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by Bent FUGLEDE

Introduction.

The harmonic morphisms of a Riemannian manifold, M, into another, N, are the morphisms for the harmonic structures on M and N (the harmonic functions on a Riemannian manifold being those which satisfy the Laplace-Beltrami equation). These morphisms were introduced and studied by Constantinescu and Cornea [4] in the more general frame of harmonic spaces, as a natural generalization of the conformal mappings between Riemann surfaces ([1]). See also Sibony [14].

The present paper deals with harmonic morphisms between two Riemannian manifolds M and N of arbitrary (not necessarily equal) dimensions. It turns out, however, that if $\dim M < \dim N$, the only harmonic morphisms $M \to N$ are the constant mappings. Thus we are left with the case

$$\dim M \geqslant \dim N .$$

If $\dim N = 1$, say $N = \mathbf{R}$, the harmonic morphisms $M \to \mathbf{R}$ are nothing but the harmonic functions on M.

If $\dim M = \dim N = 2$, so that M and N are Riemann surfaces, then it is known that the harmonic morphisms of

([1]) We use the term « harmonic morphisms » rather than « harmonic maps » (as they were called in the case of harmonic spaces in [4]) since it is necessary (in the present frame of manifolds) to distinguish between these maps and the much wider class of harmonic maps in the sense of Eells and Sampson [7], which also plays an important role in our discussion.

M into N are the same as the *conformal* mappings $f\colon \mathrm{M} \to \mathrm{N}$ (allowing for points where $df = 0$). However, if

$$\dim \mathrm{M} = \dim \mathrm{N} \neq 2\,,$$

the only non-constant harmonic morphisms (if any) are the conformal mappings with *constant coefficient of conformality* (i.e., the local isometries up to a change of scale, Theorem 8.)

In the general case where just $\dim \mathrm{M} \geqslant \dim \mathrm{N} \geqslant 1$ we find (Theorem 7) that $f\colon \mathrm{M} \to \mathrm{N}$ is a harmonic morphism if and only if f is both a *harmonic mapping* in the sense of Eells and Sampson [7] and a *semiconformal mapping* in the sense that the restriction of f to the set of points at which $df \neq 0$ is a *conformal submersion* (see § 5 for a more explicit definition). The simple case $\mathrm{N} = \mathbf{R}^n$ is settled independently in Theorem 2 below ([2]).

The *symbol* $\sigma_a(f)$ of a harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ at a point $a \in \mathrm{M}$ is a harmonic morphism $\sigma_a(f)\colon\ \mathrm{M}_a \to \mathrm{N}_{f(a)}$ (Theorem 9). Using this we prove that every non-constant harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ is an *open mapping* (Theorem 10). This theorem extends the classical fact that every non-constant conformal mapping f between Riemann surfaces is open, even if points with $df = 0$ are allowed. Theorems 9 and 10 also treat the more general case of semiconformal mappings. The methods employed are classical.

Constantinescu and Cornea proved in [4] that every non-constant harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ between Brelot harmonic spaces M and N is open with respect to the *fine topologies* ([3]) on M and N, provided that the points of N are all *polar* (which in the case of a Riemannian manifold N amounts to $\dim \mathrm{N} \geqslant 2$). This important result was obtained by purely potential theoretic methods, and it neither implies nor follows from our result (in the case of Riemannian manifolds). It is not known whether every non-constant harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ between Brelot harmonic spaces M

([2]) Recall that, in the case $\mathrm{N} = \mathbf{R}^n$, a mapping $f\colon \mathrm{M} \to \mathbf{R}^n$ is harmonic in the sense of Eells and Sampson if and only if the components $f^1, \ldots, f^n$ are harmonic functions on M.

([3]) The *fine topology* on a harmonic space is the coarsest topology making all subharmonic functions continuous. It is finer than the initially given topology on the space.

and N is open in the initially given topologies on M and N when the points of N are polar (4).

After giving some examples of harmonic morphisms (§ 11), I close by indicating briefly an easy extension of the preceding development to what I call h-harmonic morphisms (§ 12), an example of which is the classical Kelvin transformation.

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Table of Contents.

(4) The hypothesis that the points of N be polar cannot be removed, as shown by an unpublished, simple example due to Cornea (personal communication). In this example $N = \mathbf{R}$ (so that all points of N are non-polar), while the harmonic space M is a well-behaved Brelot harmonic space (not a manifold), and the fine topology on M coincides with the initial topology, just as in the case of $N = \mathbf{R}$.

Added in proof (February 1978). — The hypothesis of the Brelot convergence axiom cannot be removed, as shown recently by an example due to H. and U. Schirmeier (personal communication).

1. Notations and preliminaries.

The Riemannian manifolds to be considered should be connected, second countable, and (for simplicity) infinitely differentiable.

The metric tensor on a Riemannian manifold M is denoted by g or g_{M}. The Laplace-Beltrami operator on M, denoted by Δ or Δ_{M}, is given in local coordinates x^i by

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i}\left(\sqrt{|g|}\, g^{ij} \frac{\partial}{\partial x^j}\right),$$

using here and elsewhere the customary summation convention, and writing

$$|g| = \det(g_{ij}).$$

A real-valued function u of class C^2, defined in an open subset of M, is called *harmonic* if $\Delta_{\mathrm{M}} u = 0$. It follows that u is of class C^∞. The constant functions are of course harmonic.

As shown by R.-M. Hervé [9, Chap. 7], the sheaf of harmonic functions in this sense turns the manifold M into a Brelot harmonic space (in the slightly extended sense adopted in Constantinescu and Cornea [5] in order to include the case M compact) (5).

A C^2-function u defined in an open set $U \subset M$ is *subharmonic* if and only if $\Delta u \geqslant 0$ [9, Prop. 34.1].

We shall make extensive use of the Carleman-Aronszajn-Cordes *uniqueness theorem* for harmonic functions: If the partial derivatives of all orders of a harmonic function u on M (using local coordinates) vanish at a point of M, then u is constant. (See [1], [6].)

According to an interesting, recent result by Greene and Wu [8], every non-compact n-dimensional manifold N admits a proper embedding ψ into $\mathbf{R}^{2n+1}$ such that the components ψ^k are harmonic on N. It follows in particular

(5) If M is $\mathfrak{P}$-harmonic space (that is, if there exists a potential > 0 on M), then M satisfies all the axioms in Hervé [9, Ch. 6]. In any event, M can be covered by open subsets which are $\mathfrak{P}$-harmonic in their induced harmonic structure. (For a general result to this effect see [5, Theorem 2.3.3].)

that one can always choose *harmonic local coordinates* for a Riemannian manifold.

From now on, let M and N denote two Riemannian manifolds as above, of dimension

$$\dim \mathrm{M} = m\,, \qquad \dim \mathrm{N} = n\,,$$

and consider a mapping

$$f\colon \mathrm{M} \to \mathrm{N}\,.$$

DEFINITION. — *A continuous mapping* $f\colon \mathrm{M} \to \mathrm{N}$ *is called a harmonic morphism if* $\varphi \circ f$ *is a harmonic function in* $f^{-1}(\mathrm{V})$ *for every function* φ *which is harmonic in an open set* $\mathrm{V} \subset \mathrm{N}$ (*such that* $f^{-1}(\mathrm{V}) \neq \varnothing$).

Every harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ is of class C^∞. In fact, using harmonic local coordinates (y^k) in N (invoking [8]), we arrange locally that the components $f^k = y^k \circ f$ become harmonic and hence C^∞ in M.

Among the general results obtained by Contantinescu and Cornea in [4, § 3] for harmonic morphisms (between harmonic spaces), we shall only make use of the following two basic properties:

A non-constant harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ « pulls back » subharmonic functions, in the sense that « harmonic » can be replaced by « subharmonic » in the above definition [4, Cor. 3.2]. In our case of Riemannian manifolds M, N we therefore have

$$(1) \qquad \Delta_{\mathrm{N}}\varphi \geqslant 0 \quad \text{in} \quad \mathrm{V} \Longrightarrow \Delta_{\mathrm{M}}(\varphi \circ f) \geqslant 0 \quad \text{in} \quad f^{-1}(\mathrm{V})$$

for any C^2-function φ defined in an open set $\mathrm{V} \subset \mathrm{N}$.

The pre-image $f^{-1}(\mathrm{P})$ of any *polar set* $\mathrm{P} \subset \mathrm{N}$ is polar in M if the harmonic morphism $f\colon \mathrm{M} \to \mathrm{N}$ is non-constant [4, Theorem 3.2] (6).

Constant mappings are of course harmonic morphisms, and so are locally isometric mappings in the case $m = n$.

(6) A subset P of a Riemannian manifold (or a harmonic space) M is called *polar* if every point of M has an open neighbourhood U in which there exists a subharmonic function s such that $s = -\infty$ in $\mathrm{P} \cap \mathrm{U}$.

Every harmonic morphism $f: M \to N$ with M compact and N non-compact, is constant. In fact, via a harmonic embedding $\psi : N \to \mathbf{R}^{2n+1}$ (invoking [8]), we obtain harmonic functions $\psi^k \circ f$ on M, and the harmonic functions on a compact Riemannian manifold M are all constant according to the maximum principle.

Returning to general Riemannian manifolds M and N, it is obvious that the notion of a harmonic morphism is purely *local.* Explicitly: The assignment, to every nonvoid open set $U \subset M$, of the set of all harmonic morphisms of the submanifold U into N, defines a *sheaf* (more precisely a complete pre-sheaf).

If L denotes a further Riemannian manifold, then the composition of a harmonic morphism $\varphi : L \to M$ with a harmonic morphism $f: M \to N$ determines a harmonic morphism $f \circ \varphi : L \to N$, as it should be.

2. The case $N = \mathbf{R}^n$.

THEOREM — *For any m-dimensional Riemannian manifold* M *and any Euclidean space* $\mathbf{R}^n$ *(with its standard Riemannian structure), the following properties of a mapping* $f: M \to \mathbf{R}^n$ *are equivalent:*

1) $f: M \to \mathbf{R}^n$ *is a harmonic morphism.*

2) *For every harmonic polynomial* H *on* $\mathbf{R}^n$ *of degree* 1 *or* 2, $H \circ f$ *is a harmonic function on* M.

3) *The components* $f_1, \ldots, f_n$ *of* f *are harmonic in* M, *and their gradients are mutually orthogonal and of equal length* $\lambda(x)$ *at each point* $x \in M$:

$$g_M(\nabla f_k, \nabla f_l) = \lambda^2 \, \delta_{kl}$$

for every $k, l = 1, \ldots, n$.

4) f *is* C^2, *and there exists a function* $\lambda \geqslant 0$ *on* M *such that*

$$\Delta_M(v \circ f) = \lambda^2[(\Delta v) \circ f]$$

for every C^2*-function* v *on* $\mathbf{R}^n$.

If f is a non-constant harmonic morphism, then $m \geqslant n$, and the functions λ in 3) and 4) are uniquely determined and identical, and λ^2 is C^∞. Moreover, the first differential $df\colon M_x \to \mathbf{R}^n$ is surjective at any point $x \in M$ at which $df \neq 0$, hence in a dense, open subset of M (by the uniqueness theorem [1], [6]). We call λ the dilatation of f.

Proof. — The implication 1) $\Longrightarrow$ 2) is trivial. To prove that 2) $\Longrightarrow$ 3), take first $H(y) = y_k$ $(k = 1,\dots,n)$ to show that each f_k is harmonic (in particular C^∞). Next take $H(y) = y_k y_l$ and $H(y) = y_k^2 - y_l^2$ with $k \neq l$ to show that

$$2g_M(\nabla f_k,\nabla f_l) = \Delta_M(f_k f_l) - f_k\,\Delta_M f_l - f_l\,\Delta_M f_k = 0$$

and similarly that the function $\lambda \geqslant 0$ given by

$$\lambda^2 = g_M(\nabla f_k,\nabla f_k)$$

is independent of $k = 1, \dots, n$.

The implication 3) $\Longrightarrow$ 4) is straightforward on account of the elementary identity

$$\Delta_M(\varphi \circ f) = g_M(\nabla f_k,\nabla f_l)\left(\frac{\partial^2\varphi}{\partial y_k\,\partial y_l}\circ f\right) + (\Delta_M f_k)\left(\frac{\partial\varphi}{\partial y_k}\circ f\right)$$

(with summation over $k, l = 1, \dots, n$ in the former term on the right, and over $k = 1, \dots, n$ in the latter).

Finally, 4) $\Longrightarrow$ 1) because property 4) can be localized as follows: For any open set $V \subset \mathbf{R}^n$ and any $\varphi \in C^2(V)$, the stated relation holds in $f^{-1}(V)$. In fact, for any given $x \in f^{-1}(V)$, φ agrees in some neighbourhood $W \subset V$ of $f(x)$ with a function $\psi \in C^2(\mathbf{R}^n)$, and so we have $\varphi \circ f = \psi \circ f$ and $(\Delta\varphi) \circ f = (\Delta\psi) \circ f$ in the neighbourhood $f^{-1}(W)$ of x, showing that the desired relation holds in $f^{-1}(V)$.

At any point $x \in M$, $\nabla f_1, \dots, \nabla f_n$ span the orthogonal complement $K_x^\perp$ of $K_x = \ker df$ (at x) within the tangent space M_x. Hence 3) shows that if $df \neq 0$ at x, then $\lambda(x) > 0$, and the orthogonal vectors $\nabla f_1(x), \dots, \nabla f_n(x)$ are linearly independent, so that $K_x^\perp$ is n-dimensional and df surjective at x.

Remark. — In the case $M \subset \mathbf{R}^n = N$ the equivalence of 1) and 3) in the above theorem can be traced back essentially to Cioranesco [3]. The above simple proof of the theorem does not seem to extend to the case of a general n-dimensional Riemannian target manifold N except for $n = 2$. For although it is possible, by virtue of [8], to choose local coordinates (y^k) in N so that each y^k is harmonic in N, one cannot arrange that also $y^k y^l$ and $(y^k)^2 - (y^l)^2$ be likewise harmonic for $k \neq l$, except if N is locally conformally Euclidean, and even locally isometrically Euclidean if $n \neq 2$. (In fact, the stated properties of the local coordinates y^k would mean that the coordinate mapping from N into $\mathbf{R}^n$ would be a harmonic morphism by 2) of the above theorem, and hence a local isometry up to a change of scale (if $n \neq 2$) by Theorem 8 below).

In the sequel we shall see that, nevertheless, the above theorem extends, after deletion of Property 2), to the case of a general Riemannian manifold N of any dimension n. This extension will be accomplished in Lemmas 3 and 4 and in Theorem 7.

Examples. — 1) The *multiplication of quaternions* is a harmonic morphism $\mathbf{H} \times \mathbf{H} \to \mathbf{H}$ (when the quaternion field $\mathbf{H}$ is identified in the standard way with $\mathbf{R}^4$ as a Riemannian manifold). The dilatation is given by

$$\lambda(x, y)^2 = |x^2| + |y|^2.$$

The verification is simple. Similarly with $\mathbf{C}$ (or $\mathbf{R}$) in place of $\mathbf{H}$.

2) Every *holomorphic function* $f: \mathrm{M} \to \mathbf{C}$, defined in a domain M in $\mathbf{C}^m = \mathbf{R}^{2m}$, is a harmonic morphism of M into $\mathbf{C} = \mathbf{R}^2$ with dilatation $\geqslant 0$ given by

$$\lambda^2 = \sum_{j=1}^{m} \left|\frac{\partial f}{\partial z_j}\right|^2.$$

This follows e.g., from 2) of the above theorem since f and f^2 are complex harmonic functions on M. (See also § 11.1.)

3. On the regular points for a harmonic morphism.

Definition. — *A regular point for a* C^1*-mapping* $f: \mathrm{M} \to \mathrm{N}$ *is a point of* M *at which the differential* df *is surjective. The open set of all regular points for* f *is denoted by* M′.

LEMMA. — *The set* M' *of regular points for a non-constant harmonic morphism* $f: M \to N$ *is dense in* M.

Proof. — If $n = 1$, we may assume that $N = \mathbf{R}$ (with its standard Riemannian metric), and hence that $f: M \to \mathbf{R}$ is a non-constant harmonic function on M. By the uniqueness theorem [1], [6], f cannot be constant in a neighbourhood U of a point of M, and hence $df = 0$ cannot hold in all of U.

Now let $n \geqslant 2$, and consider a non-void open subset U of M. Let r denote the highest rank of df attained in U, and let $x \in U$ be such that df has rank r at x. Choose an open neighbourhood $V \subset U$ of x in M such that df has rank r at every point of V and further that the restriction $f|V$ is a submersion of V onto an r-dimensional submanifold $f(V)$ of N. According to [4, Theorem 3.5], $f(V)$ is a fine neighbourhood of $f(x)$ in N, and hence $r = n$ (7).

We prefer, however, to give also another proof that $r = n$, using less advanced tools. Consider an embedded submanifold $P \subset f(V)$ of dimension

$$p = \min(r, n-2).$$

Since $p \leqslant n-2$, P is a polar subset of N, and hence $f^{-1}(P)$ is polar in M by [4, Theorem 3.2] quoted above. Clearly $V \cap f^{-1}(P)$ contains an embedded submanifold of M of dimension $p + (m - r)$, and since this submanifold is polar in M, we obtain

$$p + m - r \leqslant m - 2,$$

showing that $p \leqslant r - 2$. In particular $p < r$, and hence $p = n - 2$ by the definition of p. Consequently

$$n - 2 \leqslant r - 2,$$

and so $n = r$ since trivially $n \geqslant r$.

(7) By a *fine neighbourhood* W of a point $y \in N$ is understood a neighbourhood of y in the fine topology on N (the coarsest topology making all subharmonic functions continuous). Equivalently, $N \setminus W$ should be « thin » (effilé) at y in the sense of Brelot. An r-dimensional embedded submanifold of an n-dimensional Riemannian manifold N has no finely interior points except if $r = n$. In our alternative proof of the conclusion $r = n$ above we shall use instead the equally well-known fact that an r-dimensional embedded submanifold of N is *polar* if and only if $r \leqslant n - 2$.

COROLLARY. — *If* $m < n$ (*that is, if* $\dim M < \dim N$), *then every harmonic morphism of* M *into* N *is constant.*

Remark. — Once Theorem 7 below is established, the above Lemma 3 becomes obvious in view of the uniqueness theorem, as explained above in the case $n = 1$, and earlier in the case $N = \mathbf{R}^n$ treated in Theorem 2. In fact, Theorem 7 implies that every *critical* (= non-regular) point $x \in M$ for a harmonic morphism $f : M \to N$ has rank 0, that is, $df = 0$ at x. The lemma may be sharpened as follows:

The set M_0 *of all critical points for a non-constant harmonic morphism* $f : M \to N$ *is polar in* M.

Since every critical point for f has rank 0, it follows from Sard [13, Theorem 2] that $E = f(M_0)$ has Hausdorff measure $\mu_\alpha(E) = 0$ for every $\alpha > 0$ (here we use that f is C^∞). When $n \geqslant 3$, we may take $\alpha < n - 2$ and conclude that E is polar in N. As mentioned in § 1, this implies that $M_0 \subset f^{-1}(E)$ is indeed polar in M.

An alternative, more elementary proof, valid for any dimensions m and n, can be given, using the implicit function theorem after having reduced locally to the case of a harmonic function $u = \varphi \circ f$, where φ is chosen harmonic and non-constant in an open subset of N. And it turns out that the set of critical points for any non-constant harmonic function u on M can be covered by a countable family of $(m - 2)$-dimensional submanifolds imbedded in M, and consequently the critical set is polar.

If the manifolds M and N are *real analytic*, then so is the harmonic morphism f, and hence the set $E = f(M_0)$ of critical values for f is *countable*, cf. Kellogg [10, p. 276].

4. A first characterization of harmonic morphisms.

LEMMA. — *A* C^2*-mapping* $f : M \to N$ *is a harmonic morphism if and only if there exists a function* $\lambda \geqslant 0$ *on* M (*necessarily unique and such that* λ^2 *is* C^∞) *with the property that*

$$(2) \qquad \Delta_M(\varphi \circ f) = \lambda^2[(\Delta_N \varphi) \circ f]$$

for all C^2*-functions* $\varphi : N \to \mathbf{R}$.

Remark. — The stated condition can be localized so as to apply to functions v defined merely in open subsets V of N. This is shown just like in the special case $N = \mathbf{R}^n$, see the proof of 4) $\Longrightarrow$ 1) in Theorem 2 above.

Proof of the lemma. — The « if part » is obvious in view of the above remark. The uniqueness of λ and smoothness of λ^2 are established by choosing, for any given point $x \in M$, a C^∞ function w_x in some open neighbourhood W_x of $f(x)$ so that $\Delta_N w_x > 0$ in W_x (8). Then λ is determined uniquely in the open neighbourhood $f^{-1}(W_x)$ of x by $\lambda = \lambda_x$, where

$$(3) \qquad \lambda_x^2 = \frac{\Delta_M(w_x \circ f)}{(\Delta_N w_x) \circ f} \qquad \text{in} \qquad f^{-1}(W_x),$$

and this function λ_x^2 is C^∞ because w_x and the harmonic morphism f are C^∞.

The « only if part » is obvious (with $\lambda = 0$) if f is constant. Suppose therefore that f is non-constant. We shall prove below that, for every regular point $p \in M'$ there are open neighbourhoods U_p and V_p of p in M and of $f(p)$ in N, and moreover a function $\mu_p \geqslant 0$ on U_p such that $f(U_p) \subset V_p$ and that (2) holds in U_p (with $\lambda = \mu_p$) for all $v \in C^2(V_p)$ (hence for all $v \in C^2(N)$).

Suppose for a moment that this has been achieved. It follows then, for any $x \in M$ and $p \in M'$, that

$$(4) \qquad \lambda_x = \mu_p \qquad \text{in} \qquad f^{-1}(W_x) \cap U_p,$$

in particular at the point $p \in M'$ if $p \in f^{-1}(W_x)$. (Use the above remark, taking $v = w_x$ restricted to $V = W_x \cap V_p$.) From this we obtain for any $x, y \in M$

$$\lambda_x(p) = \mu_p(p) = \lambda_y(p)$$

(8) In terms of a local coordinate system $(y^1, \ldots, y^n)$ for N centered at $f(x)$ it suffices to define

$$w_x(y^1, \ldots, y^n) = \sum_{k=1}^{n} (y^k)^2.$$

Then $\Delta_N w_x$ takes at $f(x)$ the value $2 \sum_{k=1}^{n} g_N^{kk}(f(x)) > 0$, and hence $\Delta_N w_x > 0$ in some neighbourhood of $f(x)$.

for $p \in f^{-1}(W_x) \cap f^{-1}(W_y) \cap M'$; and hence

$$\lambda_x = \lambda_y \quad \text{in} \quad f^{-1}(W_x) \cap f^{-1}(W_y),$$

because M' is dense in M according to Lemma 3. This shows that all the functions $\lambda_x (x \in M)$ have a common extension to a function $\lambda \geqslant 0$ on all of M and such that λ^2 is C^∞. And this function λ likewise extends any of the functions $\mu_p (p \in M')$ in view of (4). For any $\varphi \in C^2(N)$ we therefore have (2) in M' (namely in the abovementioned neighbourhood U_p of any given point $p \in M'$). Finally (2) extends by continuity to all of M, again by Lemma 3.

It remains to establish the above assertion concerning an arbitrary given point p of M'. Since df is surjective at p, a suitable choice of open neighbourhoods U_p of p in M and V_p of $f(p)$ in N will allow us, via local coordinates, to reduce the situation to the case where M and N are open subsets of $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively, (though not with the standard metrics), such that

$$M = N \times Z$$

for some open set $Z \subset \mathbf{R}^{m-n}$, and further that

$$f(x) = f(y,z) = y$$

for every $x = (y,z) \in M$.

To complete the proof we shall show that under these particular circumstances, there does exist a function $\lambda \geqslant 0$ on M such that (2) holds for every $\varphi \in C^2(N)$ (under the hypothesis that $f \colon M \to N$ is a harmonic morphism).

For any function $y \longmapsto \varphi(y)$ on N, $\varphi \circ f$ is the same function φ, but now viewed as a function of $x = (y,z) \in M$ not depending on $z \in Z$. For *fixed* $z \in Z$, $\Delta_M(\varphi \circ f)$ is the result of the action on φ of an elliptic differential operator L_z of the form

$$L_z\varphi = a^{kl} \frac{\partial^2 \varphi}{\partial y^k \, \partial y^l} + b^k \frac{\partial \varphi}{\partial y^k}$$

(with summation over k, $l = 1, \ldots, n$). The coefficients $a^{kl} = g_M^{kl}(\,.\,,z)$ and b^k are C^∞-functions of $y \in N$ for our fixed $z \in Z$. Thus we have

$$\Delta_M(\varphi \circ f) = (L_z \varphi) \circ f. \tag{5}$$

Now L_z and Δ_N are both linear elliptic second order operators with C^∞ coefficients and with no term without derivatives. Since f is supposed to be a harmonic morphism we infer from (1), § 1, that $\Delta_N \varphi \geqslant 0$ implies $\Delta_M(\varphi \circ f) \geqslant 0$, and hence $L_z \varphi \geqslant 0$ by (5). It follows therefore from the argument given by Bony [2, § 5] in a quite similar situation that there is a continuous function ρ_z on N (depending on the temporarily fixed point $z \in Z$) such that

$$L_z \varphi = \rho_z \Delta_N \varphi$$

for all $\varphi \in C^2(N)$, and hence by (5)

$$\Delta_M(\varphi \circ f) = (\rho_z \Delta_N \varphi) \circ f = (\rho_z \circ f)[(\Delta_N \varphi) \circ f]$$

for all $\varphi \in C^2(N)$. Clearly $\rho_z \geqslant 0$ since $(g_M^{kl}(y,z))$ and $(g_N^{kl}(y))$ (both with $k,l = 1,\ldots,n$) are positive definite matrices, and they determine the leading terms in $\Delta_M(\varphi \circ f)$ and $(\Delta_N \varphi) \circ f$, respectively. The function $\lambda = \sqrt{\rho_z \circ f}$ depends on $y \in N$ and also on $z \in Z$, thus altogether on $x = (y,z) \in M$, and we have established (2) for this function $\lambda \geqslant 0$ under the circumstances specified above. This completes the proof of the lemma.

5. Semiconformal mappings.

Definition. — *Suppose that* $m \geqslant n$, *that is,*

$$\dim M \geqslant \dim N.$$

A C^1*-mapping* $f: M \to N$ *is called semiconformal if the restriction of* f *to the set of points of* M *at which* $df \neq 0$ *is a (horizontally) conformal submersion.*

Explicitly, this means that, for any point $x \in M$ at which $df \neq 0$, the *restriction* of df to the orthogonal complement $K_x^\perp$ of $K_x = \ker df$ within the tangent space M_x should be *conformal and surjective.*

DEFINITION. — *The dilatation* $\lambda : \mathrm{M} \to \mathbf{R}$ *of a semiconformal mapping* $f\colon \mathrm{M} \to \mathrm{N}$ *is the coefficient of conformality of the above restriction, interpreted as* 0 *at points where* $df = 0$ *. Thus*

$$\lambda(x) = \|df\|,$$

the operator norm of df *at* x *.*

Altogether, a semiconformal mapping with dilatation λ is a C^1-mapping $f\colon \mathrm{M} \to \mathrm{N}$ such that df is surjective whenever $\neq 0$, and that $df|\, \mathrm{K}_x^\perp$ simply multiplies distances by $\lambda(x)$:

$$g_{\mathrm{N}}^{f(x)}\,(df(\mathrm{X}_1),df(\mathrm{X}_2)) = \lambda(x)^2 g_{\mathrm{M}}^x(\mathrm{X}_1,\mathrm{X}_2) \tag{6}$$

for every pair of tangent vectors X_1, $\mathrm{X}_2 \in \mathrm{K}_x^\perp$. Clearly $\lambda = \|df\|$ is a continuous function on M.

If $n = 1$, every C^1-mapping $f\colon \mathrm{M} \to \mathrm{N}$ is semiconformal (with $\lambda^2 = g_{\mathrm{M}}(\nabla f,\nabla f)$ if $\mathrm{N} = \mathbf{R}$).

If $n = m$, then $f\colon \mathrm{M} \to \mathrm{N}$ is semiconformal if and only if df is conformal off the points where $df = 0$.

In the general case $m \geqslant n$ we have the following dual characterization of semiconformal mappings and their dilatations:

LEMMA. — *A* C^1*-mapping* $f\colon \mathrm{M} \to \mathrm{N}$ *is semiconformal with the dilatation* $\lambda \geqslant 0$ *if and only if*

$$g_{\mathrm{M}}(\nabla f^k,\nabla f^l) = \lambda^2 . (g_{\mathrm{N}}^{kl} \circ f) \tag{7}$$

for $k, l = 1, \ldots, n$, *whereby the* $f^k = y^k \circ f$ *are the components of* f *in terms of (any) local coordinates* $y^1, \ldots, y^n$ *for* N, *and* $\nabla = \nabla_{\mathrm{M}}$ *denotes the gradient operator on* M.

Proof. — Clearly (7) holds at a point $x \in \mathrm{M}$ with $\lambda(x) = 0$ if and only if $\nabla f^1 = \cdots = \nabla f^n = 0$, which means $df = 0$ at x. It therefore remains to consider the case where the points of M are *regular*, and where $\lambda > 0$.

Since $\nabla f^1(x), \ldots, \nabla f^n(x)$ span $\mathrm{K}_x^\perp$, we see from (6) that f is semiconformal with the dilatation λ if and only if

$$g_{\mathrm{N}}(df(\nabla f^k),df(\nabla f^l)) \circ f = \lambda^2 g_{\mathrm{M}}(\nabla f^k,\nabla f^l) \tag{8}$$

for all $k, l = 1, \dots, n$. Now the contravariant components of $df(\nabla f^k)$ are $g_M(\nabla f^\alpha, \nabla f^k)$, $\alpha = 1, \dots, n$. Hence (8) reads

$$(9) \quad (g^N_{\alpha\beta} \circ f) g_M(\nabla f^\alpha, \nabla f^k) g_M(\nabla f^\beta, \nabla f^l) = \lambda^2 g_M(\nabla f^k, \nabla f^l),$$

where the $g^N_{\alpha\beta}$ denote the covariant components of g_N, and we sum over $\alpha, \beta = 1, \dots, n$.

If we introduce the symmetric $n \times n$ matrices

$$G = (g^N_{kl} \circ f), \qquad A = (g_M(\nabla f^k, \nabla f^l)),$$

then (9) takes the form

$$(10) \qquad AGA = \lambda^2 A .$$

Now, if f is semiconformal with dilatation λ (> 0), then df is surjective (at each point $x \in M$), and hence $\nabla f^1, \dots, \nabla f^n$ are linearly independent (at x). It follows that their Grammian with respect to the inner product g_M (at x) is $\neq 0$, that is, $\det A \neq 0$. Hence (10) implies

$$(11) \qquad A = \lambda^2 G^{-1}$$

because also G is invertible (at x). Conversely (11) trivially implies (10).

Since the elements of G^{-1} are the contravariant components of $g_N^{f(x)}$, (11) is a reformulation of the desired condition (7), and the proof is complete.

Without recourse to local coordinates in N, the lemma reads as follows :

Corollary. — *A C^1-mapping $f : M \to N$ is semiconformal with the dilatation λ ($\geqslant 0$) if and only if*

$$g_M(\nabla_M(v \circ f), \nabla_M(w \circ f)) = \lambda^2 [g_N(\nabla_N v, \nabla_N w) \circ f]$$

for every pair of C^1-functions v, w on N.

Remark. — As in the case of Lemma 4, the condition stated in the above corollary extends to a localized version, in which v and w may be defined just in open subsets of N. Also note that, by polarization, it suffices to consider pairs (v, w) with $v = w$.

6. Harmonic mappings.

DEFINITION. — *The tension field* $\tau(f)$ *of a* C^2*-mapping* $f: M \to N$ *is the vector field along* f *which to each point* $x \in M$ *assigns the tangent vector, denoted* $\tau(f)(x)(\in N_{f(x)})$, *whose contravariant components* $\tau^k(f)(x)$ *in terms of local coordinates* $(y^1,\dots,y^n)$ *in* N *are defined by*

$$\tau^k(f) = \Delta_M f^k + g_M(\nabla f^\alpha, \nabla f^\beta)(\Gamma^k_{\alpha\beta} \circ f) \tag{12}$$

with summation over α, $\beta = 1, \dots, n$. *Here* $f^k = y^k \circ f$, *and the* $\Gamma^k_{\alpha\beta}$ *denote the Christoffel symbols for the target manifold* N.

DEFINITION. — *A harmonic mapping is a* C^2*-mapping* $f: M \to N$ *such that* $\tau(f) = 0$.

For a study of harmonic mappings and their role in differential topology see Eells and Sampson [7].

A harmonic mapping $M \to \mathbf{R}$ is the same as a harmonic function on M.

Remark. — Let $\varphi : L \to M$ be a harmonic morphism and $f: M \to N$ a C^2-mapping. Then φ is semiconformal according to Theorem 7 below. Denoting by λ the dilatation of φ, one finds

$$\tau(f \circ \varphi) = \lambda^2[\tau(f) \circ \varphi]$$

by use of Lemma 4 and the corollary to Lemma 5. Hence $f \circ \varphi$ is a harmonic mapping if f is one. (Generally, the composition of harmonic mappings does not lead to harmonic mappings, see [7, Chap. I, § 5]).

In particular, the harmonic mappings $f: M \to N$ depend only on the harmonic structure on M, in the sense that two Riemannian metrics on a manifold M determine the same harmonic mappings of M into another Riemannian manifold N if and only if the two metrics determine the same harmonic sheaf on M (that is, the same harmonic functions in open subsets of M). And the rather restrictive condition for this is explicitated in § 8, first corollary.

Since the identity mapping $id\colon \mathrm{N} \to \mathrm{N}$ is harmonic [7, p. 128], we have [9]

$$(13) \qquad \tau^k(id) = \Delta y^k + g^{\alpha\beta}\Gamma^k_{\alpha\beta} = 0\,.$$

Lemma. — *Let $f\colon \mathrm{M} \to \mathrm{N}$ be a semiconformal C^2-mapping with dilatation λ. The tension field $\tau(f)$ is then given in terms of local coordinates (y^k) in N by*

$$\tau^k(f) = \Delta_{\mathrm{M}} f^k - \lambda^2[(\Delta_{\mathrm{N}} y^k) \circ f].$$

In particular, $\tau^k(f) = \Delta_{\mathrm{M}} f^k$ if the local coordinates (y^k) in N are harmonic functions.

Proof. — Inserting (7) in (12), we obtain

$$\tau^k(f) = \Delta_{\mathrm{M}} f^k + \lambda^2(g^{\alpha\beta}_{\mathrm{N}} \circ f)(\Gamma^k_{\alpha\beta} \circ f),$$

whence the stated expression for $\tau^k(f)$ in view of (13).

7. The connection between harmonic morphisms, harmonic mappings, and semiconformal mappings.

Theorem. — *A mapping $f\colon \mathrm{M} \to \mathrm{N}$ is a harmonic morphism if and only if f is a semiconformal, harmonic mapping. In the affirmative case the dilatation λ of f is determined by* (2) *in Lemma 4.*

(9) A direct proof of (13) runs as follows (using the summation convention throughout, and writing x in place of y, and for brevity $\partial_\alpha = \partial/\partial x_\alpha$, etc.) : By the definition of $\Gamma^k_{\alpha\beta}$ we have

$$\begin{aligned} g^{\alpha\beta}\Gamma^k_{\alpha\beta} &= \frac{1}{2}\, g^{\alpha\beta} g^{k\gamma}\, (\partial_\alpha g_{\beta\gamma} + \partial_\beta g_{\alpha\gamma} - \partial_\gamma g_{\alpha\beta}) \\ &= -\frac{1}{2}\, (\partial_\alpha g^{k\alpha} + \partial_\beta g^{k\beta} + g^{k\gamma}\partial_\gamma \log|g|) \\ &= -\Delta x^k\,. \end{aligned}$$

As to the second inequality note that, for every α, γ,

$$g^{\alpha\beta}\, dg_{\beta\gamma} + g_{\beta\gamma}\, dg^{\alpha\beta} = d(g^{\alpha\beta} g_{\beta\gamma}) = d\delta^\alpha_\gamma = 0\,,$$

and hence, for any k and α,

$$g^{k\gamma} g^{\alpha\beta}\, dg_{\beta\gamma} = -\, g^{k\gamma} g_{\beta\gamma}\, dg^{\alpha\beta} = -\,\delta^k_\beta\, dg^{\alpha\beta} = -\, dg^{k\alpha}\,,$$

showing that, e.g., $g^{\alpha\beta} g^{k\gamma}\,\partial_\alpha g_{\beta\gamma} = -\,\partial_\alpha g^{k\alpha}$. Finally, as to $g^{\alpha\beta}\,\partial_\gamma g_{\alpha\beta}$, use the identity

$$d|g| = \begin{vmatrix} dg_{11}, & g_{12}, & \cdots & g_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ dg_{n1}, & g_{n2}, & \cdots & g_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} g_{11}, & g_{12}, & \cdots & dg_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ g_{n1}, & g_{n2}, & \cdots & dg_{nn} \end{vmatrix} = |g| g^{\alpha\beta}\, dg_{\alpha\beta}\,.$$

In other words, for any C^2-mapping $f\colon M \to N$, (2) in Lemma 4 is equivalent to the conjunction of (7) in Lemma 5 and $\tau(f) = 0$ (see § 6), using local coordinates (y^k) in N, and writing $f^k = y^k \circ f$.

Proof. — Suppose first that $f\colon M \to N$ is a harmonic morphism (hence C^∞), and let $\lambda \geqslant 0$ denote the function on M determined by (2) in Lemma 4.

To verify (7) in Lemma 5, note that

$$\Delta_M(f^k f^l) = f^k \Delta_M f^l + f^l \Delta_M f^k + 2 g_M(\nabla f^k, \nabla f^l), \tag{14}$$
$$\Delta_N(y^k y^l) = y^k \Delta_N y^l + y^l \Delta_N y^k + 2 g_N^{kl}. \tag{15}$$

Compose with f in (15), multiply by λ^2, and subtract from (14). Then (7) comes out as a consequence of (2), and so f is semiconformal according to Lemma 5.

To show that $\tau(f) = 0$, apply once more (2) in Lemma 4 to $\varphi = y^k$, and invoke Lemma 6.

Conversely, suppose that $f\colon M \to N$ is a harmonic mapping (hence C^2) which is also semiconformal, and let λ denote the dilatation of f. Then (7) in Lemma 5 holds, and Lemma 6 applies. For any C^2-function φ on N we have the identity

$$\Delta_N \varphi = g_N^{kl} \frac{\partial^2 \varphi}{\partial y^k\, \partial y^l} + (\Delta_N y^k) \cdot \frac{\partial \varphi}{\partial y^k} \tag{16}$$

with summation over $k, l = 1, \ldots, n$. Similarly, after some calculation,

$$\Delta_M(\varphi \circ f) = g_M(\nabla f^k, \nabla f^l)\left(\frac{\partial^2 \varphi}{\partial y^k\, \partial y^l} \circ f\right) + (\Delta_M f^k)\left(\frac{\partial \varphi}{\partial y^k} \circ f\right). \tag{17}$$

The desired relation (2) in Lemma 4 now follows by composing with f in (16), multiplying by λ^2, and comparing with (17). (Use (7), and insert $\tau^k f = 0$ in Lemma 6.)

Corollary. — *A harmonic morphism is the same as a semiconformal mapping $f\colon M \to N$ whose components $f^k = y^k \circ f$ in terms of harmonic local coordinates (y^k) in Y are harmonic in M.*

This follows from the theorem together with Lemma 6. Alternatively, just proceed as in the above proof. The existence of harmonic local coordinates in N follows from Greene and Wu [8], as mentioned in § 1.

8. The case of equal dimensions.

THEOREM. — *Let* $\dim M = \dim N = n$.

a) If $n = 2$, *the harmonic morphisms* $M \to N$ *are precisely the (semi)conformal* C^2*-mappings of* M *into* N (10).

b) If $n \neq 2$, *the harmonic morphisms* $M \to N$ *are precisely the (semi)conformal* C^2*-mappings with constant dilatation (= coefficient of conformality)* (11).

Proof. — We proceed in two steps.

1) Consider any C^2-mapping $f\colon M \to N$ such that df is *bijective* at every point of M. In local questions we may then suppose that M and N are one and the same domain in $\mathbf{R}^n$ and that $f(x) = x$ for all $x \in M$. However, M is endowed with one Riemannian metric, g_M, and N with another, g_N. For any C^2-function φ on N we have from(16)

$$\Delta_N \varphi = g_N^{kl} \frac{\partial^2 \varphi}{\partial x^k \, \partial x^l} + \left(g_N^{kl} \frac{1}{2} \frac{\partial \log |g_N|}{\partial x^k} + \frac{\partial g_N^{kl}}{\partial x^k} \right) \frac{\partial \varphi}{\partial x^l}$$

and similarly for $\Delta_M(\varphi \circ f) = \Delta_M \varphi$.

Suppose first that $f = id : M \to N$ is a harmonic morphism. Then, by Lemma 4,

$$\Delta_M \varphi = \lambda^2 \, \Delta_N \varphi ,$$

from which we get by comparing coefficients

$$g_M^{kl} = \lambda^2 g_N^{kl} , \tag{18}$$

and a corresponding proportionality relation between the coefficients to $\partial \varphi / \partial x^l$ in $\Delta_M \varphi$ and in $\Delta_N \varphi$. In this latter

(10) That is, the mappings f which are conformal off the points where $df = 0$.

(11) In other words, the only non-constant harmonic morphisms $f\colon M \to N$ in the case $m = n \neq 2$ are the conformal mappings with constant coefficient of conformality, that is, the local isometries (up to a change of scale).

relation we insert (18) and the following consequence thereof:

$$|g_M| = \lambda^{-2n}|g_N|. \tag{19}$$

The result is simply

$$(n-2)\frac{\partial \log \lambda}{\partial x^k} = 0$$

for $k = 1, \ldots, n$, showing that λ is a constant if $n \neq 2$. By the preceding theorem, f is conformal (since $m = n$), and λ is the dilatation, or coefficient of conformality, of f.

Conversely, suppose that the immersion $f\colon M \to N$ is conformal (and C^2). Then we have again (18) by Lemma 5, and hence (19) holds as before. If furthermore the dilatation $\lambda(>0)$ of f is constant, or if $n = 2$, then the above calculations may be reversed, showing that (2) in Lemma 4 is fulfilled, so that f is indeed a harmonic morphism.

2) In the general case we may apply what was obtained above to the restriction f' of f to the open set M' of all regular points for f (cf. § 3).

First let $f\colon M \to N$ be a harmonic morphism. By the preceding theorem, f is semiconformal, and its dilatation $\lambda \geqslant 0$ is the function determined by (2) in Lemma 4. Suppose that $n \neq 2$. According to Step 1 above, $\lambda = \|df\|$ is constant and >0 in each component of M'. (We leave out the trivial case of a constant mapping f.) Thus $\nabla\lambda^2 = 0$ in M', and hence in all of M by Lemma 3. This shows that, actually, λ^2 is constant in all of M.

As to the opposite direction, consider separately the two cases *a*) and *b*).

Ad *a*) Let $n = 2$, and suppose that $f\colon M \to N$ is semiconformal and of class C^2. Then we may suppose that M and N are open subsets of $\mathbf{C}$ with its standard Riemannian structure [12]. Without reference to Step 1 we may now argue as follows:

Let M_+, M_-, and M_0 denote the subsets of M in which

[12] This is because every 2-dimensional Riemannian manifold is conformally flat, that is, may be mapped locally conformally and diffeomorphically into $\mathbf{C}$. And every conformal diffeomorphism between 2-dimensional Riemannian manifolds is a harmonic morphism (along with its inverse) according to what was proved above in Step 1.

the determinant of df is >0, <0, and $=0$, respectively. Then M_+ and M_- are open, and f is holomorphic in M_+, antiholomorphic in M_-, and constant in the interior of the closed set M_0. Hence

$$\Delta f = 0 \quad \text{in} \quad M_+ \cup M_- \cup \operatorname{int} M_0 ,$$

that is, in the dense complement of the boundary of M_0. By continuity, $\Delta f = 0$ in all of M, and therefore $\partial f/\partial z$ and $\partial f/\partial \bar{z}$ are likewise complex harmonic in M. If M_+ is not void, the harmonic function $\partial f/\partial \bar{z}$ on M equals 0 in M_+ and hence in all of M, showing that $M_+ = M$, so that f is holomorphic in all of M. Similarly, f is antiholomorphic in M if $M_- \neq \emptyset$. Finally, f is constant in M if $M^+ = M_- = \emptyset$. The proof of *a*) is thus completed by remarking that every holomorphic (or antiholomorphic) function f, defined in a domain of $\mathbf{C}$, defines a harmonic morphism into $\mathbf{C}$ (cf. § 2, Example 2).

Ad *b*) Let $n \neq 2$. If $f\colon M \to N$ is a semiconformal C^2-mapping with constant dilatation λ, then either $\lambda = 0$, and so f is constant, in particular a harmonic morphism, or else $\lambda > 0$, in which case df is bijective at every point of M, so that we are back in Case 1.

Corollary. — *Two Riemannian metrics g and g' on the same manifold M determine the same harmonic sheaf on M (that is, the same harmonic functions in open subsets of M) if and only if they are proportional: $g' = \lambda^2 g$, and with a constant $\lambda\ (>0)$ in the case $\dim M \neq 2$.*

Corollary. — *a*) *A mapping f of a Riemann surface* ([13]) *into another is a harmonic morphism (or equivalently: C^2 and semiconformal) if and only if f is either holomorphic or antiholomorphic.*

b) *When $n \neq 2$, the non-constant harmonic morphisms*

([13]) It is not necessary to fix a particular Riemannian metric on a Riemann surface M since the harmonic sheaf on M is completely determined by the natural requirement that the holomorphic functions in open subsets of M should be complex harmonic, that is, of harmonic real and imaginary part.

of a domain $M \subset \mathbf{R}^n$ *into* $\mathbf{R}^n$ *are precisely the (restrictions to* M *of the) similarities, that is, the affine conformal mappings of* $\mathbf{R}^n$ *into itself.*

Here *a*) is well known; it follows from the proof of *a*) in Step 2 of the above proof. And *b*) follows from *b*) of the theorem since it is easy to show that every locally isometric mapping $f\colon M \to \mathbf{R}^n$ must be affine (because f preserves local, hence also global Euclidean distances).

9. The symbol of a semiconformal mapping.

Consider a smooth (for simplicity C^∞) mapping $f\colon M \to N$ of a Riemannian manifold M of dimension m into a Riemannian manifold N of dimension n, and let a point $a \in M$ be given.

The *order* $O_a(f)$ of f at a is defined in terms of local coordinates (x^i) in M and (y^k) in N, centered at a and $f(a)$, respectively, as follows:

$O_a(f)$ is the smallest among those integers $p \geqslant 1$ such that, for some $k = 1, \ldots, n$, the k'th component

$$f^k = y^k \circ f \qquad \text{of} \qquad f,$$

expressed as a function of the local coordinates (x^i) of x, has a non identically vanishing p'th order differential

$$d^p f^k(0)(\xi) = \sum_{|\alpha| = p} \frac{p\,!}{\alpha\,!} D_\alpha f^k(0) \xi^\alpha \neq 0 .$$

Here the multiindex $\alpha = (\alpha_1, \ldots, \alpha_m)$ of order

$$|\alpha| = \alpha_1 + \cdots + \alpha_m = p$$

ranges over all m-tuples of integers $\alpha_i \geqslant 0$ with the sum p. Moreover $\alpha\,! = \alpha_1\,! \ldots \alpha_m\,!$, and

$$D_\alpha = \left(\frac{\partial}{\partial x^1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x^m}\right)^{\alpha_m}; \qquad \xi^\alpha = (\xi^1)^{\alpha_1} \ldots (\xi^m)^{\alpha_m},$$

where $\xi^1, \ldots, \xi^m$ are the contravariant components of the generic vector $\xi \in M_a$. We write $O_a(f) = \infty$ if $d^p f^k(0)$ vanishes identically for all $k = 1, 2, \ldots, n$ and all $p \geqslant 1$.

DEFINITION. — *The symbol* $\sigma_a(f)$ *of a* C^∞*-mapping* $f: M \to N$ *at a point* $a \in M$ *at which* f *has finite order* p *is the mapping*

$$\sigma_a(f): M_a \to N_{f(a)}$$

defined in terms of local coordinate systems (x^i) *and* (y^k) *for* M *and* N, *centered at* a *and* $f(a)$, *respectively, as follows: The contravariant components* $\sigma_a^k(f)$ *of* $\sigma_a(f)$ *are*

$$\sigma_a^k(f)(\xi) = \frac{1}{p!} d^p f^k(0)(\xi) = \sum_{|\alpha|=p} \frac{1}{\alpha!} D_\alpha f^k(0)\xi^\alpha,$$

with the notations and conventions explained above.

It is easily verified that the notions of order $O_a(f)$ and symbol $\sigma_a(f)$ of f at a are invariant under changes of local coordinates in M and N centered at a and $f(a)$, respectively. (In contrast to this, the differentials $d^p f(0)$, defined by the same formula as above, have no invariant meaning for $p > O_a(f)$.)

In the case $N = \mathbf{R}$ of a C^∞-function $f: M \to \mathbf{R}$ of finite ordre p at $a \in M$, the symbol $\sigma_a(f)$ is given by

$$\sigma_a(f) = \frac{1}{p!} d^p f(0).$$

For any C^∞-function $f: M \to \mathbf{R}$, any $a \in M$, any $i = 1, \ldots, m$, and any integer $p \geqslant 1$ one easily obtains the identity

$$\frac{\partial}{\partial \xi^i} \frac{1}{p!} d^p f(0)(\xi) = \frac{1}{(p-1)!} d^{p-1}(D_i f)(0)(\xi) \tag{20}$$

in terms of local coordinates $(x^1, \ldots, x^m)$ in M centered at a. Here $D_i = \partial/\partial x^i$.

THEOREM. — *a) If* $f: M \to N$ *is semiconformal and of class* C^∞ *with the dilatation* λ, *and if* f *is of finite order* $O_a(f) = p$ *at a point* $a \in M$, *then the symbol* $\sigma_a(f): M_a \to N_{f(a)}$ *is semiconformal with the dilatation* $\sqrt{\sigma_a(\lambda^2)}$.

b) If $f: M \to N$ *is a harmonic morphism, then* f *is of finite order at every point* $a \in M$, *and the symbol* $\sigma_a(f)$ *is a harmonic morphism of* M_a *into* $N_{f(a)}$.

Proof. — Ad a) The assertion is evident if $p = 1$, noting that then $\sigma_a(f) = df$ (at a). In the case $p > 1$ choose local coordinate systems (x^i) and (y^k) for M and N, centered at a and $f(a)$, respectively, and so that moreover

$$(21) \qquad g_{\mathrm{M}}^{ij}(a) = \delta^{ij}, \qquad g_{\mathrm{N}}^{kl}(f(a)) = \delta^{kl}$$

for $i, j = 1, \ldots, m$, $k, l = 1, \ldots, n$. Thus the tangent spaces M_a and $\mathrm{N}_{f(a)}$ can be identified with $\mathbf{R}^m$ and $\mathbf{R}^n$ with their Euclidean metrics. In the sequel all differentials refer to the given point $a \in \mathrm{M}$, or rather to $0 \in \mathbf{R}^m$, since all calculations will be performed in terms of the local coordinate systems chosen above. In particular, $f^k = y^k \circ f$ is considered as a function of the local coordinates in M.

For any $i = 1, \ldots, m$ and $k = 1, \ldots, n$ we have

$$\mathrm{O}_a(\mathrm{D}_i f^k) \geqslant \mathrm{O}_a(f^k) - 1 \geqslant p - 1,$$

and hence we obtain from (20), by Taylor's formula,

$$\begin{aligned} \mathrm{D}_i f^k &= \frac{1}{(p-1)\,!}\, d^{p-1}\mathrm{D}_i f^k + \mathrm{O}(r^p) \\ &= \frac{1}{p\,!}\, \mathrm{D}_i d^p f^k + \mathrm{O}(r^p) \end{aligned}$$

for $r \to 0$. Here r denotes the Euclidean norm of the generic point of M (near a), identified with a point of $\mathbf{R}^m$ (near 0) via our local coordinate system in M. (We now write D_i also for $\partial/\partial\xi^i$.)

Since $\mathrm{D}_i d^p f^k$ is a homogeneous polynomial of degree $p - 1$ in $(\xi^1, \ldots, \xi^m)$, it follows in view of Lemma 5 together with (21) that

$$\begin{aligned} \lambda^2 \delta^{kl} &= \sum_{i=1}^{m} \mathrm{D}_i f^k \,.\, \mathrm{D}_i f^l \\ &= \frac{1}{p\,!^2} \sum_{i=1}^{m} \mathrm{D}_i d^p f^k \mathrm{D}_i d^p f^l + \mathrm{O}(r^{2p-1}) \end{aligned}$$

for $k, l = 1, \ldots, n$. Here the leading term in the last expression is a homogeneous polynomial of degree $2p = 2$. Since λ^2 is of class C^∞ along with f, it follows that $d^q(\lambda^2) = 0$ for $0 \leqslant q < 2p - 2$, and that

$$\frac{1}{(2p-2)\,!}\, \delta^{kl} d^{2p-2}(\lambda^2) = \frac{1}{p\,!^2} \sum_{i=1}^{m} \mathrm{D}_i d^p f^k \,.\, \mathrm{D}_i d^p f^l.$$

This shows that indeed the mapping $\sigma_a(f)$ is semiconformal with dilatation μ given by

$$\mu^2 = \frac{1}{(2p-2)!} d^{2p-2}(\lambda^2).$$

Since $O_a(f) = p$, there is an index k such that $d^p f^k \not\equiv 0$. Being homogeneous of degree $p > 0$, $d^p f^k$ cannot be constant and hence $D_i d^p f^k \not\equiv 0$ for some $i = 1, \ldots, m$. Consequently, $d^{2p-2}(\lambda^2) \not\equiv 0$, $O_a(\lambda^2) = 2p-2$, and

$$\sigma_a(\lambda^2) = \frac{1}{(2p-2)!} d^{2p-2}(\lambda^2) = \mu^2.$$

Ad *b*) A harmonic morphism $f\colon M \to N$ is C^∞, and f has finite order at every point $a \in M$ by the uniqueness theorem [1], [6]. According to [8], N admits local coordinates (y^k), centered at $f(a)$, which are harmonic in N. Replacing these harmonic coordinates by suitable linear combinations of them, we may arrange again that (21) holds. Note that $f^k = y^k \circ f$ is harmonic in M (near a):

$$\Delta_M f^k = 0\,. \tag{22}$$

As before let $O_a(f) = p$. Since f is semiconformal by Theorem 7, so is $\sigma_a(f)$ according to Part *a*), and it remains, by the corollary to Theorem 7 (or just by Theorem 2) to show that the coordinates of $\sigma_a(f)$ are harmonic functions in M_a. Since M_a is identified with $\mathbf{R}^m$ with its Euclidean metric, we shall thus prove that $\Delta d^p f^k = 0$ in $\mathbf{R}^m$ for $k = 1, \ldots, n$, where Δ denotes the classical Laplace operator on $\mathbf{R}^m$. We may assume that $p \geqslant 2$, since df^k is a linear form, hence harmonic.

By a two-fold application of (20), and by summation over $i = 1, \ldots, m$, we get

$$\Delta \frac{1}{p!} d^p f^k = \frac{1}{(p-2)!} d^{p-2}\, \Delta f^k. \tag{23}$$

Next $\Delta f^k = \sum_{i=1}^{m} D_i^2 f^k$ should be compared with

$$\Delta_M f^k = \sum_{i,\,j=1}^{m} g_M^{ij} D_{ij} f^k + \{(g_M^{ij}.1/2 D_i \log|g_M| + D_i g_M^{ij}) D_j f^k\}.$$

Inserting

$$f^k = \frac{1}{p\,!} d^p f^k + O(r^{p+1})$$

$$D_j f^k = \frac{1}{(p-1)\,!} d^{p-1} D_j f^k + O(r^p),$$

etc., and $g_M^{ij} = \delta^{ij} + O(r)$, $\log |g_M| = O(r)$, etc., we obtain from (22)

$$0 = \Delta_M f^k = \frac{1}{(p-2)\,!} d^{p-2}\, \Delta f^k + O(r^{p-1}).$$

Since $d^{p-2}\,\Delta f^k$ is homogeneous of degree $p-2$, it must vanish identically, and we conclude from (23) that indeed $\Delta d^p f^k = 0$.

10. Openness of semiconformal mappings.

LEMMA. — *Let* φ *be a semiconformal* C^2*-mapping of* $\mathbf{R}^m \setminus \{0\}$ *into* $\mathbf{R}^n$ $(m \geq n \geq 2)$ *such that* φ *is positive homogeneous of some degree* $p \in \mathbf{R}\setminus\{0\}$:

$$\varphi(\rho x) = \rho^p \varphi(x)$$

for all $\rho > 0$ *and* $x \in \mathbf{R}^m \setminus \{0\}$. *Further suppose that* φ *is normalized as follows:*

$$\sup_{|x|=1} |\varphi(x)| = 1\,. \tag{24}$$

Then there exists, for every given $z_0 \in \mathbf{R}^n \setminus \{0\}$, *a continuous mapping* ψ *of* $\mathbf{R}^n \setminus \mathbf{R}_+ z_0$ *into* $\mathbf{R}^m \setminus \{0\}$ *such that* $\varphi \circ \psi = id$ *and*

$$|\psi(y)| = |y|^{1/p}$$

on $\mathbf{R}^n \setminus \mathbf{R}_+ z_0$, *and consequently*

$$|\varphi(x)| = |x|^p$$

for every x *in the range of* ψ.

Proof. — For any dimension q the unit sphere in $\mathbf{R}^q$ will be denoted by S^{q-1}. By the homogeneity of φ it suffices to construct for each $z_0 \in S^{n-1}$ a continuous mapping

$$\psi : S^{n-1} \setminus \{z_0\} \to S^{m-1}$$

such that $\varphi \circ \psi = id$, and next to extend ψ to $\mathbf{R}^n \setminus \mathbf{R}_+ z_0$ so as to become positive homogeneous of degree $1/p$.

Denoting by λ the dilatation of the semiconformal mapping $\varphi : \mathbf{R}^m \setminus \{0\} \to \mathbf{R}^n$, we have from Lemma 5 (or Theorem 2):

$$\nabla \varphi_k . \nabla \varphi_l = \lambda^2 \, \delta_{kl} \tag{25}$$

for $k, l = 1, \ldots, n$.

In view of the homogeneity of φ, (24) extends to

$$\sup_{x \in \mathbf{R}^m \setminus \{0\}} \frac{|\varphi(x)|}{|x|^p} = 1 .$$

The supremum is attained on the cone $\mathbf{R}_+ \Gamma \setminus \{0\}$, where

$$\Gamma = \{x \in S^m \mid |\varphi(x)| = 1\} \tag{26}$$

is non-void and compact by (24). Since $|x|^{2p} - |\varphi(x)|^2 \geqslant 0$ in $\mathbf{R}^m \setminus \{0\}$, with equality on Γ, we have

$$1/2 \, \nabla(|x|^{2p} - |\varphi(x)|^2) = p|x|^{2p-2}x - \sum_{k=1}^{n} \varphi_k(x) \nabla \varphi_k(x) = 0$$

for $x \in \Gamma$. When combined with (25), this shows that $\lambda(x) > 0$ for $x \in \Gamma$, and further that

$$p^2 = p^2|x|^{4p-2} = \lambda(x)^2 \sum_{k=1}^{n} \varphi_k(x)^2 = \lambda(x)^2 |\varphi(x)|^2 = \lambda(x)^2,$$

that is,

$$\lambda(x) = |p| \qquad \text{for} \qquad x \in \Gamma . \tag{27}$$

Choose a point $e \in \Gamma$. Since $|\varphi(e)| = 1$, we may arrange (after an orthonormal change of coordinates in $\mathbf{R}^n$) that

$$\varphi(e) = (0, \ldots, 0, 1). \tag{28}$$

For any $t \in \mathbf{R}$ and any point

$$u = (u_1, \ldots, u_{n-1}) \in S^{n-2} \quad (\subset \mathbf{R}^{n-1})$$

write

$$\begin{aligned} Y(t,u) &= Y(t, u_1, \ldots, u_{n-1}) \\ &= (u_1 \sin t, \ldots, u_{n-1} \sin t, \cos t). \end{aligned} \tag{29}$$

When considered as a function just of $t \in \mathbf{R}$ for fixed $u \in S^{n-2}$ we shall usually write

$$y(t) = (y_1(t), \ldots, y_n(t)) = Y(t,u).$$

Let us consider for fixed $u \in S^{n-2}$ the following first order ordinary differential system with the unknown function $x = x(t) = (x_1(t), \ldots, x_m(t))$, writing $\dot{x} = dx/dt$, etc:

$$(30) \qquad \begin{aligned} \lambda(x)^2 \dot{x} &= \sum_{k=1}^{n} \dot{y}_k \nabla \varphi_k(x) \\ &= \cos t \sum_{k=1}^{n-1} u_k \nabla \varphi_k(x) - \sin t \nabla \varphi_n(x) \end{aligned}$$

with the initial condition $x(0) = e$. Since $e \in \Gamma$, it follows from (27) that $\lambda(e) > 0$, and hence that $\lambda > 0$ in a neighbourhood of $x(0)$. Our initial value problem is therefore non-singular and determines a unique solution

$$x(t) = X(t,u) = X(t,u_1, \ldots, u_{n-1})$$

along which $\lambda(x) > 0$, all in some maximal open interval $I(u) \subset \mathbf{R}$ containing $t = 0$. It is well known that the mapping $(t,u) \longmapsto X(t,u)$ is continuous.

For $t \in I(u)$ we have from (30) and (25)

$$\begin{aligned} \frac{d}{dt} \varphi_k(x(t)) &= \nabla \varphi_k(x(t)) \cdot \dot{x}(t) \\ &= \lambda(x(t))^{-2} \sum_{l=1}^{n} \dot{y}_l(t) \nabla \varphi_k(x(t)) \cdot \nabla \varphi_l(x(t)) \\ &= \dot{y}_k(t), \end{aligned}$$

showing that $\varphi_k(x(t)) = y_k(t)$ because this holds for $t = 0$ in view of (28), (29), and $x(0) = e$. Thus

$$(31) \qquad \varphi(x(t)) = y(t) \qquad \text{for} \qquad t \in I(u).$$

Using Euler's identity for positive homogeneous functions:

$$x \cdot \nabla \varphi^k(x) = p \varphi^k(x),$$

we further obtain from (30) combined with (31):

$$\begin{aligned} \lambda(x(t))^2 x(t) \cdot \dot{x}(t) &= \sum_{k=1}^{n} \dot{y}_k(t) x(t) \cdot \nabla \varphi_k(x(t)) \\ &= p \sum_{k=1}^{n} \dot{y}_k(t) \varphi_k(x(t)) \\ &= p y(t) \cdot \dot{y}(t) = 0 \end{aligned}$$

since $|y(t)|^2 = 1$. It follows that $x(t).\dot{x}(t) = 0$, and so

$$|x(t)| = 1 \qquad \text{for} \qquad t \in \mathrm{I}(u), \tag{32}$$

because this holds for $t = 0$.

From (31) and (32) we conclude that $x(t) \in \Gamma$ for $t \in \mathrm{I}(u)$, again since $|y(t)| = 1$. Hence by (27)

$$\lambda(x(t)) = |p| \qquad \text{for} \qquad t \in \mathrm{I}(u).$$

Having thus found that $\lambda(x(t))$ remains constant > 0, we derive from the theory of first order ordinary differential systems that

$$\mathrm{I}(u) = \mathbf{R}$$

since $x(t)$ moves on the compact sphere S^{m-1} by (32).

Altogether we have now constructed, for any given point $e \in \Gamma$, a continuous mapping X, viz.

$$(t,u) \longmapsto \mathrm{X}(t,u), \tag{33}$$

of $\mathbf{R} \times \mathrm{S}^{n-2}$ into S^{m-1} with range in Γ and such that

$$\varphi \circ \mathrm{X} = \mathrm{Y}, \tag{34}$$

$$\mathrm{X}(0,u) = e. \tag{35}$$

Since Y, defined in (29), maps $\mathbf{R} \times \mathrm{S}^{n-2}$ onto all of S^{n-1}, while X maps $\mathbf{R} \times \mathrm{S}^{n-2}$ into Γ, it follows from (26) and (34) that

$$\varphi(\Gamma) = \mathrm{S}^{n-1}.$$

As mentioned in the beginning of the proof, a point $z_0 \in \mathrm{S}^{n-1}$ is given. Since $\varphi(\Gamma) = \mathrm{S}^{n-1}$, we may choose the point $e \in \Gamma$ (the starting point for the preceding construction) in such a way that

$$\varphi(e) = -z_0. \tag{36}$$

With this choice of e we consider the mappings Y and X from (29) and (33), but from now on with t confined to the interval

$$0 < t < \pi,$$

as far as Y is concerned. Then Y becomes a homeomorphism of $]0,\pi[\times \mathrm{S}^{n-2}$ onto $\mathrm{S}^{n-1} \setminus \{z_0, -z_0\}$, and so we have a continuous mapping

$$\psi = \mathrm{X} \circ \mathrm{Y}^{-1}$$

of $S^{n-1}\setminus\{z_0,-z_0\}$ into $\Gamma \subset S^{m-1}$ with the desired property

$$\varphi \circ \psi = id\,, \tag{37}$$

according to (34), although so far merely on $S^{n-1}\setminus\{z_0,-z_0\}$.
It remains only to extend ψ by continuity to the point

$$-\, z_0 = \varphi(e) = (0,\ldots,0,1),$$

cf. (28) and (36), by putting $\psi(\varphi(e)) = e$, for then (37) holds also at $-\, z_0$:

$$(\varphi \circ \psi)(-z_0) = \varphi(\psi(\varphi(e))) = \varphi(e).$$

To verify the continuity of this extension of ψ to $-\, z_0$, note first that

$$Y^{-1}y = (t,u) \quad (\in\,]0,\pi[\, \times S^{n-2})$$

need not converge (in $\mathbf{R} \times S^{n-2}$) when $y \to -\, z_0$ (through points of $S^{n-1}\setminus\{z_0,-z_0\}$), but we do have $t \to 0$ on account of (29) which shows that $\cos t = Y_n(t,u) = y_n \to 1$ as $y \to -\, z_0 = (0,\ldots,0,1)$. Consequently

$$\psi(y) = X \circ Y^{-1}(y) = X(t,u) \to X(0,u) = e$$

by (35) and the uniform continuity of the mapping (33) on the compact set $[0,\pi] \times S^{n-2}$.

Theorem. — *a) If* $\dim N \geqslant 2$ *then every semiconformal* C^∞*-mapping* $f\colon M \to N$ *is open at any point of* M *at which* f *has finite order.*

b) Every non-constant harmonic morphism $f\colon M \to N$ *is an open mapping. In particular,* $f(M)$ *is open in* N.

Proof. — Ad *a*) Choose local coordinates in M and N as in the proof of *a*) in Theorem 9. Let $p = O_a(f)$ denote the finite order of the semiconformal mapping f at the given point $a \in M$. According to Theorem 9 the symbol $\sigma_a(f)$ ($\not\equiv 0$) is a semiconformal mapping of $\mathbf{R}^m$ into $\mathbf{R}^n$ (with the Euclidean metrics). Write

$$\varphi = \alpha\sigma_a(f),$$

where the constant $\alpha > 0$ is so chosen that (24) is fulfilled. Since $\sigma_a(f)$, and hence φ, is homogeneous of degree $p \geqslant 1$, the above lemma applies to φ. The components φ_k of φ are the polynomials

$$\varphi_k(x) = \frac{\alpha}{p\,!}\, d^p f^k(0)(x), \qquad k = 1\,, \ldots, n\,.$$

For any dimension q we denote by B^q the closed unit ball in $\mathbf{R}^q$. Via the local coordinates chosen, M and N may be identified with open neighbourhoods of 0 in $\mathbf{R}^m$ and $\mathbf{R}^n$, respectively (but with metrics that need not be Euclidean except at 0). Now let $r > 0$ be given so that the ball rB^m is contained in M. We shall prove that $f(rB^m)$ is a neighbourhood of $f(0) = 0$ in $\mathbf{R}^n$. By Taylor's formula there is a constant β such that

$$(38) \qquad |\alpha f(x) - \varphi(x)| \leqslant \beta |x|^{p+1} \qquad \text{for} \qquad x \in rB^m.$$

We propose to show that

$$f(rB^m) \supset \frac{\delta}{\alpha} B^n,$$

where we write

$$(39) \qquad \delta = \frac{2}{3} \min\left(r^p, \frac{1}{(4\beta)^p}\right).$$

Thus let

$$(40) \qquad y_0 \in \frac{\delta}{\alpha} B^n$$

be given. Since $f(0) = 0$, we shall assume that $y_0 \neq 0$. Now consider the continuous mapping ψ constructed in the lemma, taking $z_0 = -y_0$. Further write

$$(41) \qquad g = \alpha f \circ \psi\,.$$

Since $\varphi \circ \psi = id$ and $|\psi(y)| = |y|^{1/p}$ (on $\mathbf{R}^n \setminus \mathbf{R}_+ z_0$), we have from (38), applied to $x = \psi(y)$ for any $y \in r^p B^n \setminus \mathbf{R}^+ z_0$ (hence $\psi(y) \in rB^m$)

$$(42) \qquad |g(y) - y| \leqslant \beta |\psi(y)|^{p+1} = \beta |y|^{1+1/p}.$$

Now consider the closed ball $K \subset \mathbf{R}^n \setminus \mathbf{R}_+ z_0$ of radius $\alpha|y_0|/2$ and center αy_0. By (40) and (39),

$$|y| \leqslant \frac{3}{2} \alpha |y_0| \leqslant \frac{3}{2} \delta \leqslant r^p$$

on K , and hence we have from (42) and (39) for all $y \in K$:

$$|g(y)-y| \leqslant \beta|y|^{1+1/p} \leqslant \beta \cdot \frac{3}{2}\alpha|y_0| \cdot \left(\frac{3}{2}\delta\right)^{1/p} \leqslant \frac{3}{8}\alpha|y_0|,$$

which is less than the radius $\alpha|y_0|1/2$ of K. It follows therefore from a well-known version of Rouché's theorem and Kronecker's existence theorem that there exists $y \in K$ with $g(y) = \alpha y_0$ (the center of K), and hence according to (41)

$$f(\psi(y)) = \frac{1}{\alpha} g(y) = y_0 .$$

The point $x = \psi(y) \in rB^m$ thus satisfies $f(x) = y_0$.

Ad *b*) The case $n \geqslant 2$ is covered by Part *a*) in view of Theorems 7 and 9. In the remaining case $n = 1$ we may assume that $N = \mathbf{R}$ (with the standard metric), so that f is simply a harmonic function on $M : \Delta_M f = 0$. Every neighbourhood of a point $a \in M$ contains a connected open neighbourhood U of a . Since $f(U) \subset \mathbf{R}$ is connected, it is an interval containing $f(a)$, and indeed a neighbourhood of $f(a)$; for $f(U)$ cannot reduce to $\{f(a)\}$ according to the uniqueness theorem [1], [6], nor can $f(a)$ be an end-point of $f(U)$ on account of the maximum principle for harmonic functions on M .

Remarks. — 1) Part *a*) of the theorem fails to hold for $n = 1$, as shown e.g. by the mapping $f(x) = |x|^2$ of $\mathbf{R}^m$ into $\mathbf{R}$, which is trivially semiconformal and has finite order 2 at the only singular point $x = 0$, but is not open at that point.

2) In the more general frame of Brelot harmonic spaces it was shown by Constantinescu and Cornea in [4, Theorems 3.3 and 3.4] that every *injective* harmonic morphism $f: M \to N$ is open and that f^{-1} is a harmonic morphism of $f(M)$ into M . However, in our case of Riemannian manifolds M and N , we obviously must have $\dim M = \dim N$ if $f: M \to N$ is injective, and so the stated results from [4] become well known in the manifold case in view of the classification in Theorem 8 above.

11. Supplementary results and examples.

In this section $\mathbf{R}^n$ is always endowed with its standard Riemannian metric and hence with its classical harmonic structure. Also, $\mathbf{C}$ is identified, as a Riemannian manifold, with $\mathbf{R}^2$.

11.1. *Holomorphic mappings.* Let M and N be complex (analytic) manifolds with

$$\dim_{\mathbf{C}} \mathrm{M} = m \,, \qquad \dim_{\mathbf{C}} \mathrm{N} = n \,.$$

Suppose that M, and similarly N, is endowed with a C^∞ Hermitian metric h_{M} with the property that *every holomorphic function on* M *is complex harmonic* with respect to the Riemannian metric

$$ds_{\mathrm{M}}^2 = \sum_{j,k=1}^{m} h_{jk}^{\mathrm{M}}\, dz^j\, \overline{dz^k}$$

on M (14). Explicitly, this requirement means that

$$\sum_{j=1}^{m} \bar{\partial}_j (h_{\mathrm{M}}^{jk} \det (h_{\mathrm{M}}^{jk})) = 0$$

for $k = 1, \ldots, m$, where the matrix (h_{M}^{jk}) is inverse to (h_{jk}^{M}).

Under this assumption on M and N, a *holomorphic mapping* $f\colon \mathrm{M} \to \mathrm{N}$ is a harmonic morphism if and only if f is semiconformal, or explicitly: if and only if there exists a function $\lambda \geqslant 0$ on M (the dilatation of f) such that

$$\sum_{j,k=1}^{m} h_{\mathrm{M}}^{jk}\, \overline{\partial_j f^\alpha}\, \partial_k f^\beta = \lambda^2 h_{\mathrm{N}}^{\alpha\beta} \circ f$$

for every $\alpha, \beta = 1, \ldots, n$, whereby $f^\alpha = w^\alpha \circ f$ denote the components of the holomorphic mapping f in terms of local holomorphic coordinates w^α in N.

This result is a particular case of the corollary to Theorem 7 (15).

(14) It is well known that every *Kähler manifold* has this property. (The converse is false except for $n \leqslant 2$.) More generally, any holomorphic differential form on a Kähler manifold is harmonic.

(15) It might be added at this point that every holomorphic mapping $f\colon \mathrm{M} \to \mathrm{N}$ between Kähler manifolds is a *harmonic mapping* [7, p. 118, third example].

11.2. *Riemannian submersions.* Let again M and N be arbitrary Riemannian manifolds, and consider a Riemannian submersion $f: M \to N$, or slightly more generally a semi-conformal mapping with constant dilatation $\lambda > 0$. It is known that f is a harmonic mapping if and only if each fiber $f^{-1}(f(x))$, $x \in M$, is *minimal* (as a submanifold of M) [7, p. 127 f.]. Invoking Theorem 7, we see that this condition of minimal fibers is also (necessary and) sufficient for f to be a harmonic *morphism.*

Examples. — 1) For any two Riemannian manifolds M and N, the *projections* $M \times N \to M$ and $M \times N \to N$ are harmonic morphisms with dilatation $\lambda = 1$.

2) *Multiplication on a Lie group* G endowed with a bi-invariant metric. When $G \times G$ is given the product metric

$$g_{G\times G}((X_1,Y_1),(X_2,Y_2)) = g_G(X_1,X_2) + g_G(Y_1,Y_2),$$

the multiplication mapping $G \times G \to G$ is a harmonic morphism with the constant dilatation $\lambda = \sqrt{2}$. (Like the preceding example, this can also be easily verified directly.)

Specific examples: $G = SO(n)$ (the special orthogonal group), and $G = S^3$ (the unit quaternions).

12. Extension to *h*-harmonic morphisms.

Let again M and N denote two Riemannian manifolds of dimensions m and n, respectively. In this section we shall suppose that a C^∞-function $h > 0$ is given on M.

Definition. — *A C^2-function $f: U \to \mathbf{R}$, defined in an open set $U \subset M$, is called h-harmonic if*

$$\Delta_M u + 2g_M(\nabla \log h, \nabla u) = 0. \tag{43}$$

Clearly the constant functions on M are h-harmonic.

Remarks. — 1) If h is harmonic: $\Delta_M h = 0$, then (43) is equivalent to

$$\Delta_M(hu) = 0$$

in U. Thus the h-harmonic functions are obtained from the harmonic functions in U in this case simply by division by h, as it is standard in potential theory.

2) If $m \neq 2$, we may perform a conformal change of metric on M so as to reduce the notion of h-harmonic functions (with respect to the given metric g_M on M) to that of a harmonic function with respect to the new metric $g^{(h)}$ on M given by

$$(44) \qquad g_{ij}^{(h)} = h^{\frac{4}{m-2}} g_{ij}^{M} .$$

In fact, the Laplace-Beltrami operator $\Delta^{(h)}$ on M with respect to the metric $g^{(h)}$ is found to be

$$\Delta^{(h)} u = h^{-\frac{4}{m-2}} (\Delta_M u + 2 g_M(\nabla \log h, \nabla u)).$$

DÉFINITION. — *An h-harmonic morphism is a continuous mapping f: M → N such that, for every harmonic function φ in an open set $V \subset N$, the composite function $\varphi \circ f$ is h-harmonic in $f^{-1}(V)$.*

Remark. — This notion is of interest only for $m \geqslant 3$. In fact, for $m = n = 2$ the argument given in the proof of Theorem 8*a* shows that there are no non-constant h-harmonic morphisms M → N except in the previous case where h is constant ($= 1$).

In the sequel we shall therefore assume that $m \neq 2$. Then f: M → N is an h-harmonic morphism if and only if f is a harmonic morphism with respect to the modified metric (44) on M (and the given metric on N). Hence we can carry over mutatis mutandis the general results of the preceding sections.

While Lemma 3 carries over to h-harmonic morphisms as it stands, (2) in Lemma 4 should now be replaced by

$$\Delta_M(\varphi \circ f) + 2 g_M (\nabla \log h , \nabla (\varphi \circ f)) = \lambda^2 [(\Delta_N \varphi) \circ f].$$

The notion of a semiconformal mapping f: M → N (§ 5) is the same for both of the conformally equivalent metrics g_M and $g^{(h)}$ on M, but the dilatations λ and $\lambda^{(h)}$ of such a mapping with respect to g_M and $g^{(h)}$, respectively, are related by

$$(45) \qquad \lambda^{(h)} = h^{-\frac{2}{m-2}} \lambda .$$

From the corollary to Theorem 7 and from b) in Theorem 8 we now obtain the following result, which implies that, for $m = n \neq 2$, a mapping $f : \mathrm{M} \to \mathrm{N}$ is an h-harmonic morphism for *some* $h > 0$ if and only if f is a conformal C^2-mapping.

THEOREM. — *a) If $m \geqslant n$ and $m \neq 2$, a mapping $f : \mathrm{M} \to \mathrm{N}$ is an h-harmonic morphism if and only if f is semiconformal and the components $f^k = y^k \circ f$ of f are h-harmonic functions for some (and hence any) choice of harmonic local coordinates (y^k) in N.*

b) If $m = n \neq 2$, then $f : \mathrm{M} \to \mathrm{N}$ is a non-constant h-harmonic morphism if and only if f is a conformal C^2-mapping with dilatation

$$\lambda = \text{const.}\, h^{\frac{2}{n-2}}.$$

Remark. — The fact that what we call an h-harmonic morphism between domains in $\mathbf{R}^3$ must necessarily be conformal (if not constant) is mentioned in Kellogg [10, p. 235].

We close by two examples of h-harmonic morphisms well known from potential theory, one with $m > n$, the other with $m = n$.

Example 1 (Axial symmetrization in $\mathbf{R}^4$). — Let $\mathrm{M} \subset \mathbf{R}^4$ be the domain in $\mathbf{R}^4$ consisting of all $x = (x_1, \ldots, x_4)$ such that

$$\rho = (x_1^2 + x_2^2 + x_3^2)^{1/2} > 0,$$

and define $f : \mathrm{M} \to \mathbf{R}^2$ by

$$f(x) = (f_1(x), f_2(x)) = (\rho, x_4).$$

Then f is clearly a Riemannian submersion, that is, a semiconformal mapping with dilatation $\lambda = 1$. Since the functions 1 and x_4/ρ are harmonic on M, f_1 and f_2 are h-harmonic functions on M when we take

$$h = \frac{1}{\rho} \quad (\text{on } \mathrm{M}).$$

It follows therefore from Part a) of the above theorem that

$f\colon \mathrm{M} \to \mathbf{R}$ is an h-harmonic morphism for this function h. There is an immediate extension to the case $m > 4$, $n = m - 2$, taking for ρ and h the above functions of x_1, x_2, x_3.

Example 2 (The Kelvin transformation). — Let M be a domain in $\mathbf{R}^n$. As shown by Liouville [11], the only non-constant, smooth conformal mappings of M into $\mathbf{R}^n$ with $n \geqslant 3$ are the Möbius transformations, that is, a *similarity* (see the second corollary to Theorem 8) possibly composed with an *inversion* in a sphere. If the sphere has centre $a \in \mathbf{R}^n$ and radius ρ, then the inversion $f\colon \mathbf{R}^n \setminus \{a\} \to \mathbf{R}^n \setminus \{a\}$ is given by

$$f(x) = a + \frac{\rho^2}{|x - a|^2}(x - a).$$

The dilatation λ of this conformal mapping f is given (for any dimension n) by

$$\lambda(x) = \frac{\rho^2}{|x - a|^2}\,. \tag{46}$$

According to Part *b*) of the above theorem, f is an h-harmonic morphism with

$$h(x) = \frac{1}{|x - a|^{n-2}}\,,$$

and this morphism is the classical *Kelvin transformation* in $\mathbf{R}^n$ (see Kellogg [11, p. 232] for the typical case $n = 3$). Note that also in this case, h is harmonic in M $(= \mathbf{R}^n \setminus \{a\})$, cf. the first remark to the definition of h-harmonic functions.

Remark. — Stated in more modern terms, Liouville's proof essentially amounts to establishing that a Riemannian metric of the form $g_{ij} = \lambda^2\, \delta_{ij}$ (with $\lambda > 0$ a scalar of class C^3) has curvature tensor 0 if and only if λ is either a constant or else of the form (46), and this fact is easily obtained by a straightforward calculation. Another classical proof is based on the conformal invariance of the lines of curvature on a surface. Liouville's proof requires that f be of class C^4. The problem of weakening the smoothness hypothesis was solved completely by Rešetnjak [12] who showed that C^1 (and even less) will do.

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