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ON THE MARTIN COMPACTIFICATION OF A BOUNDED LIPSCHITZ DOMAIN IN A RIEMANNIAN MANIFOLD (1)

by J.-C. TAYLOR

0. Introduction.

Let D be a bounded Lipschitz domain in $\mathbf{R}^n$. In [5] Hunt and Wheeden showed that, for the Laplacian, the Martin compactification of D is $\overline{D}$. In this article the same result is obtained for a class $\Lambda(\gamma, \mu; D)$ of second order elliptic operators L defined on D.

The operator $L = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i} + c(x)$ is in $\Lambda(\gamma, \mu; D)$ if the coefficients are uniformly Hölder continuous on D and such that:

(1) $$1/\gamma \|\xi\|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \leqslant \gamma\|\xi\|^2$$

for all $\xi \in \mathbf{R}^n$ and $x \in D$; for all $x \in D$,

(2) $$\sum_{i=1}^{n} b_i^2(x) \leqslant \mu ;$$

and

(3) $$-\mu \leqslant c(x) \leqslant 0.$$

While the proof of this theorem is essentially the same as

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that given in [5] for the Laplacian, some necessary technical modifications are imposed since the coefficients are variable. In addition, greater use is made of the abstract theory of the Martin compactification. Besides Serrin's result on the Harnack inequality [12], the main technical tool used is Miller's result on the existence of a universal barrier for operators $L \in \Lambda(\gamma, \mu; D)$. It is worth noting that the proof given here is not valid for self-adjoint operators $L = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(\sum_{j=1}^{n} a_{ij}(x)\frac{\partial}{\partial x_j}\right)$ with $(a_{ij}(x))$ uniformly elliptic and measurable simply because there is no universal barrier available for this class of operators on D. Presumably the theorem is still true in this case.

Let X be a Riemannian manifold and let $M \subset X$ be open, relatively compact, connected with non-void Lipschitz boundary. By considering the Laplace-Beltrami operator Δ in a suitable coordinate neighbourhood of an arbitrary boundary point and applying the above result, one can prove that M is the Martin compactification of $\overline{M}$ relative to Δ (*).

An interesting consequence of this result is the fact that if M is the interior of a compact Riemannian manifold with boundary N (in the sense that the metric of M is the metric of N restricted to M) then N is in fact the Martin compactification of M relative to Δ and hence is unique. As pointed out to the author by W. Browder, this sort of thing does not happen in the category of $\mathscr{C}^\infty$-manifolds. In fact, if $n \geqslant 6$, results of Stallings [13] imply that in each dimension there exist non-diffeomorphic compact $\mathscr{C}^\infty$-manifolds with boundary with diffeomorphic interiors.

The article begins with a detailed proof of Carleson's lemma based on the original argument in [3]. An examination of the details of this proof then permits one to obtain this result for any $L \in \Lambda(\gamma, \mu; D)$.

In section three, following [5], it is shown that for $z_0 \in \partial D$ and h_1, h_2 any two Bouligand functions associated with z_0 there exists a constant c with $ch_1 \leqslant h_2$. The theorem about the Martin compactification is established in section four and then applied to Riemannian manifolds.

(*) Added in proof: also proved by S. Ito for C^3 boundaries [18].

1. Carleson's lemma [3].

A domain $D \subset \mathbf{R}^n$ will be said to be a *Lipschitz domain* if, for each $z_0 \in \partial D$, the following condition is satisfied: for some neighbourhood U of z_0 there is a $\mathscr{C}^1$-diffeomorphism w of U with an open set in $\mathbf{R}^n$ and a Lipschitz map f: $\mathbf{R}^{n-1} \to \mathbf{R}$ such that

$$U \cap D = \{z \in \mathbf{R}^n | w_n(z) > f(w_1(z), \ldots, w_{n-1}(z))\},$$

where $w(z) = (w_1(z), \ldots, w_n(z))$, for all $z \in U$.

It is not hard to see that this condition is satisfied if w is replaced by any other $\mathscr{C}^1$-diffeomorphism (with U replaced by a smaller set in general) and hence that one can define a Lipschitz domain in any $\mathscr{C}^k$-real manifold, $k \geqslant 1$. Of course, if f is sufficiently differentiable then the domain in question is a sub-manifold with boundary.

Let D be a bounded Lipschitz domain in $\mathbf{R}^n$ and let $z_0 \in \partial D$. By a suitable choice of coordinates one can assume that $z_0 = 0$ and that if $\mathbf{R}^n$ is identified with $\mathbf{R}^{n-1} \times \mathbf{R}$ then there is a Lipschitz function $f: \mathbf{R}^{n-1} \to \mathbf{R}$ with $f(0) = 0$ whose graph determines the boundary of D in a neighbourhood of the origin. More specifically, it can and will be assumed that $D' = \{(x, y)| \, |x| \leqslant 1, |y| \leqslant 12A\} \cap D$ coincides with $\{(x, y)| \, |x| \leqslant 1, f(x) \leqslant y \leqslant 12A\}$, where (i) A is a constant such that $|f(x) - f(x')| \leqslant A|x - x'|$, $\forall x, x' \in \mathbf{R}^{n-1}$; (ii) $|x| = \max_{1 \leqslant i \leqslant n-1} |x_i|$ and (iii) $z \in \mathbf{R}^n$ is written as (x, y) with $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$.

When D is as above it will be said to be in the canonical position

Let $B_1 = 41A$ and let $r_0 = (3A)/(4B_1)$. Then, if

$$|x| \leqslant 1/8, \qquad f(x) + r_0 B_1 \leqslant A.$$

Denote by Γ the graph of f. A *closed r-cell* C in Γ with centre $z_0 = (x_0, f(x_0))$ is a set of the form

$$\Gamma \cap \{(x,y)| \, |x - x_0| \leqslant r\}.$$

An *open r-cell* in Γ is similarly defined.

Then, if $|x_0| \leqslant 1/8$ and $r \leqslant r_0$, there is a smallest integer $n_0 = n_0(r)$ such that $(*)$ $2A \leqslant f(x_0) + 2^{n_0}rB_1 \leqslant 5A$.

Consider the cylinder B_n of centre z_0 width $2^{n+1}r$ and length

$$(2^{n+1}r)B_1 \text{ i.e. } \{(x, y)| \, |x - x_0| \leqslant 2^n r, |y - f(x_0)| \leqslant B_1 2^n r\}$$

where $0 \leqslant n \leqslant n_0$. Let $D_n = D \cap B_n$. Then ∂D_n consists of three parts: $\alpha_n = \partial D_n \cap \Gamma$; $\beta_n = \partial D_n \cap \{z| \, |x - x_0| = 2^n r\}$; and $\gamma_n = \partial D_n \cap \{z|y = f(x_0) + 2^n r B_1\}$.

Let $y' > f(x_0)$ and let $\gamma' = \{z| \, y - f(x_0) \geqslant B_1|x - x_0|; y = y'\}$. Then $\gamma' \subset D$ if $y' < 12A$.

Lemma 1.1. — *There is a constant* $K_2 = K_2(A)$ *such that for all* $u \geqslant 0$ *and harmonic on* D *one has:*

1) *if* $6A \geqslant y'$, $u(z) \leqslant K_2 u(x_0, y') \forall z \in \gamma'$; *and*

2) *if* $2A \geqslant y' > y''$ *and* $y' - f(x_0) = 2(y'' - f(x_0))$ *then* $u(x_0, y'') \leqslant K_2 u(x_0, y')$.

Proof. — Map the appropriate cone by a homothety of magnitude $(y' - f(x_0))^{-1}$ and centre z_0 onto

$$\{(x, y) \,|2 \geqslant y \geqslant B_1|x|\}.$$

The inequalities are then immediate consequences of Harnack's inequality. Note that $y' - f(x_0) \leqslant 7A$.

Consider now a point $\Phi = (\xi, f(\xi))$ in β_n, $0 < n \leqslant n_0$ and let $y_n = f(x_0) + 2^n r B_1$. Since $B_1 = 41A$ and $(*)$ implies $2^n r B_1 \leqslant (5 + 1/8)A$, it follows that $2^n r \leqslant 1/8$. Consequently, $|\xi| \leqslant 1/4$ and so, using $(*)$ once more, one has that

$$y_n - f(\xi) \leqslant (5 + 1/2)A.$$

As a result, the cone

$$\{(x, y) \,|2(y_n - f(\xi)) \geqslant y - f(\xi) \geqslant 16A \,|x - \xi|\} \subset D.$$

Fix $\delta > 0$. By a homothety of centre $\Phi = (\xi, f(\xi))$ and magnitude $(y_n - f(\xi))^{-1}$ the above cone can be mapped onto $\{(x, y) \,|2 \geqslant y \geqslant 16A|x|\}$. The point (ξ, y_n) goes to $(0, 1)$ and $\{(\xi, y) \,|y_n \geqslant y \geqslant f(\xi) + \delta 2^n r\}$ goes into

$$\left\{(0, t) \,|1 \geqslant t \geqslant \frac{\delta}{B_1 + A}\right\}$$

since $y_n - f(\xi) \leqslant 2^n r(B_1 + A)$. Hence, if $u \geqslant 0$ is harmonic on D there is a constant $K_3 = K_3(\delta, B_1)$ such that, for all $n < n_0(r) + 1$, $u(\xi, y) \leqslant K_3 u(\xi, y_n)$ if $f(\xi) + \delta 2^n r \leqslant y \leqslant y_n$. Combining this with lemma 1.1 completes the proof of the following result.

LEMMA 1.2. — *Given* $r \leqslant r_0$, *there exists a constant*

$$K_4 = K_4(\delta, A)$$

such that, for all $u \geqslant 0$ *harmonic on* D *and for all*

$$n < n_0(r) + 1,$$

one has

$$(**)\ u(\xi, y) \leqslant K_4 u(x_0, y_n),$$

whenever $|x_0| \leqslant 1/8$, $|\xi - x_0| = 2^n r$ *and*

$$f(\xi) + \delta 2^n r \leqslant y \leqslant y_n.$$

A domain D in $\mathbf{R}^n$ will be said to be « *standard* » if there is a Lipschitz function $f: \mathbf{R}^{n-1} \to \mathbf{R}$ with $f(0) = 0$ and $|f(x) - f(x)| \leqslant A|x - x'|$ such that (i) the

$$\text{set } C = \{(x, f(x)) |\ 1/2 < |x| < 3/2\} \cup \{(x, y)\ ||x| = 1/2, f(x) \leqslant y \leqslant 1/2B_1\} \cup \{(x, y) |\ |x| \leqslant 1/2, |y| = 1/2B_1\}$$

is a connected component of $\partial D \cap \{(x, y) |\ |x| \leqslant 3/2\}$ and (ii) $D \cap \{(x, y) \mid |x| < 3/2, |y| < 18A\} = \{(x, y) \mid 1/2 < |x| < 3/2, y < f(x) < 18A\} \cup \{(x, y) \mid |x| \leqslant 1/2, 1/2B_1 < y < 18A\}$.

LEMMA 1.3. — *Let* D *be a standard domain and let* w *be the solution to the Dirichlet problem for* D *corresponding to the boundary value* 1_B, *where* $B = C \cap \{(x, y) |\ |x| \leqslant 1/2\}$.

Let $\varepsilon > 0$. *Then, providing* $|x_0| = 1$ *there is a constant* $\delta(\varepsilon)$ *such that*

$$w(z) < \varepsilon \quad \text{if} \quad z \in D \cap \{z |\ |z - z_0| < \delta\},$$

where $z_0 = (x_0, f(x_0))$.

Proof. — Let δ_1 be the minimum Euclidean distance of x from x', where $|x| = 1$, $|x'| = 1/2$. Let v be the solution to the Dirichlet problem on $D \cap \{z |\ |z - z_0| < \delta_1\}$ corresponding to the characteristic function of

$$\overline{D} \cap \{z | \, |z - z_0| = \delta_1\}.$$

Then there exists δ, $0 < \delta < \delta_1$ such that $v(z) < \varepsilon$ on $D \cap \{z | \, |z - z_0| < \delta\}$. Since $v \geqslant w$, this completes the proof.

Lemma 1.4. — *Let* $X \subset \Gamma \cap \{z | \, |x| \leqslant 1/8\}$ *be a Borel set that contains, as a dense subset, an open* r*-cell* C, $r \leqslant r_0$. *Denote by* u *the harmonic measure of* X. *Then there is a constant* K_5, *depending only on* A, *such that*

$$u(x, y) \leqslant K_5 u(0{,}5A),$$

for all $z = (x, y)$ *with* $|x| = 1/2$ *and* $f(x) < y \leqslant 5A$ *or* $|x| \leqslant 1/2$ *and* $y = 5A$.

Proof. — Let $z_0 = (x_0, y_0)$ be the centre of the cell C. Since the compact set $\{(x, y) | \, |x| \leqslant 1, \, 2A \leqslant y \leqslant 5A\} \subset D$ it suffices to prove that $u(x, y) \leqslant K_5 u(x_0, y_{n_0})$ for

$$(x, y) \in \beta_{n_0} \cup \gamma_{n_0},$$

where $n_0 = n_0(r)$ and the sets β_{n_0}, γ_{n_0} are defined in terms of the cylinder B_{n_0} whose centre is the centre of the open cell C.

By induction on n it is proved that there is a constant K_5 depending only on A such that for all

$$(x, y) \in \beta_n \cup \gamma_n, \qquad u(x, y) \leqslant K_5 u(x_0, y_n).$$

First consider the cylinder $\{(x, y) | \, |x| \leqslant 1, \, |y| \leqslant 2B_1\}$. Let b be the value at $(0, B_1)$ of the harmonic measure for this cylinder of $\{(x, y) | \, |x| \leqslant 1, \, y = -2B_1\}$. Then

$$u(x_0, y_1) \geqslant b$$

and so $u(x, y) \leqslant 1 \leqslant 1/b \; u(x_0, y_1)$, for all $(x, y) \in D$.

Assume that $u(x, y) \leqslant K_5 u(x_0, y_k)$ for all $(x, y) \in \beta_k \cup \gamma_k$ if $0 \leqslant k \leqslant n - 1 < n_0$ with K_5 depending only on A. From lemmas 1.1 and 1.2 it follows that

$$u(x, y) \leqslant \max \{K_2, K_4\} u(x_0, y_n)$$

for all

$$(x, y) \in (\beta_n \cup \gamma_n)\setminus\{(x, y)|f(x) < y \leqslant \delta 2^n r + f(x)\}.$$

Since K_5 can be assumed $\geqslant \max\{K_2, K_4\}$ it suffices to consider what happens on

$$\beta_n \cap \{(x, y)|f(x) < y \leqslant \delta 2^n + f(x)\} = \beta'_n(\delta).$$

Let $D_{n-1} = D\setminus\{(x, y)|\ |x - x_0| \leqslant 2^{n-1}r, f(x) \leqslant y \leqslant 2^{n-1}r\}$. Then there is a unique « standard » Lipschitz domain D' such that a homothety T of magnitude $2^n r \leqslant 1/8$ (see the proof of lemma 1.2) and centre the origin followed by a translation of the origin to the centre of the cell C maps D' onto D_{n-1}.

Denote by φ the harmonic function on D_{n-1} with boundary value $1_{\beta_{n-1}\cup\gamma_{n-1}}$. On D_{n-1}

$$u \leqslant [K_5 u(0, y_{n-1})]\varphi \leqslant [K_2 K_5 u(0, y_n)]\varphi.$$

Let $\delta = \delta(1/K_2)$ where, for any $\varepsilon > 0$, $\delta(\varepsilon)$ is the constant in lemma 1.3. Hence, by the above inequality and that lemma 1.3, $u(x, y) \leqslant K_5 u(0, y_n)$ on $\beta'_n(\delta)$.

Corollary 1.5. — *Let* $h \geqslant 0$ *be any Borel function on* ∂D *such that* $\{h \neq 0\} \subset \Gamma \cap \{z|\ |x| \leqslant 1/8\}$. *Denote by* H_h *the corresponding solution to the Dirichlet problem. Then, there is a constant* K_5, *depending only on* A, *such that*

$$(*) \quad H_h(x, y) \leqslant K_5 H_h(0, 5A),$$

for all $z = (x, y)$ *with* $|x| = 1/2$ *and* $f(x) < y \leqslant 5A$ *or* $|x| \leqslant 1/2$ *and* $y = 5A$.

Proof. — Let $\mathcal{S}$ be the set of Borel sets

$$A \subset \Gamma \cap \left\{z|\ |x| \leqslant \frac{1}{8}\right\} = E$$

such that $(*)$ holds for $h = 1_A$. It suffices to show that $\mathcal{S}$ consists of all the Borel subsets of E.

Let $\mathcal{A}$ be the set of finite unions of sets X of the type considered in lemma 1.4. Then $\mathcal{A}$ is an algebra of sets contained in $\mathcal{S}$. Since $\mathcal{S}$ is a monotone class it contains the σ-algebra $\mathcal{B}$ generated by $\mathcal{A}$ (cf. [8] IT 19). Clearly $\mathcal{B}$

contains all open sets and hence $\mathscr{B}$ consists of all Borel subsets of E.

This essentially completes the proof of the following result (Carleson's lemma).

THEOREM 1.6. (Carleson). — *Let* $u \geqslant 0$ *be harmonic on* D *and such that* $\lim_{z \to z_0} u(z) = 0$ *for all*

$$z_0 \in \partial D \setminus \left(\Gamma \cap \left\{ z \mid |x| \leqslant \frac{1}{32} \right\} \right).$$

Then there is a constant K *such that* $u(z) \leqslant \mathrm{K}u(0,5\mathrm{A})$ *for*

$$z \in D \setminus \left\{ (x, y) \mid |x| \leqslant \frac{1}{2}, f(x) \leqslant y \leqslant 5\mathrm{A} \right\}.$$

Proof. — There is a Lipschitz function g on $\mathbf{R}^{n-1}$ such that $g \geqslant f$, $g(x) = f(x)$ if $|x| \geqslant \frac{1}{8}$, $|x| \leqslant \frac{1}{8}$ implies $g(x) \leqslant \frac{\mathrm{A}}{8}$, and $g(x) = \frac{\mathrm{A}}{16}$ if $|x| \leqslant \frac{1}{16}$. Further, it can be assumed that the Lipschitz constant for g depends only on A and not on the particular f or point $z_0 \in \partial D$.

Let $D_0 = D \setminus \left\{ (x, y) \mid |x| \leqslant \frac{1}{8}, \frac{-\mathrm{A}}{8} \leqslant y \leqslant g(x) \right\}$. Then D_0 is again a Lipschitz domain and the bounded Borel function h on ∂D_0 defined by $h(z_1) = \lim_{z \to z_1} u(z)$ satisfies the hypotheses of corollary 1.5. Since $u | D_0 = \mathrm{H}_h$ the result follows.

2. Carleson's lemma for uniformly elliptic operators.

As before, D will denote a bounded domain in $\mathbf{R}^n$ with Lipschitz boundary. Let $\Lambda_0 = \Lambda_0(\gamma, \mu, \alpha; D)$ denote the set of second order partial differential operators L on D of the form

$$\mathrm{L} = \sum_{i,j=1}^{n} a_{ij}(z) \frac{\partial^2}{\partial z_i \partial z_j} + \sum_{i=1}^{n} b_i(z) \frac{\partial}{\partial z} + c(z),$$

with coefficients that are locally Hölder continuous and such that

(1) $$1/\gamma\|\xi\|^2 \leqslant \sum_{i,j=1}^{n} a_{ij}(z)\xi_i\xi_j \leqslant \gamma\|\xi\|^2$$

for all $\xi \in \mathbf{R}^n$ and $z \in D$;

(2) $$\left[\sum_{i=1}^{n} b_i^2(z)\right] \leqslant \mu;$$

and

(3) $$0 \geqslant c(z) \geqslant -\mu.$$

In addition it will be assumed that there exists a modulus of continuity Φ for the matrix-valued function

$$z \to (a_{ij}(z)) = A(z)$$

with $\int_0^\infty \Phi(s)s^{-1}\,ds = \alpha$ and $d\Phi(s) \geqslant 0$. The distance from $A(z)$ to $A(z')$ i.e. $\|A(z) - A(z')\|$ is given by the operator norm.

If T is a homothety (or a diffeomorphism) of $\mathbf{R}^n$, to each $L \in \Lambda_0(\gamma, \mu, \alpha; D)$ there corresponds a unique partial differential operator L' on $D' = T(D)$ such that for all $u' \in \mathscr{C}^2(D')$, if $u = u' \circ T$ then $Lu(z) = L'u'(w)$ whenever $w = T(z)$. For convenience, L' will be called *the image* of L under T. If $w = \lambda(z - z_0) + z_0$, it has the form

$$L' = \sum_{i,j}^{n} \lambda^2 a_{ij}(T^{-1}(w)) \frac{\partial^2}{\partial w_i \partial w_j} + \sum_{i=j}^{n} \lambda b_i(T^{-1}(w)) \frac{\partial}{\partial w_i} + c(T^{-1}(w)).$$

The following lemma is then almost immediate.

Lemma 2.1. — *Let* $0 < \lambda_0 \leqslant 1$. *Assume*

$$T(z) = \lambda(z - z_0) + z_0$$

with $\lambda \geqslant \lambda_0$. *If* $L \in \Lambda_0(\gamma, \mu, \alpha; D)$ *and* L' *is its image under* T, *then* $L'' = (1/\lambda^2)L' \in \Lambda_0(\gamma, \mu/\lambda_0^2, \alpha; T(D))$.

Proof. — It suffices to observe that if Φ is a modulus of continuity for L then $\psi(s) = \Phi(s/\lambda)$ is a modulus of continuity for L'' with $\int_0^\infty \Phi(s)s^{-1}\,ds = \int_0^\infty \psi(s)s^{-1}\,ds$.

Serrin's result [12] on the existence of Harnack inequalities yields the following result.

LEMMA 2.2. — *Let* $L \in \Lambda_0(\gamma, \mu, \alpha; D)$ *and* $u \in \mathscr{C}^2(D)$ *be such that* $u \geqslant 0$ *and* $Lu = 0$. *Then, if* $A \subset D$ *is compact there exists a constant* $\varkappa$, *depending on* γ, μ, α *and the choice of a neighbourhood of* A *in* D, *with*

$$u(z) \leqslant \varkappa u(z') \forall z, z' \in A.$$

Another fairly obvious fact is the following one.

LEMMA 2.3. — *Let* T *be an orthogonal transformation with* $Tz = w$ *and* $w_i = \sum_{j=1}^{n} \sigma_{ij} z_j$. *If* $L \in \Lambda_0(\gamma, \mu, \alpha; D)$ *then* L' (*its image under* T) $\in \Lambda_0(\gamma, \mu, \alpha; T(D))$.

Proof. — The coefficients $a'_{ij}(w)$, $b'_i(w)$ and $c'(w)$ are

$$\sum_{k,l=1}^{n} a_{kl}(T^{-1}(w))\sigma_{il}\sigma_{jk}, \qquad \sum_{k=1}^{n} b_k(T^{-1}(w))\sigma_{ik}$$

and $c(T^{-1}(w))$ respectively. Further,

$$\|A'(T(z) - A'(T(z'))\| = \|A(z) - A(z')\|$$

and so the condition on the modulus of continuity is automatically satisfied.

This fact then implies that to study the behaviour at $z_0 \in \partial D$ of functions u for which $Lu = 0$, $L \in \Lambda_0(\gamma, \mu, \alpha; D)$, it suffices to consider the case where $z_0 = 0$ and where the boundary is determined locally by the graph of a Lipschitz function $f: \mathbf{R}^{n-1} \to \mathbf{R}$ with $f(0) = 0$ and

$$|f(x) - f(x')| \leqslant A|x - x'|.$$

As before any $z \in \mathbf{R}^n$ will be *written* as $z = (x, y)$ with $x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R}$.

Let $B \geqslant A$ and let $L \in \Lambda_0(\gamma, \mu, \alpha; D)$.

Miller's universal barrier [9] implies that, for given γ, μ and B, there exists a strictly positive function u on

$$\{(x, y) | y \geqslant -B|x|\} \setminus \{0, 0\}$$

which is $\mathscr{C}^2$ on the interior and such that

(i) $\lim_{z \to 0} u(z) = 0$.

(ii) $Lu \leqslant 0$ on $D \cap \{(x, y)|y > -B|x|\}$ for any

$$L \in \Lambda_0(\gamma, \mu, \alpha; D).$$

By using this universal barrier u the classical Dirichlet problem on D relative to any $L \in \Lambda_0(\gamma, \mu, \alpha; D)$ can be solved.

PROPOSITION 2.4. — *Assume* $L \in \Lambda_0(\gamma, \mu, \alpha; D)$ *and that for each* $z_0 \in \partial D$ *there is an unbounded cone* C *with*

$$C \cap \overline{D} = \{z_0\}.$$

Let $\Phi \in \mathscr{C}(\partial D)$. *Then there is a unique continuous function* u *on* $\overline{D}$ *with* $u \in \mathscr{C}^2(D)$ *and* (i) $Lu = 0$, (ii) $u|\partial D = \Phi$.

Proof. — In case $c = 0$ a proof was given by Miller in [9] (even admitting a singularity for $\sum_{i=1}^{n} b_i^2(x)$ at ∂D).

When $-\mu \leqslant c \leqslant 0$ on D then $L1 \leqslant 0$. i.e. 1 is super-harmonic. Let e be the greatest harmonic minorant of 1. Miller's argument carries over once it is shown that e converges to 1 at every boundary point. For this, it suffices to exhibit, for each $z_0 \in \partial D$, an L-subharmonic function (see section 3) $v \leqslant 1$ on D with $\lim_{z \to z_0, z \in D} v(z) = 1$.

For example, $v(z) = e^{f(z)}$ where $f(z) = f_L(z)$ is the unique solution on D of the Dirichlet problem : $w \mid \partial D = 0$, $Mw = -\mu$ where $Mw = Lw - cw$ (see Theorem 4 [9]).

Remark. — The proposition is true for arbitrary bounded Lipschitz domains. It suffices to note that, for sufficiently small δ, the sets N_δ used by Miller on p. 99 of [9] satisfy the hypotheses above. His argument then carries over for $\Phi = 1$ providing that the appropriate constant is replaced by that constant times the solution of the Dirichlet problem for N_δ with boundary value 1.

LEMMA 2.5. — *Let* D *be a Lipschitz domain such that for each* $z \in \partial D$ *there is an unbounded cone* $C(z)$ *with*

$$C(z) \cap \overline{D} = \{z\}$$

and let A *be such that, for all* $z \in \partial D$, *in a suitable coordinate system at* z, $\{(x, y) | Ay \leqslant -|x|\} \subset C(z)$.

Denote by B *a neighbourhood of a point* $z_0 \in \partial D$. *Let* u_L *be the harmonic measure of* $B \cap \partial D$ *relative to*

$$L \in \Lambda_0(\gamma, \mu, \alpha; D).$$

Then, there is a neighbourhood $U = U(\gamma, \mu, \alpha, A, B)$ *of* z_0 *such that*

$$u_L(z) \geqslant 1/2 \ \forall z \in U \cap D.$$

Proof. — It is clear from the proof of Theorem 3 in [9] that the function $f = f_L$ tends to zero uniformly at ∂D in $L \in \Lambda_0$. Hence, for $\varepsilon > 0$ there is $\delta > 0$, $\delta = \delta(\gamma, \mu, \alpha, A)$ with $e_L(z) > 1 - \varepsilon$ if the distance of z from ∂D is at most δ (here e_L is the solution of the Dirichlet problem relative to L with boundary value 1).

Further, the universal barrier of Miller shows that there is a neighbourhood $U' = U'(\gamma, \mu, \alpha, A, B)$ of z_0 with $v_L(z) < \varepsilon$ if $z \in U' \cap D$, where v_L is the solution of the Dirichlet problem with boundary value 1_E, $E = \partial D \setminus B$. Now since $u_L = e_L - v_L$ it follows that $u_L(z) \geqslant 1 - 2\varepsilon$ on

$$U = U' \cap \{z | z - z_0|\} < \delta.$$

With the aid of these lemmas the arguments used in paragraph one to prove theorem 1.6 can be applied to

$$L \in \Lambda_0(\gamma, \mu, \alpha; D)$$

i.e. Carleson's lemma holds for such operators. The domain D is, as in theorem 1.6, determined in the neighbourhood of 0 by a Lipschitz function f with $f(0) = 0$.

THEOREM 2.6. (Carleson's Lemma). — *Let* $u \geqslant 0 \in \mathscr{C}^2(D)$ *be such that* $Lu = 0$, $L \in \Lambda_0(\gamma, \mu, \alpha; D)$, *and such that*

$$\lim_{z \to z_0} u(z) = 0$$

for all $z_0 \in \partial D \setminus (\Gamma \cap \{z | \, |x| \leqslant 1/32\})$.

Then, there is a constant $K = K(\gamma, \mu, \alpha, A)$ *for which* $u(z) \leqslant Ku(0{,}5A)$ *for all* $z \in D \setminus \{(x, y) | \, |x| \leqslant 1/2,$

$$f(x) \leqslant y \leqslant 5A\}.$$

Proof. — It suffices as before to prove the corollary 1.6. It is a consequence of the lemmas 1.1, 1.2, 1.3 and 1.4.

Lemma 1.1 with $Lu = 0$ replacing $\Delta u = 0$, is a consequence of lemmas 2.1 and 2.2 since the magnitude λ of the homothety is $\geqslant 1/7A$.

Lemma 1.2 holds by virtue of lemmas 2.1 and 2.2 since again the homotheties involved have magnitude $\lambda \geqslant 1/6A$.

Lemma 1.3 presents no problems in view of Miller's universal barrier.

It remains to consider lemma 1.4. Lemma 2.5 can be applied with D the image of $\{(x, y) | \, |x - x_0| \leqslant r, f(x) \leqslant y \leqslant 2B_1\}$ under the homothety of magnitude

$$1/r \geqslant 1/r_0 = 4B_1/3A (\geqslant 1)$$

and centre $(x_0, f(x_0))$ and with B the image of X under this homothety. Combining this with lemma 2.2 one finds that there is a constant $b = b(\gamma, \mu, \alpha, A, r)$

$$u(x, y) \leqslant 1/bu(x_0, y_1)$$

for all $u(x, y) \in D$.

The rest of the argument in the proof of lemma 1.4 holds without change, since lemmas 1.1, 1.2 and 1.3 are already established.

For easy application of Carleson's lemma to the Martin boundary it is useful to have it in the following form.

PROPOSITION 2.7. — *Let* $D \subset \mathbf{R}^n$ *be a bounded Lipschitz domain and let* $z_1 \in D$. *Denote by* U *a bounded open neighbourhood of* $z_0 \in \partial D$.

Then there exists a constant K *and a neighbourhood* V *of* z_0 *with* $\overline{V} \subset U$ *such that:*

$$u(z) \leqslant Ku(z_1) \quad \forall z \in D \setminus \overline{U}$$

whenever $u \geqslant 0$ *is* L-*hyperharmonic on* D, L-*harmonic on* $D \setminus \overline{V}$ *and, for all* $z' \in \partial D \setminus V$, $\lim_{z \to z'} u(z) = 0$. (See paragraph 3 for a definition of L-hyperharmonic.)

Proof. — As before let $z_0 = 0$ and assume D is in the canonical position. By using a suitable homothety centred at z_0 one can assume $U \supset \{(x, y) | \, |x| \leqslant 1/2, f(x) \leqslant y \leqslant 5A\}$.

Let $\overline{V} = \{(x, y) | |x| \leqslant 1/32, |y| \leqslant A/32\}$. Then, as in the final argument of the proof of theorem 7.1, one can replace f by a Lipschitz function g with $g \geqslant f$, $g(x) = f(x)$ if $|x| \geqslant 1/8$, $g(x) \leqslant A/8$ if $|x| \leqslant 1/8$ and $g(x) = A/16$ if $|x| \leqslant 1/16$. This defines a Lipschitz domain D' with $u|D'$ L-harmonic and of the form to which theorem 2.6 applies. The result follows.

3. The lemma of Hunt and Wheeden.

Let D be a bounded Lipschitz domain and let L be a second order elliptic operator on L. A function $u \in \mathscr{C}^2(0)$, $0 \subset D$ will be said to be L-*harmonic on* 0 if $Lu = 0$.

Assume that $L \in \Lambda(\gamma, \mu; D')$ where the Lipschitz domain $D' = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 12A\} \cap \{(x, y) \mid f(x) \leqslant y \leqslant 12A\}$ and $f: \mathbf{R}^{n-1} \to \mathbf{R}$ is a Lipschitz function with $f(0) = 0$ and $|f(x) - f(x')| \leqslant A|x - x'|$ for all $x, x' \in \mathbf{R}^{n-1}$. Let

$$B_n = \{(x, y) \mid |x| \leqslant 1/2^n, |y| \leqslant 12A/2^n\}$$

and let $z_n = (0, 12A/2^n)$. The *boundary Harnack principle* (Ancona [16]) states that if $L \in \Lambda(\gamma, \mu; D)$ and if $u, v \geqslant 0$ are L-harmonic functions on D' such that

$$\lim_{z \to \bar{z}} u(z) = \lim_{z \to \bar{z}} v(z) = 0$$

for all $\bar{z} \neq (0, 0)$ in the graph of f then there exists a constant $C = C(A, \gamma, \mu)$ such that

$$u(z)/u(z_n) \leqslant Cv(z)/v(z_n)$$

for all $z \in \partial B_n \cap D'$.

This result has as a corollary (see Ancona [16]) the following theorem, where a positive L-harmonic function $u \neq 0$ on D is called a *Bouligand function associated with* $z_0 \in \partial D$ if $\lim_{z \to z'} u(z) = 0$ for all $z' \in \partial D' \setminus \{z_0\}$.

THEOREM 3.1. (Lemma of Hunt and Wheeden for $L \in \Lambda(\gamma, \mu; D)$). — *Let* D *be a bounded Lipschitz domain and let* $z_0 \in \partial D$. *Let* h_1 *and* h_2 *be any two Bouligand functions associated with* z_0. *Then there is a constant* c *with* $ch_1 \leqslant h_2$.

Proof (Ancona [16]). — D can be assumed to be in standard position with $z_0 = 0$. Let $x_0 \in D \setminus D'$. Then applying the Boundary Harnack principle there is a constant $C = C(A, \gamma, \mu)$ such that

$$h_1(z)/h_1(z_n) \leqslant C h_2(z)/h_2(z_n)$$

for all $z \in D \setminus B_n$.

Hence, by symmetry, $h_1(z_n)/h_2(z_n) \leqslant C h_1(x_0)/h_2(x_0) = C_1$. Consequently, $h_1(z) \leqslant CC_1 h_2(z)$ for all $z \in D \setminus B_n$ and hence $h_1 \leqslant CC_1 h$ on D.

It is not known whether this lemma holds for arbitrary Lipschitz domains if $L \in \Lambda_0(\gamma, \mu, \alpha; D) = \Lambda_0$. However, for this class of operators it remains true if the domain is of class $\mathscr{C}^{1,1}$ and even for domains which are of this class except at isolated points. In what follows, for domains of class $\mathscr{C}^{1,1}$, Hunt and Wheeden's proof for the Laplacian will be pushed through for operators $L \in \Lambda_0$. This is feasible because the L-harmonic measures μ_L for a fixed point x_0 satisfy the following « Harnack » inequality: there exists a measure λ on ∂D and a constant c such that for all

$$L \in \Lambda_0 = \Lambda_0(\gamma, \mu, \alpha; D) \ 1/c\, \lambda \leqslant \mu_L \leqslant c\lambda$$

providing D is star-like.

First, as in Hunt and Wheeden, it will be shown that it suffices to consider a star-like domain. Let $L \in \Lambda_0(\gamma, \mu, \alpha; D)$. Mme Hervé [4] showed that the L-harmonic functions define a harmonic sheaf $\mathscr{H}_L$ on D in the sense of Brelot.

Consequently, L-hyperharmonic functions can be defined in terms of the harmonic measures associated with $\mathscr{H}_L$. Further, as indicated in [4] (see also the proof of proposition 2.4) there exist non-constant hyperharmonic functions on D that are non-negative. Hence, potentials (relative to $\mathscr{H}_L$) exist on D.

If $E \subset D$ and $u \geqslant 0$ is hyperharmonic one defines $R_E u = \inf \{v \mid v \geqslant 0,\ v \geqslant u$ on E, v is hyperharmonic$\}$ and $\hat{R}_E u$ as the lower semi-continuous regularization of this function. Then $\hat{R}_E u$ is hyperharmonic and for each $x \in D$ there is a unique measure μ on D such that

$$\hat{R}_E u(x) = \int u \, d\mu.$$

This measure will be denoted by $\mu(dz) = \hat{R}_E(x, dz)$ — the usual notations being $\mu = \varepsilon_x^E$ and $\hat{R}_E u(x) = \hat{R}_u^E(x)$. From the general theory of harmonic spaces one knows that $\hat{R}_E(x, \ )$ is carried by $\partial(D \setminus E)$.

Fix $z_0 \in \partial D$. As usual one can assume $z_0 = 0$ and that with respect to suitable coordinates, the domain D is in the canonical position of paragraph one. The point $(0, 2A)$ is such that for any $(x, f(x))$ with $|x| \leqslant 1$ the line segment joining these two points lies in D. Hence, if

$$D' = \{(x, y) \mid |x| \leqslant 1, |y| \leqslant 12A\} \cap D$$

then D is a star-like set with $(0, 2A)$ as « centre ».

Let h_1, h_2 be two Bouligand functions for D associated with $z_0 = 0$. Then, if $k_i = h_i - \hat{R}_{\complement D'} h_i$ $(i = 1, 2)$ k_1 and k_2 are two Bouligand functions for D' associated with 0. Assume that there is a constant $c > 0$ with $ck_1 \leqslant k_2$.

Then, $ch_1 \leqslant h_2 + \hat{R}_{\complement D'} h_1$ on D. Now $\hat{R}_{\complement D'} h_1$ is a potential on D because h_1 is a Bouligand function associated with $z_0 = 0$. Hence, $ch_1 \leqslant h_2$.

This shows that it suffices to prove the lemma of Hunt and Wheeden for the case of a Lipschitz domain like D' and for $z_0 = 0$. To begin with two lemmas will be proved.

Let $0 < t < 1$ and let $z_1 = (0, 3tb/4)$. Denote by $C(t)$ the closed cell in the graph Γ of f of width $tb/(6A)$ and centre 0. Let $T = T_t$ be the homothety of centre $(0, b)$ and magnitude $s = 1 - 3t/4$ and let $L'_t \in \Lambda_0(\gamma, 16\mu, \alpha; D')$ be such that the image of $s^{-2}L'_t$ under T coincides with $L|T(D')$, $L \in \Lambda_0(\gamma, \mu, \alpha; D')$ (note that $1/4 \leqslant s \leqslant 1$). Further, let $u'_{C(t)}$ be the L'_t-harmonic measure of $C(t)$ for D' and let $u_{C(t)}$ be the L-harmonic measure of $C(t)$ for D'.

Lemma 3.2. — *There is a constant* $K_1 = K_1(\gamma, \mu, \alpha; A)$ *such that for all* $u \geqslant 0$, *L-harmonic on* D',

$$u \circ T \geqslant K_1 u(z_1) u'_{C(t)} \text{ on } D'.$$

Proof. — The homothety T maps D' onto $T(D') \subset D'$ and carries $C(t)$ onto

$$C' \subset \{(x, y) \mid |x| \leqslant tb/(6A)\} \setminus \{(x, y) \mid |y - 3tb/4| > A|x|\}.$$

A further homothety of magnitude $4/(tb)$ and centre 0 shows that there is a constant $K_1 = K_1(\gamma, \mu, \alpha; A)$ with $u(T(z)) \geqslant K_1 u(z_1)$ for all $z \in C(t)$ (note that $4/(tb) \geqslant 4/b$ and so, in view of Lemma 2.1, the Harnack inequality applies).

Let $u_{C'}$ be the L-harmonic measure of $C' = T(C(t))$ for $T(D')$. Then $u'_C = u_{C'} \circ T$ and $u \geqslant K_1 u(z_1) u_{C'}$ on $T(D')$ implies the desired result.

LEMMA 3.3. — *Let* C *be the closed cell in the graph* Γ *of width* $2r$, $0 < r \leqslant 1$ *and centre* 0.
Let

$$B(a, r) = \{(x, y) \mid |x| \leqslant r, \qquad f(x) \leqslant y \leqslant a\} \subset D'$$

where $a/r > A$. *Then, there is a constant* $K_2 = K_2(\gamma, \mu, \alpha, A, a/r)$ *such that* $u_C(z) \geqslant K_2$ *for all* $z \in B(a/2, r/2)$.

Proof. — It suffices to consider the situation when $r = 1$. For $0 < r < 1$, by a homothety of magnitude $1/r \geqslant 1$ and centre 0 one returns to this case.

It follows from lemma 2.5 that there exists

$$\delta > 0, \qquad \delta = \delta(\gamma, \mu, \alpha, A)$$

such that $u_C(z) \geqslant 1/2$ if the distance of z from C' is less than δ, where C' is the closed cell in the graph of width r and centre 0. The result follows by using the Harnack inequalities.

Consider now h a harmonic function on D' that is a Bouligand function associated with 0.

Let $B(t) = \{(x, y) \mid 12A|x| \leqslant tb, \ 4|y| \leqslant 3tb\}$. Then, if $0 < t < 1$ and $s = \dfrac{4 - 3t}{4}$ one has

$$(1) \qquad h(T(z)) \geqslant K_1 h(3tb/4) u'_{C(t)}(z),$$

for all $z \in D'$. Further, since h is a Bouligand function, it follows from Carleson's lemma that there is a constant $K_3 = K_3(\gamma, \mu, \alpha; A)$ such that

$$(2) \qquad h(z) \leqslant K_3 h(3tb/4) \quad \forall z \in D' \setminus B(t/2).$$

Hence, by lemma 3.3,

$$(3) \quad h(z) \leqslant K_3 . h(3tb/4)(1/K_2) u_{C(t)}(z), \quad \forall z \in D' \setminus B(t).$$

Assume $h(0, b) = 1$. It follows from (1) and (3) that

$$K_1 h(3tb/4) u'_{C(t)}(0, b) \leqslant 1 \leqslant (K_3/K_2) h(3tb/4) u_{C(t)}(0, b).$$

Assume that there is a constant $M > 0$ such that, for all $t \in (0, 1]$, $u'_{C(t)}(0, b) \geqslant M u_{C(t)}(0, b)$. Choose the constant $\beta(t)$ so that the function $v'_t = \beta(t) u'_{C(t)}$ takes the value 1 at $(0, b)$. Then, given the last assumption, there is a constant K such that

$$h(T(z)) \geqslant K v'_t(z) \qquad \text{for all} \qquad z \in D'.$$

The functions v'_t are L'_t-harmonic on D'. Further, the family $(v'_t)_{0<t<1}$ is locally uniformly bounded since, for all t, $L'_t \in \Lambda_0(\gamma, \mu, \alpha; D')$ which implies that the constants involved in the Harnack inequalities [12] can be chosen to be independent of t.

Consequently, locally the Schauder estimates imply that for some α, $0 < \alpha < 1$, $\sup\limits_t \|v'_t\|_{2,\alpha} \leqslant M$. Hence, there is a sequence (t_n) decreasing to zero with (v_{t_n}) converging in $\| \ \|_2$ to a function $v \in \mathscr{C}^2(D')$. It is clear that $0 = \lim\limits_{n\to\infty} L'_{t_n} v'_{t_n} = Lv$. Hence, there is a constant K such that for any Bouligand function h associated to z_0 with $h(0, b) = 1$, $h \geqslant Kv$.

This essentially completes the proof of the following result.

THEOREM 3.4. — *Let* $L \in \Lambda_0(\gamma, \mu, \alpha; D')$ *and assume that for a fixed point* $x_0 \in D'$ *there is a measure* λ *with support on* ∂D *and a constant* N *such that*

$$\left(\frac{1}{N}\right)\lambda \leqslant \mu_t \leqslant N\lambda$$

for all t, $0 \leqslant t \leqslant 1$, *where* μ_t *is the* L'_t*-harmonic measure associated with* x_0.

Let $z_0 \in \partial D'$ *and denote by* h *an arbitrary Bouligand function associated to* z_0. *Let* z_1 *be a fixed point of* D'.

Then there is a positive L*-harmonic function* v *and a constant* K *(both independent of* h*) such that*

$$K v h(z_1) \leqslant h.$$

Proof. — The « Harnack » inequality for the harmonic measures implies that

$$u'_{C(t)}(0, b) \geqslant \frac{1}{N^2} u_{C(t)}(0, b).$$

To conclude it will be indicated how Serrin's results in [12] imply that the measure theoretic assumption of theorem 3.4 is verified if D' is of class $\mathscr{C}^{1,1}$. Clearly, if D itself is of class $\mathscr{C}^{1,1}$ the « box-like » subdomain D' can be replaced with an analogous $\mathscr{C}^{1,1}$ subdomain by « rounding off the corners ».

In case D' is a ball B then it follows immediately from Serrin's results that for operators $L \in \Lambda_0(\gamma, \mu, \alpha; B)$ there is a constant N such that

$$\left(\frac{1}{N}\right)\lambda \leqslant \mu_L \leqslant N\lambda$$

if μ_L is the L-harmonic measure associated with the centre of the ball and λ is surface measure on the sphere.

If D' is starlike and of class $\mathscr{C}^2$ then it is determined by a $\mathscr{C}^2$-function on a sphere. Consequently, there is an obvious « radial » $\mathscr{C}^2$-diffeomorphism of D onto a ball whose first and second derivatives are bounded. As the formula below shows such a diffeomorphism maps the class $\Lambda_0(\gamma, \mu, \alpha; D')$ into $\Lambda_0(\gamma', \mu', \alpha'; B)$ and consequently the desired property of the harmonic measures follows immediately. If

$$Lu(x) = \sum_{i,j=1}^{n} a_{ij}(x) \frac{\partial^2 u}{\partial x_i \, \partial x_j} + \sum_{i=1}^{n} b_i(x) \frac{\partial u}{\partial x_i} + u(x)c(x),$$

$x \in D$ and T is a $\mathscr{C}^2$-diffeomorphism of D onto D_0 then the image L_0 of L under T is given by

$$L_0\varphi(y) = \sum_{k,l=1}^{n} a^0_{kl}(y) \frac{\partial^2 \varphi}{\partial y_k \, \partial y_l} + \sum_{k=1}^{n} b^0_k(y) \frac{\partial \varphi}{\partial y_k} + \varphi(y)c^0(y),$$

where $a^0_{kl}(y) = \sum_{i,j} a_{ij}(x) \frac{\partial y_k}{\partial x_i}(x) \frac{\partial y_l}{\partial x_j}(x)$

$$b^0_k(y) = \sum_{i} b_i(x) \frac{\partial y_k}{\partial x_i}(x) + \sum_{i,j} a_{ij}(x) \frac{\partial^2 y_k}{\partial x_i \, \partial x_j}(x)$$

$$c^0(y) = c(x) \quad \text{and} \quad y = T(x).$$

This argument extends to star-like domains D′ of class $\mathscr{C}^{1,1}$ simply because a $\mathscr{C}^{1,1}$ function is the uniform limit of a sequence of $\mathscr{C}^2$-functions whose first and second derivatives are uniformly bounded. As a consequence, there is a « good » exhaustion of D′ by $\mathscr{C}^2$-starlike domains D_n. Namely, for each n there is a measure λ_n on ∂D_n and a constant N (independent of n) such that

$$\frac{1}{N}\lambda_n \leqslant \mu_{n,L} \leqslant N\lambda_n$$

where $\mu_{n,L}$ is the L-harmonic measure on D_n associated to the centre x_0 of the starlike domain D′.

Since μ_L is the vague-limit of the measures $(\mu_{n,L})$ and $\lambda_n(\partial D_n) \leqslant N\mu_{n,L}(\partial D_n) \leqslant N$ there is a measure λ on $\partial D'$ with

$$\left(\frac{1}{N}\right)\lambda \leqslant \mu_L \leqslant N\lambda.$$

This completes the proof of the following result.

Corollary 3.5. — *Let* D *be a bounded domain of class* $\mathscr{C}^{1,1}$ *and* $z_1 \in D$ *and let* $L \in \Lambda_0(\gamma, \mu, \alpha; D)$. *Then there is a positive* L*-harmonic function* v *and a constant* K *such that for any* L*-Bouligand function* h *associated to* z_0.

$$Kvh(z_1) \leqslant h.$$

Example 3.6 (due to Ancona). — The following example of a domain with a « point » shows that corollary 3.5 is true for domains that are worse than $\mathscr{C}^{1,1}$. It is not known to the author whether the corollary holds for general Lipschitz domains.

Let $\varphi : [-1, +\infty) \to \mathbf{R}$ be a continuous function that is $\mathscr{C}^2$ on $(-1, +\infty)$ and such that (1) $\varphi(t) = \sqrt{1-t^2}$, $-1 < t < 0$, (2) $\varphi(t) > 0$ if $|t| < 1$ and (3) $\varphi(1) = 0$. Let $\overline{D}$ be the solid of revolution in $\mathbf{R}^3$ determined by $\varphi|_{[-1,1]}$ with $z = y = 0$ as the axis of revolution.

The only point of the boundary where ∂D is possibly not Lipschitz is (1,0,0). At all other points ∂D is even $\mathscr{C}^2$. Consider the sets

$$D_n = D \cap \{(x, y, z) \mid 1 - 3/2n < x < 1 - 1/2n\}$$

and

$$A_n = D \cap \{(x, y, z) \mid x = 1 - 1/n\}.$$

By rounding off the « corners » of D_n and using a diffeomorphism the following fact can be obtained: if $u, v \geqslant 0$ are L-harmonic on D_n then there is a constant C, independent of n, such that

$$u(x)/u(1 - 1/n, 0, 0) \leqslant Cv(x)/v(1 - 1/n, 0, 0)$$

for all $x \in A_n$. This result can be easily obtained from inequality (33) in [12].

Given this inequality it follows that if u, v are two Bouligand functions associated to $(1, 0, 0)$ with

$$u(0, 0, 0) = v(0, 0, 0) = 1$$

then $u \leqslant C^2 v$ on $D \cap \{(x, y, z) \mid x < 1 - 1/n\}$ for all u and hence, $u \leqslant C^2 v$.

Note that this argument (which is due to Ancona) applies in many situations e.g. to

$$D = \{(x, y, z) \mid x > 0,\ y^2 + z^2 < 1\}$$

and the point at infinity.

4. The Martin compactification of D relative to L.

M^me^ Hervé [4] showed that, for any $y \in D$, two L-potentials with point support are proportional i.e. the hypothesis of proportionality is satisfied by $\mathscr{H}_L$. Hence, according to [4] there exist Green functions G for L on D. In other words, there exists a lower semi-continuous function $G : D \times D \to \mathbf{R}^+$ continuous off the diagonal and such that, for all $y \in D$, $x \to G(x, y)$ is a potential with support $\{y\}$.

Choose $x_0 \in D$ and define $K(x, y)$ to be 1 if

$$x = y = x_0 \quad \text{and} \quad G(x, y)/G(x_0, y)$$

otherwise. Then there is a unique compactification $\tilde{D} = \tilde{D}_L$ of D such that: (1) for all $x \in D$ the function $y \to K(x, y)$ extends continuously; and (2) the extended functions separate the points of $\tilde{D}\setminus D$ (cf. [15]). This compactification is called the Martin compactification of D relative to L.

Carleson's lemma, in the form of proposition 2.7, implies the following result, due to Brelot [17].

PROPOSITION 4.1. — *There is a unique continuous function* $\pi : \tilde{D} \to \overline{D}$ *such that* $\pi(x) = x$ *for all* $x \in D$.

Proof. — It is known (cf. [2]) that $\tilde{D}$ is a compact metric space. It suffices therefore to show that if $(y_n) \subset D$ converges in $\tilde{D}$ then it converges in $\overline{D}$.

Assume (y_{n_k}) converges in $\overline{D}$ to a point y_0. Let

$$\lim_{n\to\infty} K(x, y_n) = h(x).$$

Let U be a neighbourhood of y_0 and let $x \in D\setminus\overline{U}$. Since $\lim_{k\to\infty} y_{n_k} = y_0$, $\int \hat{R}_{\overline{U}}(x, dz)K(z, y_{n_k}) = K(x, y_{n_k})$ for k sufficiently large. Since, for $x \in D\setminus\overline{U}$, the measure $\hat{R}_{\overline{U}}(x, dz)$ is carried by $\partial(D\setminus\overline{U})$ it follows from proposition 2.7 that

$$\lim_{k\to\infty} \int \hat{R}_{\overline{U}}(x, dz)K(z, y_{n_k}) = \int \hat{R}_{\overline{U}}(x, dz) \lim_{k\to\infty} K(z, y_{n_k}).$$

Hence $\hat{R}_{\overline{U}}h = h$ on $D\setminus\overline{U}$.

From this and the existence of barriers it follows that, $\lim_{x\to x'} h(x) = 0$ if $x' \in \partial D\setminus\overline{U}$. Suppose that some other subsequence $(y_{n_{k'}})$ has limit point $y'_0 \neq y_0$. It then follows that $\lim_{x\to x'} h(x) = 0$ for all $x \in \partial D$ and so $h = 0$. Hence (y_n) converges in $\overline{D}$ to a unique point.

As is well known, the Martin compactification serves for the representation of positive harmonic functions. More precisely, if $h \geqslant 0$ is L-harmonic on D there is a measure μ on $\tilde{D}\setminus D = \Delta$ such that, for all $x \in D$

$$h(x) = \int K(x, y)\, \mu(dy),$$

where

$$K(x, y) = \lim_{n\to\infty} K(x, y_n) \quad \text{if} \quad (y_n) \subset D \quad \text{and} \quad \lim_n y_n = y \in \Delta.$$

Furthermore, there is a unique measure (the *canonical* one) carried by the set of minimal points $\Delta_1 = \{y | K(., y)$ is a minimal harmonic function$\}$ (h is minimal if $0 \leqslant k \leqslant h$, k harmonic implies $k = ch$). For further details see for example [7], [2].

PROPOSITION 4.2. — *Let* B *the support of* $\mu \in M_+(\Delta)$ *and let* $h(x) = \int K(x, y)\, \mu(dy)$. *Then,* $\lim_{x \to y_0} h(x) = 0$ *if* $y_0 \notin \pi(B)$.

Proof. — It follows from the proof of proposition 4.1 that if U is a neighbourhood of $\pi(B)$ with $y_0 \in \overline{U}$ then, for each $y \in B$, $\int \hat{R}_{\overline{U}}(x, dz) K(z, y) = K(x, y)$. Hence, $h = \hat{R}_{\overline{U}} h$.

Furthermore, since $\pi(B)$ is compact, it follows (from the proof of proposition 4.1 and from proposition 2.7) that there exists a constant K_0 with $K(x, y) \leqslant K_0 \; \forall x \in D \setminus \overline{U}, \forall y \in B$. Hence, $h \leqslant K_0 \mu(\tilde{D} \setminus D)$ on $D \setminus \overline{U}$. The result follows since $h = \hat{R}_{\overline{U}} h$.

COROLLARY 4.3. — *Let* h *be a Bouligand function associated with* $y_0 \in \partial D$. *If*

$$\mu \in M^+(\tilde{D} \setminus D) \quad \text{and} \quad h(x) = \int K(x, y) \mu\,(dy)$$

then $supp\, \mu \subset \pi^{-1}\{y_0\}$.

Proof. — Assume $B \cap \pi^{-1}\{y_0\} = \varnothing$ with B compact. Let

$$h_1(x) = \int K(x, y) 1_B(y)\, \mu(dy).$$

Then $h_1 \leqslant h$ and $\lim_{x \to y_0} h_1(x) = 0$ by proposition 4.2. Hence, $h_1 = 0$ and so $\mu(B) = h_1(x_0) = 0$.

Combining this corollary with the lemma of Hunt and Wheeden one obtains a proof of the main theorem.

THEOREM 4.4. — $\tilde{D} = \overline{D}$ *and every boundary point is minimal.*

Proof. — If $y_0 \in \partial D$ corollary 4.3 implies that there is at least one minimal point in $\pi^{-1}\{y_0\}$. First note that π is onto and so $\pi^{-1}\{y_0\}$ is never void. It follows from corollary 4.3 that, for each $y \in \pi^{-1}\{y_0\}$ every integral representation of the harmonic function $x \to K(x, y)$ involves a measure carried by $\pi^{-1}\{y_0\}$. In particular, this is true for the canonical measure, which is carried by the set of minimal points.

The harmonic functions $x \to K(x, y)$ with $\pi(y) = y_0$ are all Bouligand functions associated with y_0. At least one of them is minimal. It then follows from the lemma of Hunt and Wheeden (theorem 3.1) that they are all proportional and hence coincide since they agree at x_0. In other words, π is a continuous bijection and so is a homeomorphism.

This paragraph concludes with the following sharpened version of proposition 4.2.

PROPOSITION 4.5. — *Let* $h \geqslant 0$, L-*harmonic on* D, *be represented by the measure* μ *on* ∂D. *If* $B \subset \partial D$ *is compact then* $B \supset \operatorname{supp} \mu$ *if and only if* $\lim_{x \to y} h(x) = 0$ *for all* $y \in \partial D \setminus B$.

Further, if $\operatorname{supp} \mu \subset B$ *and* U *is a neighbourhood of* B *in* $\mathbf{R}^n$, *there exists a constant* K_0 *such that*

$$h(x) \leqslant K_0 h(x_0) \quad \textit{for all} \quad x \in D \setminus \overline{U}.$$

Proof. — The second statement has been proved above (see the proof of proposition 4.2).

It remains to show that if $\lim_{x \to y} h(x) = 0$ for all $y \in \partial D \setminus B$ then $\operatorname{supp} \mu \subset B$. It suffices by inner regularity, to prove that for any compact set $A \subset \partial D$, $A \cap B = \emptyset$ implies $\mu(A) = 0$. Let $h_1(x) = \int K(x, y) 1_A(y)\, \mu(dy)$. Then $h_1 \leqslant h$ and by proposition 4.2 $\lim_{x \to y} h_1(x) = 0$, for all $y \in B$. Hence, $h_1 = 0$ i.e. $\mu(A) = 0$.

5. The Martin compactification of an open Riemannian Manifold.

Let X denote a $\mathscr{C}^3$-manifold equipped with a $\mathscr{C}^2$-Riemannian metric and let M be an open, relatively compact, connected subset of X with a non-void Lipschitz boundary.

THEOREM 5.1. — *The Martin compactification of* M *relative to the Laplace-Beltrami operator* Δ *is the closure* $\overline{M}$ *of* M *in* X. *Further, all the Martin boundary points are minimal.*

Proof. — It suffices to establish Carleson's lemma in this context. The lemma of Hunt and Wheeden follows from purely local considerations, as is shown by its proof, and it together with Carleson's result gives a proof of the theorem (it suffices to repeat the proof of theorem 4.4).

First note that, if $\mathcal{H}$ is the harmonic sheaf on M determined by Δ, then H has a positive potential. This follows either from the abstract theory of harmonic spaces Loeb [6]) or from the theory of differential equations on manifolds (cf. [1], [11]).

Let $y_0 \in \partial M$ and let U be an open coordinate neighbourhood of y_0 in X such that $\Phi(U \cap M) = D$ where D is a bounded Lipschitz domain in $\mathbf{R}^n$ and $\Phi : U \to \mathbf{R}^n$ is a coordinate map of the manifold X.

Let $h \geqslant 0$ be a hyperharmonic function on M such that $\lim_{x \to y} h(x) = 0$, for all $y \in \partial M \setminus U$. Set $k = h - \hat{R}_{\complement U} h$ and $l = (\hat{R}_{\complement U} h) | U \cap M$. Transporting these functions to D via Φ one obtains functions k', l' on D that are solutions to $Lu \leqslant 0$, where L is a self-adjoint uniformly elliptic operator on D with $\mathscr{C}^1$-coefficients. In fact one can take L to be the image of $\Delta | D$ under Φ and $k' = k \circ \Phi^{-1}$, $l' = l \circ \Phi^{-1}$.

Applying theorem 4.4 to D and L it follows from proposition 4.5 that there is a measure μ on $\Phi(\overline{\partial U \cap M})$ such that $l'(x) = \int K'(x, y)\, \mu(dy)$, K′ being the Martin kernel for D defined by L. Let $x_1 \in U \cap M$ be such that

$$K'(\Phi(x_1), y) = 1,$$

for all $y \in \overline{D}$.

If V is an open neighbourhood of y_0 with $\overline{V} \subset U$ it follows from proposition 4.5 that there exists a constant K_1 with $l'(t) \leqslant K_1 l'(\Phi(x_1))$ for all $t \in \Phi(\overline{V} \cap M)$. Hence,

$$l(x) \leqslant K_1 l(x_1)$$

for all $x \in \overline{V} \cap M$.

Proposition 2.7, applied to D and $\Phi(V)$, implies that there is a neighbourhood 0 of $\Phi(y_0)$ with $\overline{O} \subset \Phi(V)$ and a constant K_2 with $k'(t) \leqslant K_2 k'(\Phi(x_1))$, for all $t \in D \setminus \Phi(\overline{V})$, providing (i) $\lim_{x \to y} h(x) = 0$ for all $y \in \partial M \setminus \overline{\Phi^{-1}(O)}$ and (ii) h is harmonic on $M \setminus \overline{\Phi^{-1}(O)}$.

Therefore, for a positive hyperharmonic function h on M that satisfies (i) and (ii), $h(x) \leqslant (K_1 + K_2)h(x_1)$ for all $x \in M \setminus \overline{V}$. This proves Carleson's lemma (in the form stated as proposition 2.7).

Remark. — Clearly, the above proof shows that $\overline{M}$ is the Martin compactification of M relative to any « reasonable » second order elliptic operator L defined on M, where « reasonable » means that in any coordinate neighbourhood U with $\Phi(U \cap M) = D$ a bounded Lipschitz domain the image of L under Φ belongs to $\Lambda(\gamma, \mu; D)$.

An immediate consequence of theorem 5.1 is the fact that, in an appropriate sense, an open Riemannian manifold M is the interior of at most one compact Riemannian manifold with boundary.

THEOREM 5.2. — *Let $\overline{M}$ denote a compact $\mathscr{C}^{n+2}$-manifold with boundary equipped with $\mathscr{C}^{n+1}$-Riemannian metric. Denote by M the interior of $\overline{M}$. Then $\overline{M}$ is the Martin compactification of M relative to the Laplace-Beltrami operator on M.*

Further, let $\overline{N}$ be a compact $\mathscr{C}^{n+2}$-manifold with boundary equipped with a $\mathscr{C}^{n+1}$-Riemannian metric. Let N denote the interior of $\overline{N}$ and assume $\Phi : M \to N$ is a $\mathscr{C}^{n+1}$-diffeomorphism that preserves the metrics. Then there is a unique $\mathscr{C}^{n+1}$-diffeomorphism $\overline{\Phi} : \overline{M} \to \overline{N}$ that extends Φ.

Proof. — By theorem 5.9 of [10] there is a $\mathscr{C}^{n+2}$-diffeomorphism P of a neighbourhood T in $\overline{M}$ of $\partial\overline{M}$ with $\partial\overline{M} \times (0,1)$ such that $P(x) = (x, 0)$, whenever $x \in \partial\overline{M}$. Using the map P to attach $\partial\overline{M} \times (-1, 1)$ to $\overline{M}$ one obtains a $\mathscr{C}^{n+2}$-manifold X containing $\overline{M}$ as a compact sub-manifold with boundary.

To complete the proof it suffices to show that there exists a $\mathscr{C}^{n+1}$-Riemannian metric on X that coincides with the given one on $\overline{M}$.

Let ξ be the $\mathscr{C}^{n+1}$-bundle over X obtained by replacing the fibre $\mathbf{R}^n$ of the tangent bundle by the convex cone of positive definite symmetric $n \times n$ matrices.

Let s_0 be a section of ξ over X (cf. [14], p. 58).

Denote by s a $\mathscr{C}^{n+1}$-section of ξ over $\overline{\mathrm{M}}$. For each point $x \in \partial\overline{\mathrm{M}}$ there is a neighbourhood U_x of x and $\mathscr{C}^{n+1}$-section s'_x of ξ over U_x such that $s'_x|\mathrm{U}_x \cap \mathrm{M} = s|\mathrm{U}_x \cap \mathrm{M}$. If W_x is a compact neighbourhood of x with $\mathrm{W}_x \subset \mathrm{U}_x$ there is a section s_x of ξ over X for which $s_x|\mathrm{W}_x = s'_x|\mathrm{W}_x$ and $s_x\big|\complement\, \mathrm{U}_x = s_0\big|\complement\, \mathrm{U}_x$.

Let V_x be a compact neighbourhood of x with $\mathrm{V}_x \subset \mathring{\mathrm{W}}_x$ and assume $\bigcup_{i=1}^{n} \mathrm{V}_{x_i} \supset \partial\overline{\mathrm{M}}$. Denote by $(\Phi_i)_{i\in\mathrm{I}}$ a $\mathscr{C}^{n+2}$-partition of unity subordinate to the cover $\mathring{\mathrm{W}}_{x_1}, \mathring{\mathrm{W}}_{x_2}, \ldots, \mathring{\mathrm{W}}_{x_n}$, $\mathrm{X}\Big\backslash\left(\bigcup_{i=1}^{n} \mathrm{V}_{x_i}\right)$ and set $t = \sum_{i\in\mathrm{I}} \left(\sum_{i=1}^{n} \Phi_i s_{x_i}\right)\cdot$ Then t is a section of ξ and if $x \in \left(\bigcup_{i=1}^{n} \mathring{\mathrm{W}}_{x_i}\right) \cap \mathrm{M}$, $t(x) = s(x)$.

Hence if ψ_1 and ψ_2 are appropriately chosen $\mathscr{C}^{n+1}$-functions $s_1 = \psi_1 t + \psi_2 s$ is a section of ξ over X that agress with s on $\overline{\mathrm{M}}$.

The diffeomorphism $\Phi : \mathrm{M} \to \mathrm{N}$ obviously extends as a homeomorphism $\overline{\Phi} : \overline{\mathrm{M}} \to \overline{\mathrm{N}}$. Let $z_0 \in \partial\mathrm{M}$ and $w_0 = \Phi(z_0)$. There is point $z_1 \in \mathrm{M}$ such that a geodesic neighbourhood U of z_1 in X is a neighbourhood of z_0 and a geodesic neighbourhood V of $\Phi(z_1) = w_1$ in Y (an open manifold containing $\overline{\mathrm{N}}$ as X contains $\overline{\mathrm{M}}$) is a neighbourhood of w_0. It can be assumed that $\overline{\Phi}(\mathrm{U} \cap \overline{\mathrm{M}}) \subset \mathrm{V} \cap \overline{\mathrm{N}}$. Then it is easy to see that $\overline{\Phi}$ commutes with the appropriate exponential maps and hence $\overline{\Phi}$ is a $\mathscr{C}^{n+1}$-diffeomorphism.

BIBLIOGRAPHY

[1] J. M. Bony, Majorations *a priori* et problèmes frontières elliptiques du second ordre, Sem. Choquet (Initiation à l'Analyse) 5e année 1965-1966 exposée 3.

[2] M. Brelot, On topologies and boundaries in potential theory, *Lecture Notes in Mathematics* 175 Springer-Verlag, Berlin-Heidelberg, New York, 1971.
[3] L. Carleson, On the existence of boundary values for harmonic functions in several variables, *Arkiv för Math.*, 4 (1962), 393-399.
[4] Mme R.-M. Hervé, Recherches axiomatiques sur la théorie des fonctions surharmoniques et du potentiel, *Ann. Inst. Fourier*, 12 (1962), 415-571.
[5] R. Hunt and R. Wheeden, Positive harmonic functions on Lipschitz domains, *Trans. Amer. Math. Soc.*, 147 (1970), 507-527.
[6] P. Lœb, An axiomatic treatment of pairs of elliptic differential equations, *Ann. Inst. Fourier*, 16 (2) (1966), 167-208.
[7] R. S. Martin, Minimal positive harmonic functions, *Trans. Amer. Math. Soc.*, 49 (1941), 137-172.
[8] P.-A. Meyer, Probability and potentials, Blaisdell Publishing Company, Waltham, Mass. 1966.
[9] K. Miller, Barriers on cones for uniformly elliptic operators, *Ann. di Mat. pura ed appl.* (IV), 76 (1967), 93-106.
[10] J. R. Munkres, Elementary differential Topology, *Annals of Math. Studies* 54, Princeton University Press, Princeton N.J., 1963.
[11] M. Schiffer and D. C. Spencer, Functionals of Finite Riemann Surfaces, Princeton University Press, Princeton, N.J., 1954.
[12] J. Serrin, On the Harnack inequality for linear elliptie equations, *Jour. d'Anal. Math.*, 4 (1955-1956), 297-308.
[13] J. Stallings, On infinite processes leading to differentiability in the complement of a point, publ. in Differential and Combinatorial Topology ed., by S. S. Cairns, Princeton University Press, Princeton, N.J., 1965.
[14] N. Steenrod, The Topology of Fibre Bundles, Princeton University Press, Princeton, N.J., 1951.
[15] J. C. Taylor, The Martin boundary of equivalent sheaves, *Ann. Inst. Fourier*, XX (1) (1970) 433-456.
[16] A. Ancona, Principe de Harnack à la frontière et théorème de Fatou pour un opérateur elliptique dans un domaine lipschitzien, to appear.
[17] M. Brelot, Remarques sur la variation des fonctions sous-harmoniques, *Ann. Inst. Fourier*, 2 (1950), 101-112.
[18] S. Ito, Martin boundary for linear elliptic differential operators of second order in a manifold, *J. Math. Soc. Japan*, 16 (1964), 307-334.

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