic singularity of $\omega \in \vartheta^r(M^3)$, $r \geqslant 2$, and $j^2(\omega)_p = q \in Q(3)$. Taking a parametrization of a neighborhood of p, we can suppose $p = 0$, $\omega \in \vartheta^r(\mathbf{R}^3)$. As rot (q) is hyperbolic, there exist neighborhoods $\tilde{\mu} \subset Q(3)$ of q such that if $\tilde{q} \in \tilde{\mu}$ then $\tilde{q}$ is hyperbolic. As q is hyperbolic we can take $\tilde{\mu}$ in such a way that $\tilde{q} \in \tilde{\mu}$ is topologically equivalent to q. Now it is sufficient to show that given a neighborhood V of 0, there exists a neighborhood μ of ω in $\vartheta^r(\mathbf{R})^3$ such that if $\tilde{\omega} \in \mu$, there exists $\tilde{p} \in V$ such that $j^1(\tilde{\omega})_{\tilde{p}} = 0$ and $j^2(\tilde{\omega})_{\tilde{p}} \in \tilde{\mu}$. This is an immediate consequence of the fact that 0 is a hyperbolic singularity of rot (ω) and of lemma 2.2.2.

2.4. *Proof of Theorem C.*

Let us show that sing $(d\omega) \subset$ sing (ω) and sing $(d\omega)$ is a C^{r-1} codimension three submanifold of M. By lemma 2.2.2 we can suppose dim $(M) \geqslant 4$. Taking a parametrization of a neighborhood of p we can suppose that $\omega \in \vartheta^r(\mathbf{R}^n)$, $p = 0$. Let $\mathbf{R}^n = \mathbf{R}^3 \times \mathbf{R}^{n-3}$ be a decomposition of $\mathbf{R}^n$ such that $\omega/\mathbf{R}^3 \times 0$ is simple. Let $(x_1, x_2, x_3, 0, \ldots, 0)$ be the coordinates of $\mathbf{R}^3 \times 0$ and $(0, 0, 0, x_4, \ldots, x_n)$ be the coordinates of $0 \times \mathbf{R}^{n-3}$. Then we have

$$d\omega = \sum_{1 \leqslant i < j \leqslant n} \Omega_{ij}\, dx_i \wedge dx_j.$$

As $\omega/\mathbf{R}^3 \times 0$ is simple, the map $\psi = (\Omega_{23}, \Omega_{31}, \Omega_{12}) : \mathbf{R}^n \to \mathbf{R}^3$ has rank three at 0 and in fact the « Jacobian » matrix $A(x) = \dfrac{\partial \psi}{\partial(x_1, x_2, x_3)}$ is non singular at 0. Let $k \geqslant 4$ and $\psi_k = (\Omega_{1k}, \Omega_{2k}, \Omega_{3k})$. By the integrability condition we have $d\omega \wedge d\omega = 0$ so that the scalar product $\psi . \psi_k = 0$. Taking partial derivatives with respect to (x_1, x_2, x_3) we have

$$A^t . \psi_k + A_k^t . \psi = 0,$$

where $A_k = \dfrac{\partial \psi_k}{\partial(x_1, x_2, x_3)}$ and A^t, A_k^t are the transposes of A and A_k. As A is non-singular in a neighborhood V of 0, in V we have $\psi_k = (A^t)^{-1} A_k^t \psi$, so that $\operatorname{sing}(\psi) \subset \operatorname{sing}(\psi_k)$. Now it is sufficient to show that $\operatorname{sing}(\psi) \subset \operatorname{sing}(\Omega_{ij})$ if $i, j \geqslant 4$ (this implies

$$\operatorname{sing}(d\omega/V) = \operatorname{sing}(\psi) = \{x \in V \mid \psi(x) = 0\}).$$

Let $p \in \operatorname{sing}(\psi)$. By the integrability condition we have

$$(*) \qquad \Omega_{12}\Omega_{ij} + \Omega_{2i}\Omega_{1j} + \Omega_{i1}\Omega_{2j} = 0.$$

As A is non-singular in V, there exists $1 \leqslant l \leqslant 3$ such that $\dfrac{\partial \Omega_{12}}{\partial x_l}(p) \neq 0$. Taking the partial derivative of $(*)$ with respect to x_l at p and using the fact that

$$\Omega_{1i}(p) = \Omega_{2i}(p) = \Omega_{1j}(p) = \Omega_{2j}(p) = \Omega_{12}(p) = 0$$

we have $\Omega_{ij}(p) = 0$.

Let us show that $j^1(\omega)_p = 0$ if $p \in \operatorname{sing}(d\omega)$. We can suppose $p = 0$. As $\operatorname{sing}(d\omega) \subset \operatorname{sing}(\omega)$ we have $\omega_0 = 0$, so that $\omega = l + q + R$ where l is linear, q is quadratic and $\lim\limits_{x \to 0} \dfrac{R_x}{\|x\|^2} = 0$. We want to show that $l = 0$. As

$$0 \in \operatorname{sing}(d\omega), \quad dl = 0 \quad \text{and} \quad l = df$$

where f is of degre 2. By the integrability condition we have

$df \wedge dq = 0$ and $dq \wedge dq = 0$. By remark 2.1.1 we can suppose that dq depends of three variables. The equation $df \wedge dq = 0$ implies $\frac{\partial f}{\partial x_i} \alpha_{jk} + \frac{\partial f}{\partial x_j} \alpha_{ki} + \frac{\partial f}{\partial x_k} \alpha_{ij} = 0$ where

$$dq = \sum_{1 \leqslant i < j \leqslant 3} \alpha_{ij}\, dx_i \wedge dx_j$$

and $\alpha_{ij} = 0$ if $i \geqslant 4$. If $1 \leqslant j < k \leqslant 3$, $i \geqslant 4$, we have $\frac{\partial f}{\partial x_i} \alpha_{jk} = 0$, therefore $\frac{\partial f}{\partial x_i} = 0$ if $i \geqslant 4$. This implies that f depends only of the variables (x_1, x_2, x_3). By lemma 2.2.2 we have $df = 0$, therefore $l = 0$.

Let us show the existence of $\mu \subset \vartheta^r(M)$ and

$$\xi : \mu \to \xi^{r-2}(\operatorname{sing}(d\omega), M).$$

Let V be a tubular neighborhood of $\operatorname{sing}(d\omega)$ and

$$\pi : V \to \operatorname{sing}(d\omega)$$

the projection, which we can suppose to be C^{r-1} and

$$\pi^{-1}(x) \subset \partial M \quad \text{if} \quad x \in \partial M \cap \operatorname{sing}(d\omega).$$

The fibers $\pi^{-1}(x)$ are C^{r-1} embedded 3-disks. We can consider for each $x \in \operatorname{sing}(d\omega)$ and each $\tilde{\omega} \in \Lambda^{1,r}(M)$ the C^{r-2} 2-form $d\tilde{\omega}/\pi^{-1}(x)$, which intersects transversally the zero section of $\Lambda^{2,r-2}(\pi^{-1}(x))$, if $\tilde{\omega}$ is near ω in the C^r topology $(r \geqslant 3)$. As we are considering $\Lambda^{1,r}(M)$ endowed with Whitney's topology, it is sufficient to show that for each $x \in \operatorname{sing}(d\omega)$ there exist neighborhoods U_x of x in $\operatorname{sing}(d\omega)$ and μ_x of ω in $\Lambda^{1,r}(\pi^{-1}(U_x))$ such that if $\tilde{\omega} \in \mu_x \cap \vartheta^r(M)$ then

1) If $x' \in U_x$ then $d\tilde{\omega}/\pi^{-1}(x')$ has one and only one singularity in $\pi^{-1}(x') \cap V = \pi^{-1}(x')$.

2) The projection $\pi : \operatorname{sing}(d\tilde{\omega}) \cap \pi^{-1}(U_x) \to U_x$ is a C^{r-1} diffeomorphism.

3) There exist a continuous map

$$\xi : \mu_x \cap \vartheta^r(M) \to \xi^{r-2}(U_x, \pi^{-1}(U_x)),$$

such that the image of $\xi(\tilde{\omega})$ is $\operatorname{sing}(d\tilde{\omega}) \cap \pi^{-1}(U_x)$.

To see 1, 2, 3 above, let $K \subset \operatorname{sing}(d\omega)$ be a compact neighborhood of x. Let $f\colon D^3 \times K \to \pi^{-1}(K)$ be a C^{r-1} diffeomorphism such that $f(0, x) = x$ and

$$\pi \circ f(y, x) = x, \quad x \in K, \quad y \in D^3.$$

The map $f^* : \Lambda^{1,r}(\pi^{-1}(K)) \to \Lambda^{1,r-1}(D^3 \times K)$ is continuous, therefore we can suppose that $\pi^{-1}(K) = D^3 \times K$,

$$\pi : D^3 \times K \to K$$

is $\pi(y, x) = x$, $\omega \in \Lambda^{1,r-1}(D^3 \times K)$ and $0 \times K = \operatorname{sing}(d\omega)$. Define

$$\varphi : D^3 \times K \times \Lambda^{1,r-1}(D^3 \times K) \to \Lambda^2(D^3)$$

by $\varphi(y, x, \tilde{\omega}) = d\tilde{\omega}_{(y,x)}/\pi^{-1}(x) = d\tilde{\omega}_{(y,x)}/D^3 \times x$. Then φ is C^{r-2} and the partial derivative of φ in a point $(0, x, \omega)$ in the direction of D^3 is $\partial_1\varphi(0, x, \omega) : \dot{y} \in R^3 \to L\dot{y}$, where L is the linear part of $d\omega/D^3 \times x$ at 0. But $\dot{y} \to L\dot{y}$ is non-singular because 0 is a simple singularity of $d\omega/D^3 \times x$. By the implicit function theorem there exist neighborhood U_x of x in $\operatorname{sing}(d\omega) = 0 \times K$ and μ_x of ω in $\Lambda^{1,r-1}(D^3 \times K)$ and an unique C^{r-2} map $\psi : U_x \times \mu_x \to \pi^{-1}(U_x)$ such that $\psi(x', \tilde{\omega})$ is the unique singularity of $d\tilde{\omega}/\pi^{-1}(x')$ which is, hyperbolic. Now if $\tilde{\omega} \in \vartheta^r(D^3 \times K) \cap \mu_x$, by the first part of the theorem, $\psi(x', \tilde{\omega}) \in \operatorname{sing}(d\tilde{\omega})$ and

$$\operatorname{sing}(d\tilde{\omega}/\pi^{-1}(U_x)) = \psi(U_x, \tilde{\omega}),$$

therefore $\pi : \operatorname{sing}(d\tilde{\omega}/\pi^{-1}(U_x)) \to U_x$ is a C^{r-2} diffeomorphism and $\pi^{-1} : U_x \to \operatorname{sing}(d\tilde{\omega}) \cap \pi^{-1}(U_x)$ is $\psi_{\tilde{\omega}}(x) = \psi(x, \tilde{\omega})$. If we define $\xi : \mu_x \cap \vartheta^r(M) \to \xi^{r-2}(U_x, \pi^{-1}(U_x))$ by

$$\xi(\tilde{\omega})(x') = \psi(x', \tilde{\omega})$$

then ξ is continuous and $\xi(\tilde{\omega})(U_x) = \operatorname{sing}(d\tilde{\omega}) \cap \pi^{-1}(U_x)$.

2.5. *Proof of Theorem D.*

Taking local coordinates in M and using proposition 1.3.2 we can suppose that $\omega \in \vartheta^r(\mathbf{R}^n)$, $p = 0$ and $j^2(\omega)_0 = q$ depends of the three variables x_1, x_2, x_3 only. Let $\omega = q + R$ where $\lim\limits_{x \to 0} \dfrac{R_x}{\|x\|^2} = 0$. We need one lemma.

2.5.1. Lemma. — *Let* $\omega = q + R$ *be as above. Let*

$$\mathbf{R}^n = \mathbf{R}^{n-1} \times \mathbf{R}$$

be a canonical decomposition of $\mathbf{R}^{n-1}$, *where* $\mathbf{R}^{n-1}$ *is generated by the elements* $(x, 0)$, $x = (x_1, \dots, x_{n-1})$ *and* $\mathbf{R}$ *by the elements* $(0, \dots, 0, x_n)$. *If* $\omega = \sum_{i=1}^{n} \omega_i \, dx_i$, *let* $\tilde{\omega} = \sum_{i=1}^{n-1} \omega_i(x, 0) \, dx_i$ *considered as an integrable* 1*-form in* $\mathbf{R}^{n-1}$. *Then there exist neighborhoods* $0 \in U \subset \mathbf{R}^n$ *and* $0 \in U' \subset \mathbf{R}^{n-1}$, *a real number* $\varepsilon > 0$ *and a* C^{r-3} *diffeomorphism* $f \colon U' \times (-\varepsilon, \varepsilon) \to U$ *such that* $f^*(\omega) = \tilde{\omega}$ *and* $f/U' \times 0$ *is the identity.*

Lemma 2.5.1 implies theorem D, because $\tilde{\omega}$ depends of $n - 1$ variables $(n \geqslant 4)$, $\tilde{\omega}$ is C^r and $j^2(\tilde{\omega})_0 = j^2(\omega)_0 = q$.

Proof. — We shall construct a C^{r-3} vector field X in a neighborhood V of 0 satisfying $X(x) \neq 0$ if $x \in V$,

$$i_X(d\omega) = 0$$

and $X(0) = (0, \dots, 0, 1) \in 0 \times \mathbf{R}$. By lemma 2.2.1 we have $i_X(\omega) = 0$ and $L_X(\omega) = 0$ and the trajectories of X are contained in the leaves of $\mathcal{F}(\omega)$, in sing (ω) or in sing $(d\omega)$. Let $X_t(x)$ be the local flow induced by X. If $r \geqslant 4$, by the inverse function theorem, there exist a neighborhood $0 \in U' \subset \mathbf{R}^{n-1}$ and $\varepsilon > 0$ such that the map

$$f \colon U' \times (-\varepsilon, \varepsilon) \to \mathbf{R}^n$$

defined by $f(x, t) = X_t(x)$ is a C^{r-3} diffeomorphism of $U' \times (-\varepsilon, \varepsilon)$ onto $f(U' \times (-\varepsilon, \varepsilon)) = U$. Now it is not difficult to see that $f^*(\omega) = \tilde{\omega}$ and $f/U' \times 0 = \text{identity}$.

Let us construct X. Suppose $X = (\Delta_1, \Delta_2, \Delta_3, 0, \dots, 0, 1)$. The condition $i_X(d\omega) = 0$ is equivalent to

(*) $\Delta_1 \Omega_{1j} + \Delta_2 \Omega_{2j} + \Delta_3 \Omega_{3j} + \Omega_{nj} = 0, \; j = 1, \dots, n,$

where $d\omega = \sum_{1 \leqslant i < j \leqslant n} \Omega_{ij} \, dx_i \wedge x_j$.

Assertion. — The two conditions

$$(**) \qquad \begin{aligned} -\Delta_2 \Omega_{12} + \Delta_3 \Omega_{31} &= \Omega_{1n} \\ \Delta_1 \Omega_{12} - \Delta_3 \Omega_{23} &= \Omega_{2n} \end{aligned}$$

imply the conditions (*) in a neighborhood of 0.

Proof. — Let $\operatorname{sing}(\Omega_{ij}) = \{x | \Omega_{ij}(x) = 0\}$. By the proof of theorem C, there exists a neighborhood V of 0 such that the matrix $\left(\frac{\partial \Omega_{ij}}{\partial x_k}\right)_{\substack{1\leqslant i<j\leqslant 3 \\ 1\leqslant k\leqslant}}$ is non singular in V and

$$\operatorname{sing}(d\omega) \cap V = \{x \in V | \Omega_{23}(x) = \Omega_{31}(x) = \Omega_{12}(x) = 0\}.$$

This implies in particular that the interior of $V \cap \operatorname{sing}(\Omega_{ij})$ is empty, $1 \leqslant i < j \leqslant 3$. By the integrability condition $d\omega \wedge d\omega = 0$ and $\Omega_{23}\Omega_{1n} + \Omega_{31}\Omega_{2n} + \Omega_{12}\Omega_{3n} = 0$. Substituting (**) in the above relation we have

$$\Omega_{12}(\Omega_{3n} - \Delta_1\Omega_{13} - \Delta_2\Omega_{23}) = 0,$$

therefore $\Delta_1\Omega_{13} + \Delta_2\Omega_{23} + \Omega_{n3} = 0$ in V, which is (*) for $j = 3$. Applying $d\omega \wedge d\omega = 0$ against, we have

$$\Omega_{2j}\Omega_{1n} + \Omega_{j1}\Omega_{2n} + \Omega_{12}\Omega_{1n} = 0$$

and substituting (**) we have

$$\Omega_{12}(\Omega_{jn} - \Delta_1\Omega_{1j} - \Delta_2\Omega_{2j}) + (\Omega_{23}\Omega_{1j} + \Omega_{31}\Omega_{2j})\Delta_3 = 0$$

or

$$\Omega_{12}(\Omega_{jn} - \Delta_1\Omega_{1j} - \Delta_2\Omega_{2j} - \Delta_3\Omega_{3j}) = 0$$

which implies $\Delta_1\Omega_{1j} + \Delta_2\Omega_{2j} + \Delta_3\Omega_{3j} + \Omega_{nj} = 0$ in V and the assertion is proved.

Let us show now that (**) is satisfied for some Δi's of class C^{r-3}. Let $\Sigma' = \operatorname{sing}(\Omega_{31}) \cap \operatorname{sing}(\Omega_{12}) \cap V$. Then $\Sigma' \subset \operatorname{sing}(\Omega_{1n}) \cap V$, because if $p \in \Sigma'$ we have

$$\Omega_{23}(p)\Omega_{1n}(p) = 0,$$

by the relation $d\omega \wedge d\omega = 0$ (If $\Omega_{23}(p) = 0$ then

$$p \in \operatorname{sing}(d\omega) \cap V \quad \text{and} \quad \Omega_{1n}(p) = 0).$$

As $\Sigma' \subset \operatorname{sing}(\Omega_{1n}) \cap V$, by the implicit function theorem we have $\Omega_{1n} = -\Delta_2\Omega_{12} + \Delta_3\Omega_{31}$ where Δ_2 and Δ_3 are C^{r-2}. Let $f = \Delta_{2n} + \Delta_3\Omega_{23}$. By the relations

$$\Omega_{1n} = -\Delta_2\Omega_{12} + \Delta_3\Omega_{31}$$

and $\Omega_{23}\Omega_{1n} + \Omega_{31}\Omega_{2n} + \Omega_{12}\Omega_{3n} = 0$ we have

$$\Omega_{31}f + \Omega_{12}(\Omega_{3n} - \Delta_2\Omega_{23}) = 0,$$

so that sing $(\Omega_{12}) \cap V \subset$ sing $(f) \cap V$. By the implicit function theorem there exists Δ_1 of class C^{r-3} such that $f = \Delta_1 \Omega_{12}$, which implies that $\Omega_{2n} = \Delta_1 \Omega_{12} - \Delta_3 \Omega_{23}$ and we have (**).

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