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ON SYSTEMS OF IMPRIMITIVITY ON LOCALLY COMPACT ABELIAN GROUPS WITH DENSE ACTIONS

by J. MATHEW and M. G. NADKARNI

Introduction.

Let R be a second countable locally compact Abelian group. Let $\Gamma \subseteq R$ be a dense sub-group of R with another topology under which it is a second countable locally compact Abelian group and such that the injection map of Γ into R is continuous. Let $\Gamma_0 \subseteq \Gamma \subseteq R$ be a closed sub-group in R, hence also closed in Γ. Then $\Gamma/\Gamma_0 \subset R/\Gamma_0$; Γ/Γ_0 is dense in R/Γ_0 and the injection map of Γ/Γ_0 in R/Γ_0 is continuous. Let $B = \hat{\Gamma}$, $S = \hat{R}$ and let K be the annihilator of Γ_0 in B. Then S is a dense sub-group of B and the injection map of S into B is continuous. Further $K \cap S$ is a dense sub-group of K and the injection map of $K \cap S$ into K is continuous, where the topology on $K \cap S$ is the one which it receives as a closed sub-group of S. Thus we have four pairs (Γ,R), $(\Gamma/\Gamma_0,R/\Gamma_0)$, $(K \cap S,K)$, (S,B). The first coordinate of each is dense in the second coordinate and the injection map is continuous. Hence the first coordinate acts on the second coordinate through translation.

Let H be a complex separable Hilbert space. Let (V,E) be a system of imprimitivity for $K \cap S$ based on K and acting in H. We call it $(K \cap S,K)$ system of imprimitivity. We show that every such system gives rise, in a natural fashion, to an (S,B) system of imprimitivity $(\overline{V},\overline{E})$, and, every (S,B) system of imprimitivity is equivalent to one which arises in this fashion, (section 3).

Let U, F be given by

$$\begin{aligned} U_\gamma &= \int_K (\gamma,y)E\,(dy) & \gamma &\in \Gamma/\Gamma_0 \\ V_s &= \int_{R/\Gamma_0} (-\,s,x)F\,(dx), & s &\in K \cap S\,. \end{aligned}$$

Then (U,F) is a $(\Gamma/\Gamma_0, R/\Gamma_0)$ system of imprimitivity, (section 1). We show in section 2 that (U,F) gives rise, in a natural fashion, to a (Γ,R) system of imprimitivity $(\tilde{U},\tilde{F})$ and that every (Γ,R) system of imprimitivity is equivalent to one which arises in this fashion. Finally let $\overline{U}$, $\overline{F}$ be defined by

$$\begin{aligned} \overline{U}_\gamma &= \int_B (\gamma,x)\overline{E}\,(dx) & \gamma &\in \Gamma \\ \overline{V}_s &= \int_R (-\,s,x)\overline{F}\,(dx) & s &\in S\,. \end{aligned}$$

Then $(\overline{U},\overline{F})$ is a (Γ,R) system of imprimitivity. We thus get two (Γ,R) systems of imprimitivity, $(\tilde{U},\tilde{F})$ and $(\overline{U},\overline{F})$, starting with the same $(K \cap S, K)$ system of imprimitivity (V,E). Under a mild assumption which is satisfied in many cases, probably in all the cases, we show $(\tilde{U},\tilde{F})$ and $(\overline{U},\overline{F})$ are equivalent systems of imprimitivity, (section 4). We thus show that the two way arrows shown below between equivalence classes of systems of imprimitivity are valid and that in many cases no matter which way we arrive at the lower left hand corner starting from the upper right hand corner we get the same system of imprimitivity, (up to equivalence).

$$\begin{tikzcd}
(K \cap S, K) & & & & (S,B) \\
& (V,E) \arrow[rr, shift left] \arrow[d, shift left] & & (\overline{V},\overline{E}) \arrow[ll, shift left] \arrow[d, shift left] & \\
& (U,F) \arrow[u, shift left] \arrow[dr, shift left] & & (\overline{U},\overline{F}) \arrow[u, shift left] & \\
(\Gamma/\Gamma_0, R/\Gamma_0) & & (\tilde{U},\tilde{F}) \arrow[ul, shift left] & & (\Gamma,R)
\end{tikzcd}$$

Diagram 1

This paper is a generalization of our earlier joint paper with Bagchi [1], where a special case was treated. The main steps in the present paper are same as in the special case, but, to carry out details one has to rely upon the advanced theory of locally compact Abelian groups.

For all unexplained terminology we refer to Varadarajan [3]. By a measure we mean a σ-finite non-negative measure. When we have complex valued measures we are explicit about them. We speak of cocycles relative to a mesure rather than with respect to its measure class. If G is an Abelian group and H a sub-group of G, then by a section of G with respect to H we mean a set C which intersects each coset of H in exactly one point. Every element $g \in G$ can be written uniquely in the form $g = c + h$, $c \in C$, $h \in H$. We denote c and h by $\langle g \rangle$ and $[g]$ respectively and observe that for any two elements $g_1, g_2 \in G$,

$$[g_1+g_2] = [g_1] + [\langle g_1 \rangle + g_2] \quad \text{and} \quad \langle g_1+g_2 \rangle = \langle \langle g_1 \rangle + g_2 \rangle.$$

We shall use these facts in the sequel. If ν is a measure on a locally compact Abelian group G, ν_g will mean the measure $A \to \nu(A + g)$. For a locally compact (second countable) Abelian group G, λ_G will denote the Haar measure on G, unless stated otherwise.

We use A^* to denote the adjoint of A if A is an operator, and the complex conjugate of A if A is a complex number.

1. Dual systems.

By a pair (Γ,R) we mean a pair of locally compact second countable Abelian groups Γ and R together with a one-one continuous homomorphism φ of Γ into R such that $\varphi(\Gamma)$ is dense in R. Given a pair (Γ,R) there arises another pair in a natural fashion. Consider the dual groups $\hat{\Gamma}$ and $\hat{R}$, and the map $\hat{\varphi} : \hat{R} \to \hat{\Gamma}$ defined by

$$(x,\hat{\varphi}(\hat{y})) = (\varphi(x),\hat{y}), \qquad x \in \Gamma, \qquad \hat{y} \in \hat{R}.$$

It can be shown that $\hat{\varphi}$ is a one-one continuous homomorphism of $\hat{R}$ into $\hat{\Gamma}$ and that $\hat{\varphi}(\hat{R})$ is dense in $\hat{\Gamma}$. The pair $(\hat{R},\hat{\Gamma})$ is called the dual pair of (Γ,R). It is convenient to identify Γ with $\varphi(\Gamma)$, and thus regard Γ as a dense subgroup of R. The topology on Γ need not be the one induced from R; but it is such that the injection map of Γ into R is continuous.

Let (U,P) be a (Γ,R) system of imprimitivity acting in H, a complex separable Hilbert space. Apply Stone's theorem to U to yield a spectral measure Q on $\hat{\Gamma}$, and to P to yield a representation V of $\hat{R}$:

$$U_g = \int_{\hat{\Gamma}} (-y,g) Q(dy) \qquad g \in \Gamma$$
$$V_h = \int_R (x,h) P(dx) \qquad h \in \hat{R}.$$

Some elementary computations [1, p. 291] show that

$$V_h^{-1} Q(D) V_h = Q(D + \varphi(h))$$

for each Borel set $D \subseteq \hat{\Gamma}$ and for each $h \in \hat{R}$. Hence (V,Q) is a $(\hat{R},\hat{\Gamma})$ system of imprimitivity acting in H. We call (V,Q) the dual system of (U,P). We observe that a subspace of H reduces (U,P) if and only if it reduces (V,Q).

2. Quotient systems.

Let (Γ,R) be a pair. Let $\Gamma_0 \subseteq \Gamma$ be a closed sub-group of R. Then Γ_0 is closed in Γ as well. Further $(\Gamma/\Gamma_0, R/\Gamma_0)$ is a pair which we call the quotient pair. In this section we show that:

Every $(\Gamma/\Gamma_0, R/\Gamma_0)$ system of imprimitivity (U,F) gives rise, in a natural fashion, to a (Γ,R) system of imprimitivity $(\tilde{U},\tilde{F})$; moreover every (Γ,R) system of imprimitivity is equivalent to one which arises in this fashion.

To this end we keep in view the well known representation of a system of imprimitivity in terms of quasi-invariant measures and unitary operator valued cocycles [3, Thm 9]. Lemmas 2.1 and 2.2 below then immediately establish the desired statement.

Since Γ_0 is a closed sub-group in R we can choose a section of R with respect to Γ_0 which is a Borel set [3, Thm 8.11]. Let Q be a Borel section of R with respect to Γ_0. One can define a group operation in Q by

$$c_1 \dotplus c_2 = \langle c_1 + c_2 \rangle.$$

With this operation Q is a group isomorphic to R/Γ_0, isomorphism being $c \to \bar{c}$ where $\bar{c}$ denotes the coset of Γ_0 to which c belongs. This isomorphism is furthermore a Borel isomorphism between Q and R/Γ_0. Every element $x \in R$ can be written uniquely in the form:

$$x = c + \gamma_0\,, \quad c \in Q\,, \quad \gamma_0 \in \Gamma_0\,,$$

and the map $\delta : c + \gamma_0 \to (\bar{c}, \gamma_0)$, is a Borel isomorphism between R and $R/\Gamma_0 \times \Gamma_0$. Let ν be a measure on R/Γ_0 and let $\lambda_0 (= \lambda_{\Gamma_0})$ denote the Haar measure on Γ_0. By $\tilde{\nu}$ we shall mean the measure on R defined by

$$\tilde{\nu}(A) = (\nu \times \lambda_0)(\delta(A)), \quad A \subseteq R\,.$$

LEMMA 2.1. — *If* ν *is quasi-invariant under the action of* Γ/Γ_0 *on* R/Γ_0 *then* $\tilde{\nu}$ *is quasi-invariant under the action of* Γ *on* R. *Any measure on* R, *quasi-invariant under the action of* Γ, *is equivalent to a measure of the form* $\tilde{\nu}$ *for some measure* ν *on* R/Γ_0, *quasi-invariant under* Γ/Γ_0.

Proof. — Let A be a Borel subset of R such that $\tilde{\nu}(A) = 0$. By Fubini theorem, then, for ν almost every $\bar{c} \in R/\Gamma_0$, $c \in Q$, $\lambda_0((\delta(A))_{\bar{c}}) = 0$, where $(\delta(A))_{\bar{c}}$ is the $\bar{c}$ section of $\delta(A)$. Let γ belong to Γ. Then $\gamma = \langle\gamma\rangle + [\gamma]$, where $\langle\gamma\rangle \in Q$ and $[\gamma] \in \Gamma_0$. For any $c \in Q$

$$\begin{aligned}
(\delta(A + \gamma))_{\bar{c}} &= \{\gamma_0 \in \Gamma_0 : (\bar{c}, \gamma_0) \in \delta(A + \gamma)\} \\
&= \{\gamma_0 \in \Gamma_0 : c + \gamma_0 \in A + \gamma\} \\
&= \{\gamma_0 \in \Gamma_0 : \langle c - \langle\gamma\rangle\rangle + \gamma_0 + [c - \gamma] \in A\} \\
&= \{\gamma_0 - [c - \gamma] \in \Gamma_0 : \langle c - \langle\gamma\rangle\rangle + \gamma_0 \in A\} \\
&= (\delta(A))_{\bar{c} - \overline{\langle\gamma\rangle}} - [c - \gamma].
\end{aligned}$$

By invariance of λ_0 we have

$$\lambda_0((\delta(A + \gamma))_{\bar{c}}) = \lambda_0((\delta(A))_{\bar{c} - \overline{\langle\gamma\rangle}})$$

Since $\lambda_0(\delta(A))_{\bar{c}}) = 0$ a.e. ν, by quasi-invariance of ν we have $\lambda_0((\delta(A))_{\bar{c} - \overline{\langle\gamma\rangle}}) = 0$ a.e. ν. Hence

$$\tilde{\nu}(A + \gamma) = \int_{R/\Gamma_0} \lambda_0((\delta(A + \gamma))_{\bar{c}})\nu(d\bar{c}) = 0$$

which shows that $\tilde{\nu}$ is quasi-invariant under Γ. To prove the second part of the lemma, let μ be a measure on R, quasi-invariant under the action of Γ. Assume without loss of generality that μ is finite. Let $\pi : R \to R/\Gamma_0$ be the natural homomorphism. Put $\nu = \mu \circ \pi^{-1}$. Then ν is a measure on R/Γ_0, quasi-invariant under the action of Γ/Γ_0. It can be verified that μ and $\tilde{\nu}$ are equivalent. q.e.d.

The next lemma is on cocycles. Our cocycles will be taking values in the group $\mathcal{U}(H)$ of unitary operators on a complex separable Hilbert space H. Let ν be a measure on R/Γ_0, quasi-invariant under the action of Γ/Γ_0. Let A be a $(\Gamma/\Gamma_0, R/\Gamma_0)$ cocycle relative to ν. Define $\tilde{A}$ on $\Gamma \times R$ by $\tilde{A}(\gamma,x) = A(\bar{\gamma},\bar{x})$.

LEMMA 2.2. — *$\tilde{A}$ is a (Γ,R) cocycle relative to $\tilde{\nu}$. Two $(\Gamma/\Gamma_0,R/\Gamma_0)$ cocycles A_1 and A_2 are cohomologous if and only if the corresponding (Γ,R) cocycles $\tilde{A}_1$, $\tilde{A}_2$ are cohomologous. Every (Γ,R) cocycle relative to $\tilde{\nu}$ is cohomologous to a cocycle $\tilde{A}$ for some $(\Gamma/\Gamma_0,R/\Gamma_0)$ cocycle A relative to ν.*

Proof. — The first statement is easy to verify and a proof of the second statement runs on natural lines. We prove the third statement. Let B be a (Γ,R) cocycle relative to $\tilde{\nu}$. For each $\gamma \in \Gamma$ we can modify $B(\gamma,.)$ on a $\tilde{\nu}$ null set in such a way that the resulting function, which we denote by B_1, satisfies, for all $\gamma_1, \gamma_2 \in \Gamma$,

$$B_1(\gamma_1 + \gamma_2,x) = B_1(\gamma_1,x)B_1(\gamma_2,x + \gamma_1) \quad \text{a.e. } \tilde{\nu}.$$

The new function B_1 may not be $\Gamma \times R$ Borel measurable but it is such that for any two $\xi, \eta \in H$ and for any Borel set $E \subseteq R$ of finite $\tilde{\nu}$ measure, the function

$$\gamma \to \int_E (A(\gamma,x)\xi,\eta)\, \tilde{\nu}\,(dx)$$

is continuous. Moreover, since Γ_0 is closed in R, and acts on R by translation, B_1 may be so chosen that its restriction to $\Gamma_0 \times R$ is $\Gamma_0 \times R$ Borel measurable, and, indeed a (Γ_0,R) coboundary. We may thus assume that the restriction of B_1 to $\Gamma_0 \times R$ is a strict cocycle. Define the Borel map τ on R by $\tau(x) = B_1([x],\langle x\rangle)$. Let B_2 be defined by

$$B_2(\gamma,x) = \tau(x)B_1(\gamma,x)\tau^*(x + \gamma).$$

If $h, h' \in \Gamma_0$ and $\gamma \in \Gamma$, we have a.e. $\tilde{\nu}$,

$$\begin{aligned}
B_2(\gamma+h,x+h') &= \tau(x+h')B_1(\gamma+h,x+h')\tau^*(x+\gamma+h+h') \\
&= B_1([x]+h',\langle x\rangle)B_1(\gamma+h,x+h')B_1^*([x+\gamma]+h+h',\langle x+\gamma\rangle) \\
&= B_1([x],\langle x\rangle)B_1(h',x)B_1(\gamma+h,x+h')B_1^*(h+h',x+\gamma) \\
&\qquad\qquad B_1^*([x+\gamma],\langle x+\gamma\rangle) \\
&= B_1([x],\langle x\rangle)B_1(\gamma+h+h',x)B_1^*(h+h',x+\gamma) \\
&\qquad\qquad B_1^*([x+\gamma],\langle x+\gamma\rangle) \\
&= B_1([x],\langle x\rangle)B_1(\gamma,x)B_1^*([x+\gamma],\langle x+\gamma\rangle) \\
&= \tau(x)B_1(\gamma,x)\tau^*(x+\gamma) \\
&= B_2(\gamma,x)\,.
\end{aligned}$$

Now B_2 satisfies all the properties of a (Γ,R) cocycle except that it may not be a $\Gamma \times R$ Borel function. On the other hand B_2 satisfies the cocycle identity a.e. $\tilde{\nu}$ for each pair γ_1, $\gamma_2 \in \Gamma$, and the map $\gamma \to \int_E (B_2(\gamma,x)\xi,\eta)\tilde{\nu}\,(dx)$ is continuous for any two ξ, $\eta \in H$ and for any Borel set E of finite $\tilde{\nu}$ measure. A well known lemma [3, Lemma 8.5] permits us to modify B_2, for each $\gamma \in \Gamma$, on a $\tilde{\nu}$ null set in such a way that the new function, which we call B_3, is $\Gamma \times R$ Borel and for λ_Γ almost every γ, $B_2(\gamma,x) = B_3(\gamma,x)$ a.e. $\tilde{\nu}$. We then have for each pair h, $h' \in \Gamma_0$

$$B_3(\gamma + h,x + h') = B_3(\gamma,x) \quad \text{a.e. } \lambda_\Gamma \times \tilde{\nu}\,.$$

One can now show that $\lambda_\Gamma \times \tilde{\nu}$ almost every where, B_3 is equal to an $\tilde{A}$ for a suitable $(\Gamma/\Gamma_0,R/\Gamma_0)$ cocycle A with respect to ν. Finally we note that a.e. $\lambda_\Gamma \times \tilde{\nu}$,

$$\tilde{A}(\gamma,x) = B_3(\gamma,x) = \tau(x)B(\gamma,x)\tau^*(x + \gamma).$$

Hence B is cohomologous to an $\tilde{A}$ for a suitable $(\Gamma/\Gamma_0,R/\Gamma_0)$ cocycle A relative to ν. q.e.d.

Suppose (U,F) is a $(\Gamma/\Gamma_0,R/\Gamma_0)$ system of imprimitivity which is homogeneous. Then there is a quasi-invariant measure ν and a cocycle A (relative to ν) associated with (U,F) which together describe (U,F) (up to equivalence) according to a well known formula [3, Thm 9.7]. The measure $\tilde{\nu}$ and the cocycle $\tilde{A}$ then define a (Γ,R) system of imprimitivity $(\tilde{U},\tilde{F})$ according to the same formula. Second part of lemma 2.1 and the third part of lemma 2.2 show that every homogeneous (Γ,R) system of imprimitivity is equivalent to the system of imprimitivity $(\tilde{U},\tilde{F})$ for a suitable $(\Gamma/\Gamma_0,R/\Gamma_0)$ system of imprimitivity (U,F). This proves the assertion made at the beginning of the section for homogeneous systems. For a general system of imprimitivity the result follows by decomposing the given system into its homogeneous components. We shall call the equivalence class of the system (U,F) the quotient of the equivalence class of the system $(\tilde{U},\tilde{F})$.

3. Gamelin systems.

Now let us consider the dual pairs $((R/\Gamma_0)^\wedge,(\Gamma/\Gamma_0)^\wedge)$ and $(\hat{R},\hat{\Gamma})$. Put $\hat{R} = S$, $\hat{\Gamma} = B$, $(\Gamma/\Gamma_0)^\wedge = K$, the annihilator of Γ_0 in B. Then $(R/\Gamma_0)^\wedge = K \cap S$, the annihilator of Γ_0 in S. Thus the dual pair of $(\Gamma/\Gamma_0,R/\Gamma_0)$ is $(K \cap S,K)$ and that of (Γ,R) is (S,B). In this section we prove that:

Every $(K \cap S,K)$ system of imprimitivity (V,E) gives rise, in a natural fashion, to an (S,B) system of imprimitivity $(\overline{V},\overline{E})$; moreover every (S,B) system of imprimitivity is equivalent to one such.

Lemma 3.1. — *A Borel section* C *of* S *with respect to* $K \cap S$ *is at the same time a Borel section of* B *with respect to* K.

Proof. — Let C be a Borel section of S with respect to $K \cap S$. Since S is a Borel subset of B, C is a Borel subset of B. Each coset of K can contain almost one element of C, for if $c + K = c_1 + K$ then $c - c_1 \in K \cap S$ whence $c = c_1$. Now the relative topology of Γ_0 in Γ is the same as the relative topology of Γ_0 in R. Therefore, B/K and $S/K \cap S$ are topologically isomorphic, both being duals of Γ_0. Hence, given a $b \in B$, there is a $c \in C$ such that $(b + K,\gamma_0) = (c + K \cap S,\gamma_0)$ for all $\gamma_0 \in \Gamma_0$, i.e.,

$$(b,\gamma_0) = (c,\gamma_0) \qquad \text{for all} \qquad \gamma_0 \in \Gamma_0 .$$

Thus $c - b$ annihilates Γ_0, i.e., $c - b \in K$, or $c \in b + K$. Thus every coset of K contains an element of C. Therefore C is a Borel section of B with respect to K. q.e.d.

Throughout the rest of this paper C will stand for a section of S with respect to $K \cap S$. The natural one-one correspondence $\alpha : c \to \bar{c}$ between C and $S/K \cap S$ is a Borel map. A measure on $S/K \cap S$ can be carried over to a measure on C and vice-versa. The Haar measure on $S/K \cap S$ is carried over to a measure on C which we denote by dc, i.e., dc is the measure on C defined by $\lambda_{S/K\cap S} \circ \alpha^{-1}$. This measure is invariant under the group operation on C defined by $c_1 \dot{+} c_2 = \langle c_1 + c_2 \rangle$. Every element z in B can be uniquely written in the form $z = c + y$, $c \in C$, $y \in K$, and the map $\eta : c + y \to (c,y)$ is a Borel isomorphism between B and $C \times K$. Restriction of η to S is a Borel isomorphism between S and $C \times (K \cap S)$.

Lemma 3.2. — *Let* μ *be a measure on* K, *quasi-invariant under the action of* $K \cap S$. *Then*

$$\bar{\mu} : A \to (dc \times \mu)(\eta(A)), \quad A \subseteq B,$$

is quasi-invariant under the action of S *and for any* $s \in S$,

$$(3.1) \qquad \frac{d\bar{\mu}_s}{d\bar{\mu}}(c + y) = \frac{d\mu_{[s+c]}}{d\mu}(y) \quad \text{a.e. } dc \times \mu .$$

Every measure on B, *quasi-invariant under the action of* S, *is equivalent to a measure* $\bar{\mu}$ *for some measure* μ *on* K, *quasi-invariant under* $K \cap S$.

Proof. — We omit the proof of the first part of the lemma as it is similar to the proof of lemma 4.3 on page 294 of [1]. We prove the second part. Let m be a measure on B, quasi-invariant under S. Assume, without any loss, that m is finite. Let μ on K be defined by $\mu(A) = m(\eta^{-1}(C \times A))$, $A \subseteq K$. Then μ is quasi-invariant under $K \cap S$ and $\bar{\mu}$ and m have the same null sets.

q.e.d.

Let A be a $(K \cap S, K)$ cocycle relative to μ, a measure on K quasi-invariant under $K \cap S$. Define $\bar{A}$ on $S \times B$ by

$$(3.2) \qquad \bar{A}(s, c + y) = A([s + c], y), \quad s \in S, \quad c \in C, \quad y \in K .$$

Clearly $\bar{A}$ is a Borel function on $S \times B$. Now the map $(s_1, s_2, c + y) \to ([s_1 + c], [s_2 + \langle s_1 + c\rangle], y)$ maps $\lambda_S \times \lambda_S \times \bar{\mu}$ sets of positive measure onto $\lambda_{K \cap S} \times \lambda_{K \cap S} \times \mu$ sets of positive measure. This, and the fact that

$$[s_1 + s_2 + c] = [s_1 + c] + [s_2 + \langle s_1 + c\rangle],$$

can be used to verify that $\bar{A}$ is an (S,B) cocycle relative to $\bar{\mu}$. Further, one can verify that two $(K \cap S, K)$ cocycles A_1 and A_2 are cohomologous if and only if the corresponding (S,B) cocycles $\bar{A}_1$ and $\bar{A}_2$ are cohomologous. Thus we have shown that every $(K \cap S, K)$ cocycle relative to μ gives rise, in a natural fashion, to an (S,B) cocycle relative to $\bar{\mu}$. Next we show that every (S,B) cocycle D relative to $\bar{\mu}$ is equivalent to a cocycle $\bar{A}$ for some $(K \cap S, K)$ cocycle A relative to μ. If D were a strict cocycle, one could simply take A equal to the restriction of D to $(K \cap S) \times K$ and observe that for $s \in S$, $c + y \in B$ $(c \in C, y \in K)$ we have

$$\begin{aligned} D(s,c+y) &= D^*(c,y) \quad \bar{A}(s,c+y) \quad D(\langle s+c\rangle, y+[s+c]) \\ &= \varphi(c+y) \quad \bar{A}(s,c+y)\varphi^{-1}(c+y+s) \end{aligned}$$

where we have put $D^*(c,y) = \varphi(c+y)$. Unfortunately the cocycle identity is valid only almost everywhere, and may fail to hold on the possible null sets $K \cap S$ and K. An argument involving the Fubini theorem, therefore, becomes imperative. Now

$$D(s_1+s_2,z) = D(s_1,z) \quad D(s_2,z+s_1) \quad \text{a.e.} \quad \lambda_S \times \lambda_S \times \bar{\mu}.$$

Let $s_1 = c + s$, $c \in C$, $s \in K \cap S$. Then

$$\begin{aligned} &(3.3) \quad D(c+s+s_2,z) \\ &\quad = D(c+s,z) \quad D(s_2,z+c+s), \quad \text{a.e.} \quad dc \times \lambda_{K \cap S} \times \lambda_S \times \bar{\mu}. \end{aligned}$$

By Fubini theorem, therefore, there exists an $s_0 \in K \cap S$ such that

$$\begin{aligned} &D(c+s_0+s_2,z) \\ &\qquad = D(c+s_0,z) \quad D(s_2,z+c+s_0) \quad \text{a.e.} \quad dc \times \lambda_S \times \bar{\mu}. \end{aligned}$$

By shifting the section C to $C + s_0$, if necessary, we may assume that

$$D(c+s_2,z) = D(c,z) \quad D(s_2,z+c) \quad \text{a.e.} \quad dc \times \lambda_S \times \bar{\mu}.$$

Now, with $\delta \in C$, we have

$$\begin{aligned}
(3.4) \quad & D(c+s+s_2,z) \\
&= D(-\delta+\langle c+s_2\rangle+s+[c+s_2]+\delta,z) \\
&= D([c+s_2]+\delta,z) \quad D(-\delta+\langle c+s_2\rangle+s,z+[c+s_2]+\delta) \\
&= D([c+s_2]+\delta,z) \quad D(-\delta,z+[c+s_2]+\delta) \\
&\qquad\qquad D(\langle c+s_2\rangle+s,z+[c+s_2])
\end{aligned}$$

where the second equality is valid almost everywhere $dc \times \lambda_{K\cap S} \times \lambda_S \times \bar{\mu} \times dc$, because the map

$$(c,s,s_2,\delta) \to (-\delta+\langle c+s_2\rangle+s,[c+s_2]+\delta)$$

takes $dc \times \lambda_{K\cap S} \times \lambda_S \times dc$ sets of positive measure onto $\lambda_S \times \lambda_S$ sets of positive measure. This can be seen by breaking the map as follows :

$$\begin{aligned}
(c,s,s_2,\delta) &\longrightarrow (\langle c+s_2\rangle,s,[c+s_2],\delta) \\
&\longleftrightarrow (\langle c+s_2\rangle+s,[c+s_2],\delta) \\
&\longleftrightarrow (\langle c+s_2\rangle+s+\delta,[c+s_2]-\delta)
\end{aligned}$$

and verifying the statement at each stage. Comparing the right hand side of (3.3) with the last term of (3.4) we get

$$\begin{aligned}
D(c+s,z) \quad & D(s_2,z+c+s) \\
&= D([c+s_2]+\delta,z) \quad D(-\delta,z+[c+s_2]+\delta) \\
&\qquad D(\langle c+s_2\rangle+s,z+[c+s_2])
\end{aligned}$$

where the equality holds a.e. $dc \times \lambda_{K\cap S} \times \lambda_S \times \bar{\mu} \times dc$. Therefore, a.e. with respect to the same measure,

$$\begin{aligned}
D(c+s,z) \quad D(s_2,z+c+s) \quad & D^*(\langle c+s_2\rangle+s,z+[c+s_2]) \\
&= D([c+s_2]+\delta,z) \quad D(-\delta,z+[c+s_2]+\delta).
\end{aligned}$$

The left hand side of the above identity is independent of δ, hence outside a dc null set the right hand side is independent of δ, and it is a function only on $(K \cap S) \times B$. We denote this function by D_1. If we now put $z = c_1 + y$,

$c_1 \in \mathrm{C}$, $y \in \mathrm{K}$, we get

$$\begin{aligned}&\mathrm{D}(c+s,c_1+y) \quad \mathrm{D}(s_2,y+c+c_1+s)\\&\mathrm{D}^*(\langle c+s_2\rangle+s,c_1+y+[c+s_2])\\&\qquad = \mathrm{D}_1([c+s_2],c_1+y), \quad \text{a.e.} \quad dc \times \lambda_{\mathrm{K}\cap\mathrm{S}} \times \lambda_{\mathrm{S}} \times dc \times \mu .\end{aligned}$$

By Fubini theorem we can find suitable c_1 , s , say $c_1 = c_0$ and $s = s_0$, such that a.e. $dc \times \lambda_{\mathrm{S}} \times \mu$ we have :

$$\begin{aligned}&\mathrm{D}(c+s,c_0+y) \quad \mathrm{D}(s_2,y+c+c_0+s_0)\\&\mathrm{D}^*(\langle c+s_2\rangle+s_0,c_0+y+[c+s_2]) = \mathrm{D}_1([c+s_2],c_0+y).\end{aligned}$$

If we put $\varphi(c+y) = \mathrm{D}(c+s_0,c_0+y)$, we get

$$\begin{aligned}&\varphi(c+y) \quad \mathrm{D}(s_2,y+c+c_0+s_0) \quad \varphi^*(c+y+s_2)\\&\qquad = \mathrm{D}_1([c+s_2],c_0+y), \quad \text{a.e.} \quad dc \times \lambda_{\mathrm{S}} \times \mu .\end{aligned}$$

Thus the function D_2 on $\mathrm{S} \times \mathrm{B}$ defined by

$$\mathrm{D}_2(s_2,c+y) = \mathrm{D}_1([s_2+c],y)$$

is a cocycle cohomologous to a shifted D , hence cohomologous to D . Finally we show that D_1 is a $(\mathrm{K} \cap \mathrm{S},\mathrm{K})$ cocycle relative to μ :

$$\begin{aligned}&\mathrm{D}_1([s_1+c]+[s_2+\langle s_1+c\rangle],y)\\&\quad = \mathrm{D}_1([s_1 + s_2 + c],y)\\&\quad = \mathrm{D}_2(s_1 + s_2,c + y)\\&\quad = \mathrm{D}_2(s_1,c+y) \quad \mathrm{D}_2(s_2,c+y+s_1), \quad \text{a.e.} \quad \lambda_{\mathrm{S}} \times \lambda_{\mathrm{S}} \times dc \times \mu\\&\quad = \mathrm{D}_1([s_1+c],y) \quad \mathrm{D}_1([s_2+\langle c+s_1\rangle],y+[c+s_1]).\end{aligned}$$

We now observe that the map

$$(s_1,s_2,c,y) \to ([s_1+c],[s_2+\langle s_1+c\rangle],y)$$

takes $\lambda_{\mathrm{S}}\times\lambda_{\mathrm{S}}\times dc\times\mu$ sets of full measure onto $\lambda_{\mathrm{K}\cap\mathrm{S}}\times\lambda_{\mathrm{K}\cap\mathrm{S}}\times\mu$ sets of full measure, and hence that D_1 is a $(\mathrm{K} \cap \mathrm{S},\mathrm{K})$ cocycle relative to μ . If we set $\mathrm{D}_1 = \mathrm{A}$, then clearly D is cohomologous to $\overline{\mathrm{A}}$.

Remark. — For the special case when B is compact and $\Gamma(=\hat{\mathrm{B}})$ is a dense sub-group of the real line and the cocycles are complex valued these results were proved by Gamelin [2, p. 182]. For the special case Gamelin even establishes that the cocycles have a strict version [2, p. 187]. In our context it may be mentioned that, if $K \cap S$ is countable, then every (S,B) cocycle has a strict version because every $(K \cap S,K)$ cocycle then has a strict version.

Notation. — Given a $(K \cap S,K)$ cocycle A we shall call cocycle $\bar{A}$ the Gamelin cocycle obtained from A.

Now let (V,E) be a homogeneous $(K \cap S,K)$ system of imprimitivity. Then there is a quasi-invariant measure μ on K and a $(K \cap S,K)$ cocycle A relative to μ, which together describe (V,E) (up to equivalence) according to a well known formula (cf eqn. 4.3). The measure $\bar{\mu}$ and the cocycle $\bar{A}$ then define a homogeneous (S,B) system of imprimitivity $(\bar{V},\bar{E})$ (cf eqn. 4.4). The second part of lemma 3.2 and the fact that every (S,B) cocycle is cohomologous to the cocycle $\bar{A}$ for some $(K \cap S,K)$ cocycle A, shows that every homogeneous (S,B) system of imprimitivity is equivalent to the system $(\bar{V},\bar{E})$ for a suitable $(K \cap S,K)$ system (V,E). This proves the statement made at the beginning of the section for a homogeneous system of imprimitivity. For a general system, the result follows by decomposing the given system into its homogeneous components. We shall call $(\bar{V},\bar{E})$ the Gamelin system of imprimitivity obtained from (V,E).

4. Commutativity (under a mild assumption).

Let (V,E) be a homogeneous $(K \cap S,K)$ system of imprimitivity. Let $(\bar{V},\bar{E})$ be the Gamelin system of imprimitivity obtained from (V,E). Let (U,F) and $(\bar{U},\bar{F})$ be the duals of (V,E) and $(\bar{V},\bar{E})$ respectively. Let $(\tilde{U},\tilde{F})$ be the (Γ,R) system of imprimitivity obtained from the $\Gamma/\Gamma_0,R/\Gamma_0)$ system (U,F) according to the procedure of § 2. Thus we have diagram 1 and we obtain two (Γ,R) systems of imprimitivity $(\bar{U},\bar{F})$ and $(\tilde{U},\tilde{F})$ starting from the same $(K \cap S,K)$ system of imprimitivity (V,E). Under a mild assumption we now prove that $(\bar{U},\bar{F})$ and $(\tilde{U},\tilde{F})$ are equivalent.

Recall that Q and C stand for a Borel section of R with respect to Γ_0 and a Borel section of S with respect to $K \cap S$

respectively. It is clear that the map $g \to g \circ \alpha^{-1}$ is an isometry between $L^2(C,dc)$ and $L^2(S/K \cap S$, Haar measure), where $g \in L^2(C,dc)$, and α is the natural map from C to $S/K \cap S$. For $h \in L^2(C,dc)$ we first define $\hat{h}$ on Γ_0 by $\hat{h}(\gamma_0) = \widehat{h \circ \alpha^{-1}}(\gamma_0)$, where on the right hand side we have the Fourier-Plancherel transform of $h \circ \alpha^{-1}$. $\hat{h}$ is thus defined a.e. λ_0. We extend this definition of $\hat{h}$ to the whole of R by defining $\hat{h}(r) = ((q,.)h(.))\hat{}(\gamma_0)$ where $r = q + \gamma_0$, $q \in Q$, $\gamma_0 \in \Gamma_0$. If $h \in L^1(C,dc) \cap L^2(C,dc)$ then indeed we have

$$\hat{h}(r) = \int_C (r,c)h(c)\, dc = \int_C (\gamma_0,c)(q,c)h(c)\, dc\,.$$

Suppose $h \in L^2(C,dc)$ is such that $\hat{h}|_{\Gamma_0}$, i.e., the restriction of $\hat{h}$ to Γ_0, is non-vanishing λ_0 almost every where. Then by Wiener's theorem (by passing to $S/K \cap S$ and back) the collection $\{h(\langle .+c\rangle) : c \in C\}$ spans $L^2(C,dc)$. Hence the collection $\{h(\langle .+s\rangle) : s \in S\}$ spans $L^2(C,dc)$. Let ν be a finite measure on R/Γ_0. We regard ν as a measure on Q as well, although in fact we mean the measure $\nu \circ \alpha'$, where α' is the natural map from Q to R/Γ_0. Suppose $h \in L^2(C,dc)$ is such that the function $\hat{h}$ on R is $\tilde{\nu}$ a.e. non-vanishing, where, as in § 2, $\tilde{\nu} = (\nu \times \lambda_0) \circ \delta$. By Fubini theorem, then, for ν almost every $y \in Q$, $((y,.)h(.))\hat{}|_{\Gamma_0}$ is non-vanishing λ_0 a.e. on Γ_0. Hence for ν almost every $y \in Q$, $\{(y,\langle .+s\rangle)h(\langle .+s\rangle) : s \in S\}$ spans $L^2(C,dc)$. Thus we have:

Proposition 4.1: *Suppose* $h \in L^2(C,dc)$ *is such that* $\hat{h}$ *is non-vanishing* $\tilde{\nu}$ *a.e. Then for* ν *almost every* $y \in Q$, *the collection* $\{(y,\langle .+s\rangle)h(\langle .+s\rangle) : s \in S\}$ *spans* $L^2(C,dc)$.

Let $h \in L^2(C,dc)$ and put $h^* =$ complex conjugate of h,

$$f(x+\gamma_0) = (h^*)\hat{}(x+\gamma_0), \quad x \in Q\,, \quad \gamma_0 \in \Gamma_0\,,$$
$$F_g(x+\gamma_0) = f(x+\gamma_0)f^*(x+\gamma_0-g), \quad g \in \Gamma\,,$$

where $*$ again stands for the complex conjugate.

Let ν be a finite complex valued measure on R/Γ_0 and let $\tilde{\nu}$ denote the complex valued countably additive function on the class of Borel subsets of R with compact closure, defined by

$$(4.1) \qquad \tilde{\nu}(\mathrm{A}) = \int_{\mathrm{A}} \frac{d\nu}{d|\nu|}(\pi r)|\tilde{\nu}|\,(dr)$$

where $|\nu|$ denotes the total variation measure of ν. The symbol $\mathrm{F}_g\,d\tilde{\nu}$ will denote the complex valued measure on B, defined by

$$\mathrm{A} \to \int_{\mathrm{A}} \mathrm{F}_g(r)\,\frac{d\nu}{d|\nu|}\,(\pi r)|\tilde{\nu}|\,(dr).$$

If we consider the measure on Q given by $\mathrm{A} \to \nu(\alpha'(\mathrm{A}))$, $\mathrm{A} \subseteq \mathrm{Q}$, which we continue to denote by ν, then the measure $\mathrm{F}_g\,d\tilde{\nu}$ is also given by

$$\mathrm{A} \to \int_{\mathrm{A}}\int \mathrm{F}_g(x+\gamma_0)\,\frac{d\nu}{d|\nu|}\,(x)\lambda_0(d\gamma_0)|\nu|\,(dx).$$

We shall use this form of the measure $\mathrm{F}_g\,d\tilde{\nu}$ to calculate its Fourier-Stieltjes transform.

LEMMA 4.1. — *For each* $t \in \mathrm{S}$,

$$(4.2) \qquad (\mathrm{F}_g\,d\tilde{\nu})\hat{}\,(t) = \int_{\mathrm{C}} \hat{\nu}([t+c])\ (g,\langle t+c\rangle)\ h(\langle t+c\rangle)\ h^*(c)\,dc\,.$$

Proof. — We use the Plancherel theorem (transferred to C) in the third equation below. Now

$$\begin{aligned}
&(\mathrm{F}_g\,d\tilde{\nu})\hat{}\,(t)\\
&= \int_{\mathrm{Q}}\int_{\Gamma_0} (x+\gamma_0,t)f(x+\gamma_0)f^*(x+\gamma_0-g)\,\frac{d\nu}{d|\nu|}\,(x)\lambda_0(d\gamma_0)|\nu|(dx)\\
&= \int_{\mathrm{Q}} (x,t)\int_{\Gamma_0} f(x+\gamma_0)(\gamma_0,\langle t\rangle)f^*(x+\gamma_0-g)\,\frac{d\nu}{d|\nu|}\,(x)\lambda_0(d\gamma_0)|\nu|(dx)\\
&= \int_{\mathrm{Q}} (x,t)\int_{\mathrm{C}} (x,c)\\
&\qquad h^*(c)(x,-\langle t+c\rangle)(g,\langle t+c\rangle)h(\langle t+c\rangle)\,\frac{d\nu}{d|\nu|}\,(x)dc\,|\nu|(dx)\\
&= \int_{\mathrm{C}}\left\{\int_{\mathrm{Q}}(x,[t+c])\,\frac{d\nu}{d|\nu|}\,(x)|\nu|\,(dx)\right\}(g,\langle t+c\rangle)h(\langle t+c\rangle)h^*(c)\,dc\\
&= \int_{\mathrm{C}} \hat{\nu}([t+c])(g,\langle t+c\rangle)h(\langle t+c\rangle)h^*(c)\,dc\,.
\end{aligned}$$

q.e.d.

Let (V,E) be a homogeneous $(K \cap S, K)$ system of imprimitivity. Let H be a Hilbert space of dimension same as the multiplicity of E. Let μ be a measure on K, quasi-invariant under $K \cap S$. Let A be a $(K \cap S, K, \mathscr{U}(H))$ cocycle relative to μ such that μ and A together describe a system equivalent to (V,E). There is no loss if we assume that (V,E) acts in $L^2(K,H,\mu)$ as follows: For $\varphi \in L^2(K,H,\mu)$,

$$(4.3) \quad \begin{cases} (V_s\varphi)(.) = A(s,.)\sqrt{\dfrac{d\mu_s}{d\mu}}(.)\varphi(.+s), & s \in K \cap S \\ E(\sigma)\varphi = 1_\sigma\varphi\,, \quad \sigma \subseteq K\,, \end{cases}$$

where 1_σ is the function equal to one on σ and zero outside. The system $(\overline{V},\overline{E})$ then acts on $L^2(B,H,\overline{\mu})$ as follows:
If $\varphi \in L^2(B,H,\overline{\mu})$, then

$$(4.4) \quad \begin{cases} (\overline{V}_s\varphi)(.) = \overline{A}(s,.)\sqrt{\dfrac{d\overline{\mu}_s}{d\overline{\mu}}}(.)\varphi(.+s), & s \in S \\ \overline{E}(\sigma)\varphi = 1_\sigma\varphi\,, \quad \sigma \subseteq B\,. \end{cases}$$

Let (.,.) denote the inner product in $L^2(K,H,\mu)$ which will cause no confusion with the same symbol used to denote characters. Let ((.,.)) denote the inner product in $L^2(B,H,\overline{\mu})$. If $\varphi \in L^2(B,H,\overline{\mu})$ and $c \in C$, then φ_c will denote the function in $L^2(K,H,\overline{\mu})$ defined by $\varphi_c(y) = \varphi(c+y)$, $y \in K$. A routine calculation using (3.1) and (3.2) shows that, for any two elements φ, ξ in $L^2(B,H,\overline{\mu})$,

$$(4.5) \qquad ((\overline{V}_s\varphi,\xi)) = \int_C (V_{[s+c]}\varphi_{\langle s+c\rangle}, \xi_c)\, dc\,.$$

Assume now that (U,F) is a homogeneous system of imprimitivity of multiplicity n. In case (U,F) is not homogeneous, the arguments below can be carried out for each homogeneous component. Let I denote the set of first n natural numbers if n is finite, and the set of all natural numbers if $n = \aleph_0$. Since F is homogeneous of multiplicity n there are n functions h_i, $i \in I$, in $L^2(K,H,\mu)$ such that for all $s \in S$

(i) $(V_s h_i, h_j) = 0$ if $i \neq j$,

(ii) the measures $\nu_i(.) = (F(.)h_i,h_i)$ are all equal. (We denote the common measure by ν .)

(iii) the closed linear span of $\{V_s h_i : i \in I , s \in K \cap S\}$ is $L^2(K,H,\mu)$.

Let $h \in L^2(C,dc)$.

Let $\bar{h}_i(c+x) = h_i(x)h(c)$, $c \in C$, $x \in K$.

Then $\bar{h}_i \in L^2(B,H,\bar{\mu})$. An easy computation using (4.5) shows that for all $s \in S$,

(1) $((\bar{V}_s\bar{h}_i,\bar{h}_j)) = 0$ if $i \neq j$,

(2) the measures $\bar{\nu}_i(.) = ((\bar{F}(.)\bar{h}_i,\bar{h}_i))$ are all equal. (We denote the common measure by $\bar{\nu}$.)

We now state the assumption under which we prove the equivalence of $(\bar{U},\bar{F})$ and $(\tilde{U},\tilde{F})$.

Assumption A. — There is a function h in $L^2(C,dc)$ such that for all $y \in Q$, $\{(.,y)h(.)\}^\wedge$ is non-vanishing λ_0 a.e. on Γ_0 .

This assumption is satisfied in many cases but we do not know if it is always satisfied. Next lemma is the only place where it is used. Note that if h is as in assumption A then h satisfies the condition of proposition 4.1.

LEMMA 4.2. — *If h is as in assumption A then the collection $\{\bar{V}_s\bar{h}_i : i \in I, s \in S\}$ spans $L^2(B,H,\bar{\mu})$, (i.e., under assumption A, $(\bar{U},\bar{F})$ is also homogeneous of multiplicity n).*

Proof. — Let $\varphi \in L^2(B,H,\bar{\mu})$ be such that for all $i \in I$ and $s \in S$, $((\bar{V}_s\bar{h}_i,\varphi)) = 0$. We show that φ is the null function. Now, by (4.5),

$$\begin{aligned} 0 &= ((\bar{V}_s\bar{h}_i,\varphi)) \\ &= \int_C (V_{[s+c]}\bar{h}_i,\varphi_c)h(\langle s+c\rangle)\, dc \\ &= \int_C \Big(\int_Q (y,-[s+c])B_i(y,c)\nu\,(dy)\Big)h(\langle s+c\rangle)\, dc , \end{aligned}$$

where $B_i(.,.)$ is a jointly Borel function such that for dc almost every c , $B_i(.,c)$ is the Radon-Nikodym derivative of the measure $(F(.)h_i,\varphi_c)$ with respect to ν . By Fubini theorem we have

$$0 = \int_Q (y,-[s])\Big(\int_C (y,-[\langle s\rangle+c])B_i(y,c)h(\langle\langle s\rangle+c\rangle)\, dc\Big)\nu\,(dy).$$

If we fix $\langle s\rangle$ and let $[s]$ run over $K \cap S$ we see that for ν almost every $y \in Q$,

$$0 = \int_C (y,-[\langle s\rangle + c])B_i(y,c)h(\langle\langle s\rangle + c\rangle)\, dc\,.$$

Or, since $\langle\langle s\rangle + c\rangle = \langle s + c\rangle$,

$$(4.6)\quad 0 = \int_C (y,-\langle s\rangle)(y,-c)B_i(y,c)(y,\langle s+c\rangle)h(\langle s+c\rangle)\, dc\,,$$

which now holds for all $s \in S$ and ν almost every $y \in Q$. Since h is as in assumption A, it satisfies the condition of proposition 4.1. Therefore, by (4.6), for ν almost every y, $B_i(y,.)$ is a null function. Hence dc a.e., $B_i(.,c)$ is a null function. Since this holds for every i, in view of the choice of the functions h_i, we see that dc a.e. φ_c is a null function in $L^2(K,H,\mu)$. Whence φ is null in $L^2(B,H,\bar{\mu})$.

q.e.d.

From now on h will always be according to assumption A.

Let $g \in \Gamma$, $g = \langle g\rangle + [g]$, $\langle g\rangle \in Q \cap \Gamma$, $[g] \in \Gamma_0$. Put $\langle g\rangle = u$, $[g] = \gamma_0$. Define complex valued measures $\nu_{\bar{u}}^{i,j}$ on R/Γ_0 and $\bar{\nu}_g^{i,j}$ on R as follows:

$$\nu_{\bar{u}}^{i,j}(.) = (F(.)U_{\bar{u}}h_i,h_j) = (F(.)\chi_{\bar{u}}h_i,h_j),$$
$$\bar{\nu}_g^{i,j}(.) = ((\bar{F}(.)\bar{U}_g\bar{h}_i,\bar{h}_j)) = ((\bar{F}(.)\chi_g\bar{h}_i,\bar{h}_j))\,,$$

where $\chi_{\bar{u}}$ and χ_g are the characters on K and B respectively, defined by $\bar{u}$ and g. (Here $\bar{u}$ is the natural element in Γ/Γ_0 corresponding to $u = \langle g\rangle$.).

Remark. — The complex valued measures $\nu_{\bar{u}}^{ij}$ and $\bar{\nu}_g^{ij}$ are not the translates of the complex valued measures

$$\nu_0^{ij}(=\nu^{ij}) \qquad \text{and} \qquad \bar{\nu}_0^{ij}(=\bar{\nu}^{ij}).$$

LEMMA 4.3. — *$\bar{\nu}_g^{ij}$ is absolutely continuous with respect to $\tilde{\nu}_{\bar{u}}^{ij}$ and*

$$(4.7)\qquad \frac{d\bar{\nu}_g^{ij}}{d\tilde{\nu}_{\bar{u}}^{ij}} = F_g$$

on compact subsets of R.

Proof. — For $t \in \mathrm{S}$ we have by (4.5)

$$\begin{aligned}\hat{\bar{\nu}}_g^{ij}(t) &= ((\bar{\mathrm{V}}_t\chi_g\bar{h}_i,\bar{h}_j)) \\ &= \int_{\mathrm{C}} (\mathrm{V}_{[t+c]}\chi_{\bar{u}}h_i,h_j)\chi_g(\langle t+c\rangle)h(\langle t+c\rangle)h^*(c)\,dc \\ &= (\mathrm{F}_g\,d\tilde{\nu}_{\bar{u}}^{ij})^\wedge(t).\end{aligned}$$

Thus the measures $\bar{\nu}_g^{ij}$ and $\mathrm{F}_g\,d\tilde{\nu}_{\bar{u}}^{ij}$ are the same because their Fourier-Stieltjes transforms are the same. Further on compact subsets of R we have

$$\frac{d\bar{\nu}_g^{ij}}{d\tilde{\nu}_{\bar{u}}^{ij}} = \mathrm{F}_g\,. \qquad \text{q.e.d.}$$

Remark. — We have to say that (4.7) is valid only for compact subsets of R because $\tilde{\nu}_{\bar{u}}^{ij}$ is not a complex valued measure on Borel subsets of whole of R but only on Borel subsets of a compact set. But now we define the left hand side of (4.7) on the whole of R to be equal to the right hand side.

If we take $i = j$, and $g = 0 \in \Gamma$ we get,

Corollary. — *The measures* $\bar{\nu}$ *and* $\tilde{\nu}$ *are equivalent and*

$$\frac{d\bar{\nu}}{d\tilde{\nu}} = |f|^2. \tag{4.8}$$

This corollary and lemma 4.2 together show that $\bar{\mathrm{F}}$ and $\tilde{\mathrm{F}}$ are equivalent spectral measures. It remains to show that the cocycles associated with $\bar{\mathrm{U}}$ and $\tilde{\mathrm{U}}$ are cohomologous. To this end let H_n denote the space of complex n-tuples or l^2 according as n is finite or equal to $\aleph_0$. Let $\langle .,. \rangle$ denote the inner product in H_n. Let T denote the isometric isomorphism between $\mathrm{L}^2(\mathrm{K},\mathrm{H},\mu)$ and $\mathrm{L}^2(\mathrm{R}/\Gamma_0,\mathrm{H}_n,\nu)$ defined by

$$\mathrm{TF}(\mathrm{X})h_i = (\ldots,0,\ldots,1_{\mathrm{X}},\ldots,0,\ldots), \quad i \in \mathrm{I}\,, \quad \mathrm{X} \subseteq \mathrm{R}/\Gamma_0$$

(where 1_{X} appears at i^{th} place, and zeros elsewhere). $(\mathrm{TUT}^{-1},\mathrm{TFT}^{-1})$ is a $(\Gamma/\Gamma_0,\mathrm{R}/\Gamma_0)$ system of imprimitivity acting in $\mathrm{L}^2(\mathrm{R}/\Gamma_0,\mathrm{H}_n,\nu)$ as follows: there is a $(\Gamma/\Gamma_0,\mathrm{R}/\Gamma_0,\mathcal{U}(\mathrm{H}_n))$ cocycle D relative to ν such that

$$(\mathrm{TU}_{\bar{u}}\mathrm{T}^{-1}\varphi)(y) = \mathrm{D}(-\bar{u},y)\sqrt{\frac{d\nu_{-\bar{u}}}{d\nu}}\,\varphi(y-\bar{u})\,,$$
$$\mathrm{TF}(\mathrm{X})\mathrm{T}^{-1}\varphi = 1_{\mathrm{X}}\varphi\,,$$

where $\varphi \in L^2(R/\Gamma_0, H_n, \nu)$. D is a matrix valued function whose entries d_{ij} may be computed as follows:

$$\begin{aligned}
\nu^{ij}_{\bar{u}}(X) &= (F(X)\chi_{\bar{u}}h_i, h_j) = (F(X)\chi_{\bar{u}}h_i, F(X)h_j) \\
&= (TF(X)T^{-1}T\chi_{\bar{u}}T^{-1}Th_i, TF(X)T^{-1}Th_j) \\
&= \int_X \left\langle D(-\bar{u}, y)\sqrt{\frac{d\nu_{-\bar{u}}}{d\nu}}(y)(Th_i)(y-\bar{u}),\ (Th_j)(y)\right\rangle \nu\,(dy) \\
&= \int_X d_{ij}(-\bar{u}, y)\sqrt{\frac{d\nu_{-\bar{u}}}{d\nu}}(y)\nu\,(dy)\,.
\end{aligned}$$

Thus

$$d_{ij}(\bar{u}, y) = \frac{d\nu^{ij}_{-\bar{u}}}{d\nu}\cdot\sqrt{\frac{d\nu}{d\nu_{\bar{u}}}}(y)\,.$$

Similarly let $\overline{T}$ denote the isometric isomorphism between $L^2(B, H, \bar{\mu})$ and $L^2(R, H_n, \bar{\nu})$ defined by:

$$\overline{T}\overline{F}(X)\bar{h}_i = (\ldots, 0, \ldots, 1_X, \ldots, 0, \ldots), \quad i \in I\,, \quad X \subseteq R\,.$$

Then $(\overline{T}\overline{U}\overline{T}^{-1}, \overline{T}\overline{F}\overline{T}^{-1})$ is a (Γ, R) system of imprimitivity acting in $L^2(R, H_n, \bar{\nu})$ as follows: there is a $(\Gamma, R, \mathcal{U}(H_n))$ cocycle $\overline{D}$ such that for $\varphi \in L^2(R, H_n, \bar{\nu})$,

$$\begin{gathered}
(\overline{T}\overline{U}_g\overline{T}^{-1}\varphi)(z) = \overline{D}(-g, z)\sqrt{\frac{d\bar{\nu}_{-g}}{d\bar{\nu}}}(z)\varphi(z-g), \\
\overline{T}\overline{F}(X)\overline{T}^{-1}\varphi = 1_X\varphi\,.
\end{gathered}$$

Exactly as above $\overline{D}$ is a matrix valued cocycle whose entries are

$$\bar{d}_{ij}(g, z) = \frac{d\bar{\nu}^{ij}_{-g}}{d\bar{\nu}}(z)\sqrt{\frac{d\bar{\nu}}{d\bar{\nu}_g}}(z)\,.$$

Now

$$\begin{aligned}
\bar{d}_{ij}(g, .) &= \frac{d\bar{\nu}^{ij}_{-g}}{d\tilde{\nu}^{ij}_{-\bar{u}}}\frac{d\tilde{\nu}^{ij}_{-\bar{u}}}{d\tilde{\nu}}\frac{d\tilde{\nu}}{d\bar{\nu}}\sqrt{\frac{d\bar{\nu}}{d\tilde{\nu}}\frac{d\tilde{\nu}}{d\tilde{\nu}_{\bar{u}}}\frac{d\tilde{\nu}_{\bar{u}}}{d\bar{\nu}_g}} \\
&= \frac{d\bar{\nu}^{ij}_{-g}}{d\tilde{\nu}^{ij}_{-\bar{u}}}\frac{d\tilde{\nu}}{d\bar{\nu}}\sqrt{\frac{d\bar{\nu}}{d\tilde{\nu}}\frac{d\tilde{\nu}_{\bar{u}}}{d\bar{\nu}_g}}\,\tilde{d}_{ij}(g, .)\,,
\end{aligned}$$

where

$$\tilde{d}_{ij}(g,.) = \frac{d\tilde{\nu}^{ij}_{-\bar{u}}}{d\nu}\sqrt{\frac{d\tilde{\nu}}{d\tilde{\nu}_{\bar{u}}}} = d_{ij}(\bar{u},\pi(.))$$

are the entries of the cocycle $\tilde{D}$ obtained from D as in lemma 2.2, (π being the natural homomorphism of R onto R/Γ_0). By (4.7) and (4.8) we get

$$\bar{d}_{ij}(g,.) = \frac{\sqrt{\dfrac{d\bar{\nu}}{d\tilde{\nu}}}}{f^*} \frac{f^*(.+g)}{\sqrt{\dfrac{d\bar{\nu}}{d\tilde{\nu}}(.+g)}} \tilde{d}_{ij}(g,.) .$$

Or,

$$\bar{d}_{ij}(g,x) = H(x)\, \tilde{d}_{ij}(g,x) H^{-1}(g+x), \quad x \in R$$

where

$$H(x) = \frac{\sqrt{\dfrac{d\bar{\nu}}{d\tilde{\nu}}(x)}}{f^*(x)} = \frac{\sqrt{|f(x)|^2}}{f^*(x)},$$

a function of modulus one on R. Hence $\bar{D}$ and $\tilde{D}$ are cohomologous. Thus we have proved the following theorem for a homogeneous system (V,E).

Theorem 4.1. — *Let* (V,E) *be a* $(K \cap S, K)$ *system of imprimitivity and* (U,F) *its dual. Let* $(\bar{V},\bar{E})$ *be the Gamelin system of imprimitivity obtained from* (V,E) *and let* $(\bar{U},\bar{F})$ *be its dual. Let* $(\tilde{U},\tilde{F})$ *be the* (Γ,R) *system of imprimitivity of which* (U,F) *is a quotient. Then, under assumption* A, $(\tilde{U},\tilde{F})$ *and* $(\bar{U},\bar{F})$ *are equivalent systems of imprimitivity.*

The case of general (V,E) follows on decomposing it into its homogeneous components and applying theorem 4.1 to each component.

5. On assumption « A ».

Assumption A would be redundant if we knew that given a non-zero finite measure ν on a second countable locally compact Abelian group B, there exists a Borel function f

on B such that $(f\,d\nu)^\wedge$ never vanishes. We do not know the truth or falsity of this. For compact B it is true. For non compact group one can find an f such that the set of zeros of $(f\,d\nu)^\wedge$ has Haar measure zero. We prove this statement here. There is no loss if we assume that $B = R^n \times L$ where R^n is the Euclidean n space and L is compact with a countable discrete dual $D = \{d_1, d_2, d_3 \ldots\}$. Then $\hat{B} = R^n \times D$. Assume without loss that ν is a finite non-zero measure with compact support. By using disintegration of measure we can get, for each $r \in R^n$, a measure ν_r on L such that for each Borel $A \subseteq L$, $\nu_r(A)$ is a Borel function of r and for any Borel $A \subseteq B$

$$\nu(A) = \int_{R^n} \nu_r(A_r)\mu\,(dr)$$

where A_r is the r-section of A and μ is the measure on R^n given by $\mu(\sigma) = \nu(\sigma \times L)$, $\sigma \subseteq R^n$. Let

$$Q = [0,1] \times [0,1/2] \times [0,1/4] \ldots,$$

and, let $q = (q_n)_{n=1}^{\infty}$ represent an element in Q. For each integer k, let $M_k = \left\{ q : \sum_{n=1}^{\infty} q_n \hat{\nu}_r\,(d_n + d_k) \text{ is a } \mu \text{ null function of } r \right\}$. Then M_k is closed nowhere dense in Q. (Q is given the product topology.) Hence, by Baire category theorem, we can find a $q = (q_n)_{n=1}^{\infty}$ such that $\sum_{n=1}^{\infty} \hat{\nu}_r\,(d_n + d_k)$ is, for each k, a non-null function (with respect to μ). Define f on $R^n \times L$ by $f(r,l) = \sum_{i=1}^{\infty} q_i(l, d_i)$; then

$$(f\,d\nu)^\wedge(t,d) = \int_{R^n} (t,r)\widehat{f\,d\nu_r}\,(d)\mu\,(dr)\,.$$

Now for each $d \in D$, $(f\,d\nu)^\wedge(t,d)$ is the Fourier-Stieltjes transform of a measure on R^n with compact support, hence it is analytic in t. Hence for each $d \in D$ the set of t for which $(f\,d\nu)^\wedge(t,d) = 0$ has Haar measure zero in R^n. Hence the set of zeros of $(f\,d\nu)^\wedge$ has Haar measure zero in B.

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