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INTERSECTION PROPERTIES OF BALLS IN SPACES OF COMPACT OPERATORS

by Asvald LIMA

Let A be a real or complex Banach space. The closed ball in A with center a and radius r is denoted by $B(a, r)$ and the unit ball $B(0, 1)$ by A_1 ; A^* is the dual space of A. A family $\{B(a_i, r_i)\}_{i\in\Gamma}$ of balls in A is said to have the *weak intersection property* if $\bigcap_{i\in\Gamma} B(f(a_i), r_i) \neq \emptyset$ in R or C for every $f \in A_1^*$. The notion of weak intersection property was introduced by Hustad [8]. In the real case this is equivalent to $\|a_i - a_j\| \leqslant r_i + r_j$ for all $i, j \in \Gamma$. Let $n \geqslant 3$ be a natural number. We say that A is an $E(n)$*-space* if every family of n balls in A with the weak intersection property has a non-empty intersection. In the real case this is the same as the n.2. intersection property (n.2.I.P.) studied by Lindenstrauss in [15]. Lindenstrauss proved that a real space A has the 4.2.I.P. iff it has the n.2.I.P. for all n iff A^* is isometric to an $L_1(\mu)$-space. Hustad [8] and Lima [13], [14] then showed that for a complex space, A is an $E(3)$-space iff A is an $E(n)$-space for all n iff A^* is isometric to an $L_1(\mu)$-space. In the real case, the 3.2.I.P. does not imply the 4.2.I.P. In fact, real $L_1(\mu)$-spaces have the 3.2.I.P., but not the 4.2.I.P.

We shall mainly study (real) spaces with the 3.2.I.P. and spaces with an intersection property which is weaker than $E(3)$. First, in § 1, we extend the following theorem of Hanner [5] to infinite dimensional spaces : A real Banach space has the 3.2.I.P. if and only if for every pair F_1, F_2 of disjoint faces of A_1, there exists a proper face F of A_1 such that $F_1 \subseteq F$ and $F_2 \subseteq -F$.

A Banach space A is said to have the *extreme point intersection property* (E.P.I.P.) if $\bigcap_{i=1}^{3} B(a_i, r_i) \neq \phi$ for every family $\{B(a_i, r_i)\}_{i=1}^{3}$ of three balls in A with the weak intersection property such that $B(a_1, r_1) \cap B(a_2, r_2)$ consists of one point. (Observe that $B(a_1, r_1) \cap B(a_2, r_2)$ consists of one point if and only if $\|a_1 - a_2\|^{-1}(a_1 - a_2)$ is an extreme point of A_1). Clearly every E(3)-space has the E.P.I.P. Real spaces with this property were studied in [15]. In § 2 we generalize Theorem 4.7 in [15] to the complex case. Thus we get that A has the E.P.I.P. if and only if $|f(e)| = 1$ for all extreme points e of A_1 and f of A_1^*.

The connection between the spaces with the E.P.I.P. and the CL-spaces is studied in § 3. (A is a CL-space if $A_1 = co(F \cup -F)$ for every maximal proper face F of A_1). We show that dual CL-spaces have the E.P.I.P., and if A* has the E.P.I.P. then A is "almost" a CL-space. (This is made precise in § 3).

In § 4 we show that if P is a *bicontractive projection* (i.e. $\|P\| \leqslant 1$ and $\|I - P\| \leqslant 1$) in a (real) CL-space, in an E(3)-space, or in an $L_1(\mu)$-space, then $2P - I$ is an involutive isometry. This result is a partial generalization of a theorem of Bernau and Lacey [2] and the proof is very simple.

The last three paragraphes are devoted to the study of intersection properties of spaces of linear operators, and in particular to the space of all compact operators C(Y, X) from a real Banach space Y to a real Banach space X. In Theorem 5.2 and Theorem 5.5 we show that if X* and Y* are CL-spaces, then C(Y, X) has the E.P.I.P. if and only if every extreme operator T in the unit ball of C(Y, X) is *nice*. (T is nice if T* maps extreme points of X_1^* into extreme points of Y_1^*).

Corollary 6.6 and Theorem 6.8 together with Theorem 7.1 and Theorem 7.5 show that C(Y, X) has the 3.2.I.P. if and only if Y and X have the 3.2.I.P. and either Y is an L_1-space or X has the 4.2.I.P. We also show that C(Y, X) has the 4.2.I.P. if and only if Y is an L_1-space and X has the 4.2.I.P.

The results in § § 5, 6 and 7 are strongly influenced by the work of Lazar [10]. The results we obtain are generalizations of some of the results in [10]. Also some results of Sharir and Fakhoury are generalized. (See [20], [21], [22], [23], [24] and [27].)

The notation we use is fairly standard. We write $co(\mathrm{S})$ for the convex hull of a set S, $\overline{\mathrm{S}}$ for the closure of S and $\partial_e \mathrm{C}$ for the set of extreme points of a convex set C.

If C is a set in A, the cone (C) is defined by

$$\operatorname{cone}(\mathrm{C}) = \bigcup_{\lambda \geqslant 0} \lambda \mathrm{C} .$$

The smallest face of a point x in a convex set C is given by

$$\operatorname{face}(x) = \{y \in \mathrm{C} : x = \alpha y + (1-\alpha) z \text{ for some } \alpha \in \langle 0,1] \text{ and some } z \in \mathrm{C}\}.$$

1. Spaces with the 3.2.I.P.

We will here generalize a characterization of Hanner [5] of spaces with the 3.2.I.P. to infinite-dimensional spaces. Hanner's theorem says that a finite dimensional space has the 3.2.I.P. if and only if any two disjoint faces of its unit ball are contained in disjoint parallel hyperplanes. Before we state the theorem we need some definitions and a lemma.

If $x \in \mathrm{A}_1$, $\operatorname{face}(x)$ means the smallest face of A_1 containing x. For any $x \in \mathrm{A}$, write $\mathrm{C}(x) = \operatorname{cone}\left[\operatorname{face}\left(\frac{x}{\|x\|}\right)\right]$ for $x \neq 0$ and $\mathrm{C}(0) = (0)$. Following [1] we define an ordering $<$ on A as following :

$$z < x \quad \text{means} \quad \|x\| = \|z\| + \|x - z\| .$$

LEMMA 1.1. – *Let* A *be a real or complex Banach space and let* $x , y \in \mathrm{A}$. *Then there exist* $z , u , v \in \mathrm{A}$ *such that*

$$x = z + u \qquad \|x\| = \|z\| + \|u\|$$

$$y = z + v \qquad \|y\| = \|z\| + \|v\|$$

and

$$\mathrm{C}(u) \cap \mathrm{C}(v) = (0) .$$

Proof. – Define

$$\mathrm{C} = \{z \in \mathrm{A} : z < x \quad \text{and} \quad z < y\} .$$

Let (z_γ) be a maximal totally ordered subset of C. By Lemma 2.8

in [1] (z_γ) has a least upper bound z and (z_γ) converges to z. Define

$$u_\gamma = x - z_\gamma \quad \text{and} \quad v_\gamma = y - z_\gamma .$$

Then $u = \lim u_\gamma$ and $v = \lim v_\gamma$ exist. Clearly we have

$$x = z + u \qquad \|x\| = \|z\| + \|u\|$$

$$y = z + v \qquad \|y\| = \|z\| + \|v\| .$$

Suppose $w \in C(u) \cap C(v)$. Then for some $\alpha > 0$, $\alpha w < u$ and $\alpha w < v$. Hence

$$z_\gamma < z < z + \alpha w < x$$

and

$$z_\gamma < z < z + \alpha w < y$$

for all γ. Since (z_γ) is maximal totally ordered and z is its least upper bound, we get $z = z + \alpha w$. Hence $w = 0$ and $C(u) \cap C(v) = (0)$. This completes the proof.

A real Banach space A is said to have the $R_{3,2}$-*property* if for every pair x, y of points in A, there exist $z, u, v \in A$ such that

$$x = z + u, \qquad \|x\| = \|z\| + \|u\|,$$

$$y = z + v, \qquad \|y\| = \|z\| + \|v\|,$$

and

$$\|x - y\| = \|u - v\| = \|u\| + \|v\| .$$

THEOREM 1.2. – *Let* A *be a real Banach space. The following statements are equivalent :*

(i) A^* *has the 3.2.I.P.*

(ii) A *has the 3.2.I.P.*

(iii) A *has the* $R_{3,2}$-*property*

(iv) *For every pair* F_1, F_2 *of disjoint proper faces of* A_1, *there exists a proper face* F *of* A_1 *such that* $F_1 \subseteq F$ *and* $F_2 \subseteq -F$.

(v) *For every pair* x, y *of points in* A *such that* $1 = \|x\| = \|y\|$ *and* face $(x) \cap$ face $(y) = \phi$, *there exists a proper face* F *of* A_1 *such that* $x \in F$ *and* $y \in -F$.

(vi) *For every pair* x, y *of points in* A *such that* $1 = \|x\| = \|y\|$ *and* face $(x) \cap$ face $(y) = \phi$, *we have* $\|x - y\| = 2$.

Proof. – (i) $\Longleftrightarrow$ (ii) $\Longleftrightarrow$ (iii) is proved in [12].

(iv) $\Longrightarrow$ (v) $\Longleftrightarrow$ (vi) is trivial

(vi) $\Longrightarrow$ (iii). Let $x, y \in A$ and let $z, u, v \in A$ be as in Lemma 1.1. Since $C(u) \cap C(v) = (0)$, we have

$$\text{face}\left(\frac{u}{\|u\|}\right) \cap \text{face}\left(\frac{v}{\|v\|}\right) = \emptyset ,$$

so $\left\| \dfrac{u}{\|u\|} - \dfrac{v}{\|v\|} \right\| = 2$. But then $\|u - v\| = \|u\| + \|v\|$.

(iii) $\Longrightarrow$ (iv). Let F_1 and F_2 be disjoint proper faces of A_1. Let $\Omega = \{(x, y) : x \in F_1$ and $y \in F_2\}$. We order Ω by writing $(x, y) < (u, v)$ if and only if $x \in \text{face}(u)$ and $y \in \text{face}(v)$. Since A has the $R_{3,2}$-property and $F_1 \cap F_2 = \emptyset$, we get that $\|x - y\| = 2$ for each $(x, y) \in \Omega$. For each $(x, y) \in \Omega$ define

$$K_{(x,y)} = \{e \in A_1^* : e(x) = 1 \quad \text{and} \quad e(y) = -1\}.$$

Then $K_{(x,y)} \neq \emptyset$, and $K_{(x,y)}$ is a w^*-compact face of A_1^*. It follows that if $(x, y) < (u, v)$, then

$$K_{(x,y)} \supseteq K_{(u,v)} .$$

Hence $\{K_w\}_{w \in \Omega}$ is directed by inclusion. Let

$$K = \bigcap_{w \in \Omega} K_w .$$

Then K is a non-empty w^*-compact face of A_1^*. Let $e \in K \cap \partial_e A_1^*$ and let $F = \{z \in A_1 : e(z) = 1\}$. Then F is a proper face of A_1 and $F_1 \subseteq F$ and $F_2 \subseteq -F$. The proof is complete.

Remark. – For spaces with $\dim A < \infty$ the equivalence of (ii) and (iv) was proved by Hanner in [5].

By an easy application of the Hahn-Banach theorem it follows that Theorem 1.2 (iv) is equivalent to the statement in the Abstract, i.e. if F_1 and F_2 are disjoint faces of the unit ball A_1, then there exists a hyperplane H such that $F_1 \subseteq H$ and $F_2 \subseteq -H$.

2. The extreme point intersection property.

If A is a real Banach space, write $\Gamma = \{\pm 1\}$ and if A is complex write $\Gamma = \{\theta \in \mathrm{C} : |\theta| = 1\}$. The next result is well known in the real case.

THEOREM 2.1. – *Let* A *be a real or complex Banach space, let* $e \in \partial_e \mathrm{A}_1$, let $\mathrm{S} = \{f \in \mathrm{A}_1^* : \|f\| = f(e) = 1\}$ *and let* $\Phi : \mathrm{A} \longrightarrow \mathrm{C(S)}$ *be defined by* $\Phi(x)(f) = f(x)$. *Then the following statements are equivalent :*

(i) $|f(e)| = 1$ *for all* $f \in \partial_e \mathrm{A}_1^*$.

(ii) $\partial_e \mathrm{A}_1^* \subseteq \Gamma \mathrm{S}$.

(iii) *For every* $x \in \mathrm{A}$, *there exists* $\theta \in \Gamma$ *such that* $\|x + \theta e\| = \|x\| + \|e\|$.

(iv) Φ *is an isometry into.*

Proof. – (i) $\Longleftrightarrow$ (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (iii) is trivial.

(iii) $\Longrightarrow$ (ii). Suppose for contradiction that $\partial_e \mathrm{A}_1^* \not\subseteq \Gamma . \mathrm{S}$. By Milman's theorem we then have $\mathrm{A}_1^* \not\subseteq \overline{co}(\Gamma \cdot \mathrm{S})$ (w^*-closure). Let $f \in \mathrm{A}_1^*$ be such that $f \notin \overline{co}(\Gamma \cdot \mathrm{S})$. By the Hahn-Banach theorem there exists $x \in \mathrm{A}$ such that

$$\mathrm{Re}\, f(x) > \sup \{\mathrm{Re}\, g(x) : g \in \overline{co}(\Gamma \cdot \mathrm{S})\}.$$

Let $\theta \in \Gamma$ be such that $\|x + \theta e\| = \|x\| + 1$ and let $g \in \partial_e \mathrm{A}_1^*$ be such that $\|x\| + 1 = g(x + \theta e)$. Then $\|x\| = g(x)$ and $1 = \theta g(e)$. Hence $g \in \Gamma \cdot \mathrm{S}$ and

$$g(x) = \|x\| \geqslant \mathrm{Re}\, f(x) > \mathrm{Re}\, g(x) = \|x\|.$$

This contradiction shows that $\partial_e \mathrm{A}_1^* \subseteq \Gamma \cdot \mathrm{S}$. The proof is complete.

We say that A has the *extreme point intersection property* (E.P.I.P. in short) if for every family of balls $\{\mathrm{B}(a_i, r_i)\}_{i=1}^3$ in A with the weak intersection property and such that $\mathrm{B}(a_1, r_1) \cap \mathrm{B}(a_2, r_2)$ consists of one point a, then we have $a \in \mathrm{B}(a_3, r_3)$.

A is said to have the *restricted E.P.I.P.* (R.E.P.I.P.) if the above holds whenever all $r_i = 1$.

The next result is an extension to the complex case of Theorem 4.7 in [15].

THEOREM 2.2. – *Let* A *be a real or complex Banach space. The following statements are equivalent :*

(i) A *has the E.P.I.P.*

(ii) A *has the R.E.P.I.P.*

(iii) $|f(e)| = 1$ *for all* $e \in \partial_e A_1$ *and all* $f \in \partial_e A_1^*$.

(iv) *For all* $e \in \partial_e A_1$ *and all* $x \in A$, *there exists* $\theta \in \Gamma$ *such that* $\|x + \theta e\| = \|x\| + 1$.

(v) *For all* $e \in \partial_e A_1$ *and every maximal proper face* F *of* A_1 , *there exists* $\theta \in \Gamma$ *such that* $\theta e \in F$.

Proof. – (iii) $\Longleftrightarrow$ (iv) follows from Theorem 2.1 ; (i) $\Longrightarrow$ (ii) and (v) $\Longrightarrow$ (iv) are trivial. It remains to prove (ii) $\Longrightarrow$ (iv) $\Longrightarrow$ (v) and (iii) $\Longrightarrow$ (i).

(ii) $\Longrightarrow$ (iv). Suppose for contradiction that there exist $x \in A$ and $e \in \partial_e A_1$ such that $\|x + \theta e\| < \|x\| + 1$ for all $\theta \in \Gamma$. We may assume $\|x\| = 1$. Since the map $\theta \longrightarrow \|x + \theta e\|$ is uniformly continuous, we get

$$s = \sup \{\|x + \theta e\| : \theta \in \Gamma\} < 2 .$$

Let $r = 3 - s > 1$. Then for $\theta \in \Gamma$

$$\|rx + \theta e\| \leqslant (2 - s)\|x\| + \|x + \theta e\| \leqslant 2 .$$

The balls $\{B(0,1), B(rx + e, 1), B(rx - e, 1)\}$ has the weak intersection property. In fact, if $g \in A_1^*$, let $u = rg(x)$ and $v = g(e)$. Then since $\|u + \theta v\| \leqslant 2$ for all $\theta \in \Gamma$, *er* have $|u| + |v| \leqslant 2$ and $|v| \leqslant 1$. Hence $w \in B(0,1) \cap B(u + v, 1) \cap B(u - v, 1)$ where

$$w = \begin{cases} u & \text{if} \quad |u| \leqslant 1 \\ \dfrac{u}{|u|} & \text{if} \quad |u| > 1 . \end{cases}$$

If $y \in B(rx + e, 1) \cap B(rx - e, 1)$, then

$$e = \frac{1}{2}(e + rx - y) + \frac{1}{2}(e - rx + y)$$

is a convex combination in A_1. Since $e \in \partial_e A_1$, we get $e = e + rx - y$, so $rx = y$. Hence

$$\{rx\} = B(rx + e, 1) \cap B(rx - e, 1).$$

By (ii), $rx \in B(0,1)$ so $r \leqslant 1$. This contradiction shows that (ii) $\Longrightarrow$ (iv).

(iv) $\Longrightarrow$ (v). Let $e \in \partial_e A_1$ and let F be a maximal proper face of A_1. For each $x \in F$, let

$$\Gamma_x = \{\theta \in \Gamma : \|x + \theta e\| = 2\}.$$

$\Gamma_x \neq \emptyset$ by (iv) for each $x \in F$. If $x_1, \ldots, x_n \in F$, then $x = \frac{1}{n} \sum_{i=1}^{n} x_i \in F$ and $\Gamma_x \subseteq \bigcap_{i=1}^{n} \Gamma_{x_i}$. Hence $\{\Gamma_x\}_{x \in F}$ has the finite intersection property. Since each Γ_x is compact, there exists $\theta \in \bigcap_{x \in F} \Gamma_x$. Then $co(\{\theta e\} \cup F)$ is a convex subset of the sphere of A_1. Since F is maximal, we get $\theta e \in F$.

(iii) $\Longrightarrow$ (i) Suppose $\{B(a_i, r_i)\}_{i=1}^{3}$ has the weak intersection property and that

$$B(a_1, r_1) \cap B(a_2, r_2) = \{a\}.$$

Then $(r_1 + r_2)a = r_2 a_1 + r_1 a_2$. By translation we get

$$B(0, r_1) \cap B(a_2 - a_1, r_2) = \{a - a_1\}.$$

We have

$$a - a_1 = \left(\frac{r_1}{r_1 + r_2}\right)(a_2 - a_1).$$

Hence $e = r_1^{-1}(a - a_1) \in \partial_e A_1$. Let $g \in \partial_e A_1^*$. Then we have

$$B(0, r_1) \cap B(g(a_2 - a_1), r_2) \cap B(g(a_3 - a_1), r_3) \neq \emptyset.$$

By (iii) $|g(e)| = 1$. By rotating if necessary, we may assume $g(a_2 - a_1) = r_1 + r_2$. But then

$$B(0, r_1) \cap B(g(a_2 - a_1), r_2) = \{r_1\}.$$

Hence $r_1 \in B(g(a_3 - a_1), r_3)$. Thus

$$r_3 \geqslant |g(a_3 - a_1) - r_1| = |g(a_3 - a_1) - g(a - a_1)| = |g(a_3 - a)|.$$

It follows that $a \in B(a_3, r_3)$. The proof is complete.

Remarks. – a) Complex L_1-spaces have the E.P.I.P. by [12 ; Corollary 6.8]. Preduals of complex L_1-spaces have the E.P.I.P. by Theorem 4.8 of Hustad [8]. (See also [13] and [14]). Hence Theorem 1 of Hirsberg and Lazar [7] is an easy consequence of Theorem 2.1 and Theorem 2.2. Theorem 1 of [7] says that if A is predual of a complex L_1-space and $e \in \partial_e A_1$, then the map Φ of Theorem 2.1 is an isometry. (See also [9]).

b) Suppose now that A is a complex predual of an L_1-space. Suppose $x \in A$ and $\|x\| \leqslant 1$. If $x \notin \partial_e A_1$, then $|f(x)| < 1$ for some $f \in \partial_e A_1^*$. An application of the selection theorem [19] then shows that there exists a $y \in A$ with $y \neq 0$ such that $\|x - \theta y\| \leqslant 1$ for all $\theta \in \Gamma$. Hence for $x \in A$ with $\|x\| \leqslant 1$, we have $x \notin \partial_e A_1$ if and only if $\|x - \theta y\| \leqslant 1$ for some $y \neq 0$ and all $\theta \in \Gamma$.

c) Suppose A is a C*-algebra with identity I. If A is commutative, then it is known that $|f(I)| = 1$ for all $f \in \partial_e A_1^*$. Assume conversely that $|f(I)| = 1$ for all $f \in \partial_e A_1^*$. Then by Theorem 2.1 and the Remarks following Corollary 1.6 in [18], we get that A is commutative. In particular A is commutative if and only if A has the E.P.I.P.

3. CL-spaces and semi L-summands.

Let A be a real or complex Banach space. A closed subspace J of A is called a *semi* L-*summand* if for every $x \in A$, there exists a unique $y \in J$ such that $\|x - y\| = d(x, J)$, and moreover this y satisfies $\|x\| = \|y\| + \|x - y\|$.

Semi L-summands were studied in [12].

THEOREM 3.1. – *Let* A *be a real Banach space and let* $e \in \partial_e A_1$. *Then* span(e) *is a semi* L-*summand if and only if* $|f(e)| = 1$ *for all* $f \in \partial_e A_1^*$.

Proof. – Assume first that span(e) is a semi L-summand. Then by Corollary 6.8 in [12] we get $|f(e)| = 1$ for all $f \in \partial_e A_1^*$. Next, if $|f(e)| = 1$ for all $f \in \partial_e A_1^*$, define

$$F = \{f \in A_1^* : f(e) = \|f\| = 1\}.$$

From Theorem 2.1 we get $A_1^* = co(F \cup -F)$. Let $f \in A$ and define

$$a = \inf \{f(x) : x \in F\},$$

$$b = \sup \{f(x) : x \in F\}$$

and

$$g = \frac{1}{2}(a+b)e \cdot$$

Then $g \in \mathrm{span}(e)$ is the unique element we are seeking.

Remark. – Theorem 3.1 is false in the complex case. Assume A is complex and that A^* is isometric to an $L_1(\mu)$-space. Let $e \in \partial_e A_1$ and assume span(e) is a semi L-summand. By [12 ; Theorem 6.14] span(e) is a semi L-summand in A^{**}, so $e \in \partial_e A_1^{**}$. It is known that A^{**} is isometric to $C(S)$ for some compact Hausdorff space S ([26]). Hence we may assume $e = 1 \in C(S)$. Either S is dispersed or S contains a perfect subset [9]. In both cases it is easy to see that (ii) in Theorem 5.6 of [12] is not fulfilled. This contradicts that span(e) is a semi L-summand.

PROPOSITION 3.2. – *Let* A *be a real Banach space and let* $e \in \partial_e A_1$. *The following statements are equivalent :*

(i) span(e) *is a semi* L-*summand.*

(ii) *If* $\|x\| = 1$ *and* $e \notin \mathrm{face}(x)$, *then there exists a proper face* F *of* A_1 *such that* $x \in F$ *and* $e \in -F$.

(iii) *If* G *is a proper face of* A_1 *and* $e \notin G$, *then there exists a proper face* F *of* A_1 *such that* $G \subseteq F$ *and* $e \in -F$.

Proof. – Similar to the proof of Theorem 1.2 using Theorem 3.1 and Theorem 2.1. See also Theorem 4.7 in [15].

We say that a real Banach space A is a CL-*space* if $A_1 = co(F \cup -F)$ for every maximal proper face F of A_1. A is an *almost* CL-*space* if $A_1 = \overline{co}(F \cup -F)$ for every maximal proper face F for A_1.

THEOREM 3.3. – *Let* A *be a real Banach space and let* F *be a maximal proper face of* A_1. *Then* $A_1 = co(F \cup -F)$ *if and only if for every* $x \in A$ *with* $\|x\| = 1$ *and* $\mathrm{face}(x) \cap F = \emptyset$, *we have* $x \in -F$.

Proof. – This is a special case of Corollary 2.10 in [1].

THEOREM 3.4. – *Let* A *be a real Banach space. Let* F *be a maximal proper face of* A_1 *and let* $f \in \partial_e A_1^*$ *be such that* $f = 1$ *on* F. *The following statements are equivalent :*

(i) $A_1 = \overline{co}(F \cup -F)$.

(ii) $B(0, 1-\epsilon) \subseteq co(F \cup -F)$ *for every* $\epsilon \in \langle 0, 1 \rangle$.

(iii) $\mathrm{span}(f)$ *is a semi* L-*summand.*

Proof. – (i) $\Longleftrightarrow$ (ii) follows from the Tukey-Klee-Ellis theorem [4]. (i) $\Longrightarrow$ (iii). Similar to the proof of Theorem 3.1. (iii) $\Longrightarrow$ (i) This follows from the next theorem.

THEOREM 3.5. – *Let* A *be a real or complex Banach space and let* $e \in \partial_e A_1^*$. *Let* $F = \{x \in A : \|x\| = 1 = e(x)\}$. *If* span(e) *is a semi* L-*summand then* $F \neq \emptyset$ *and* $A_1 = \overline{co}(\Gamma \cdot F)$.

Proof. – Assume A is real. (The proof in the complex case is almost identical to the real case). Let $S = \overline{co}(F \cup -F)$. By application of the Bishop-Phelps theorem [3], it follows that $S \neq \emptyset$. We shall prove $S = A_1$. Assume for contradiction that there exists an $x \in A_1 \backslash S$. By Hahn-Banach there exists an $f \in A^*$ with $\|f\| = 1$ such that

$$\|x\| \geqslant f(x) > \sup \{f(y) : y \in S\}.$$

By Theorem 3.1. and Theorem 2.1 we may assume $\|f + e\| = 2$. Choose $\delta > 0$ such that

$$(f + \delta e)(x) > \sup \{(f + \delta e)(y) : y \in S\}.$$

By the Bishop-Phelps theorem [3], there exists $g \in A^*$ such that $\|g\| = \|f + \delta e\| = 1 + \delta$, $\|(f + \delta e) - g\| < \delta$ and $g(z) = \|g\|$ for some $z \in A_1$. We may also assume

$$g(x) > \sup \{g(y) : y \in S\}.$$

We have

$$\|g - \delta e\| < \|f\| + \delta = \|f + \delta e\| = \|g\|.$$

This shows that $C(g) \cap C(e) \neq (0)$. Hence for some $\lambda \neq 0$, we have since span(e) is a semi L-summand,

$$\|g\| = \|\lambda e\| + \|g - \lambda e\| .$$

Hence $\lambda e(z) = |\lambda|$ and $(g - \lambda e)(z) = \|g - \lambda e\|$. It follows that $z \in S$ and

$$\|g\| \geqslant g(x) > \sup\{g(y) : y \in S\} \geqslant g(z) = \|g\| .$$

This contradiction shows that $A_1 = S$. The proof is complete.

Remark. – Theorem 3.5 improves Theorem 4.8 (b) of Lindenstrauss [15].

COROLLARY 3.6. – *Let* A *be a real Banach space. The statements below are related as follows* (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii) :

(i) A^* *is a* CL-*space.*

(ii) A^* *has the* E.P.I.P.

(iii) A *is an almost* CL-*space.*

Proof. – (i) $\Longrightarrow$ (ii) follows from Theorem 2.2, and (ii) $\Longrightarrow$ (iii) follows from Theorem 2.2, Theorem 3.1 and Theorem 3.5.

Remark. – In [12] we proved that every space with the 3.2.I.P. is a CL-space.

From Theorem 2.2 we get the following corollary.

COROLLARY 3.7. – *Let* A *be real or complex and let* $\dim A < \infty$. *Then the following statements are equivalent :*

(i) $A_1 = co(\Gamma \cdot F)$ *for every proper maximal face* F *of* A_1.

(ii) $|f(e)| = 1$ *for all* $e \in \partial_e A_1$ *and all* $f \in \partial_e A_1^*$.

(iii) $A_1^* = co(\Gamma \cdot F)$ *for every proper maximal face* F *of* A_1^*.

4. Bicontractive projections.

In this section P shall be a projection in a real or complex Banach space A. U shall denote the operator $U = 2P - I$. Then U is involutive i.e. $U^2 = I$ and $P = \frac{1}{2}(I + U)$.

We say that P is *bicontractive* if $\|P\| \leqslant 1$ and $\|I - P\| \leqslant 1$. Clearly P is bicontractive if U is an isometry.

In [2] Bernau and Lacey showed that in L_p-spaces, $1 \leqslant p \leqslant \infty$, and in preduals of L_1-spaces P is bicontractive if and only if U is an isometry. We will prove this result for a class of spaces which contains L_1-spaces and preduals of L_1-spaces.

The following theorem is well known and easy to prove, so we only state it.

THEOREM 4.1. – *Let* P *be a projection in a real or complex Banach spaces* A *and* $U = 2P - I$. *The following statements are equivalent :*

(i) U *is an isometry*

(ii) U^* *is an isometry*

(iii) $U(A_1) \subseteq A_1$

(iv) $U^*(A_1^*) \subseteq A_1^*$

(v) $U^*(\partial_e A_1^*) \subseteq A_1^*$

LEMMA 4.2. – *Assume* P *is a bicontractive projection in a real or complex Banach space* A. *Assume* $e \in \partial_e A_1$ *and that* span(e) *is a semi* L*-summand. Then* $\| U(e) \| = 1$.

Proof. – By Theorem 5.6 in [12], we can write $Pe = te + f$ where $t \in C$, $f \in A$ and $\|f + \theta e\| = \|f\| + |\theta|$ for all $\theta \in C$. Then we have

$$1 = \|e\| \geqslant \|Pe\| = |t| + \|f\|,$$

so

$$1 - |t| \geqslant \|f\|. \tag{4.1}$$

We also have

$$1 = \|e\| \geqslant \|e - Pe\| = \|(t-1)e + f\| = |t-1| + \|f\| \geqslant 1 - |t| + \|f\|,$$

so

$$|t| \geqslant \|f\|. \tag{4.2}$$

Since P is a projection we get

$$te + f = Pe = P^2 e = P(te + f) = t^2 e + tf + Pf.$$

Hence

$$\|f\| \geqslant \|Pf\| = \|t(1-t)e + (1-t)f\| = (|t| + \|f\|)\,|1-t| \geqslant (|t| + \|f\|)(1-|t|),$$

so

$$|t|\,\|f\| \geqslant |t|\,(1-|t|). \tag{4.3}$$

Moreover we have

$$\|f\| \geqslant \|f - Pf\| = \|tf + t(t-1)e\| = |t|(\|f\| + |1-t|) \geqslant |t|(\|f\| + 1 - |t|),$$

so

$$(1-|t|)\,\|f\| \geqslant (1-|t|)\,|t|. \tag{4.4}$$

If $|t| = 1$. then $f = 0$ and $t = 1$ by (4.1). If $t = 0$, then $f = 0$ by (4.2). Assume next that $0 < |t| < 1$. By (4.1) and (4.3) we get $\|f\| = 1 - |t|$ and by (4.2) and (4.4) we get $\|f\| = |t|$. Hence $\|f\| = |t| = \frac{1}{2}$. By the inequality preceeding formula (4.2) we get $|1-t| = 1 - |t| = \frac{1}{2}$, so $t = \frac{1}{2}$. Thus we get

$$Pe = \frac{1}{2}\left(e + \frac{f}{\|f\|}\right).$$

Hence

$$Ue = \begin{cases} -e & \text{if} \quad |t| = 0 \\ e & \text{if} \quad |t| = 1 \\ \dfrac{f}{\|f\|} & \text{if} \quad 0 < |t| < 1 . \end{cases}$$

The proof is complete.

THEOREM 4.3. – *Let* A *be a (real)* CL-*space and let* P *be a projection in* A. *Then* P *is bicontractive if and only if* U *is an isometry.*

Proof. – Assume P is bicontractive and let $x \in A$. Let F be a maximal proper face of A_1 such that $\frac{U(x)}{\|U(x)\|} \in F$. Let $e \in \partial_e A_1^*$ be such that $e = 1$ on F. Then span(e) is a semi L-summand by Theorem 3.4. By Lemma 4.2 we get

$$\| Ux \| = e(Ux) = U^*e(x) \leqslant \| U^*e \| \cdot \| x \| = \| x \| \cdot$$

Hence $U(A_1) \subseteq A_1$ and U is an isometry. The proof is complete.

The same method of proof gives the next two results.

THEOREM 4.4. – *Let* A *be a real or complex Banach space such that* span(e) *is a semi* L-*summand for all* $e \in \partial_e A_1^*$. *Then a projection* P *in* A *is bicontractive if and only if* U *is an isometry.*

THEOREM 4.5. – *Assume* A *or* A^* *is isometric to an* $L_1(\mu)$-*space. Then* U *is an isometry for every bicontractive projection* P *in* A.

Remark. – Theorem 4.5 is contained in Theorem 2.1 and Theorem 2.8 of Bernau and Lacey [2].

The last result in this section shows that the M-projections and the L-projections are the most regular bicontractive projections. A projection P is said to be an *L-projection* if

$$\|x\| = \|Px\| + \|x - Px\|$$

for all $x \in A$ and P is said to be an *M-projection* if

$$\|x\| = \max(\|Px\|, \|x - Px\|)$$

for all $x \in A$.

THEOREM 4.6. – *Let* P *be a projection in a real or complex Banach space* A. *The following statements are equivalent :*

(i) P *is an* M-*projection*.

(ii) P^* *is an* L-*projection*.

(iii) $P^*e = e$ or 0 *for all* $e \in \partial_e A_1^*$.

(iv) $U^*e = e$ or $-e$ *for all* $e \in \partial_e A_1^*$.

Proof. – (i) $\Longleftrightarrow$ (ii) is proved by Alfsen and Effros in [1].

(ii) $\Longrightarrow$ (iii) $\Longleftrightarrow$ (iv) is trivial.

(iii) $\Longrightarrow$ (ii). Let $x \in A^*$ with $\|x\| = 1$.

Choose a net (x_α) in $co(\partial_e A_1^*)$ such that $x_\alpha \longrightarrow x(w^*)$. Write

$x_\alpha = \sum_{i=1}^{n} \lambda_i e_i$ where $\lambda_i > 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $e_i \in \partial_e A_1^*$. Let $I = \{i : P^* e_i = e_i\}$ and $J = \{i : P^* e_i = 0\}$. Then $P^* x_\alpha = \sum_{i \in I} \lambda_i e_i$ and $x_\alpha - P^* x_\alpha = \sum_{i \in J} \lambda_i e_i$. If $\|x_\alpha\| = 1$ then clearly

$$1 = \|x_\alpha\| = \|P^* x_\alpha\| + \|x_\alpha - P^* x_\alpha\|.$$

If $\|x_\alpha\| < 1$, choose $f_\alpha \in A_1$, such that

$$1 \geqslant f_\alpha(x_\alpha) \geqslant \|x_\alpha\|^2.$$

Then we have

$$\sum_{i \in I} \lambda_i \geqslant \|P^* x_\alpha\| \geqslant f_\alpha(P^* x_\alpha)$$

and

$$\sum_{i \in J} \lambda_i \geqslant \|x_\alpha - P^* x_\alpha\| \geqslant f_\alpha(x_\alpha - P^* x_\alpha).$$

Hence

$$1 = \sum_{i=1}^{n} \lambda_i \geqslant \|P^* x_\alpha\| + \|x_\alpha - P^* x_\alpha\| \geqslant f_\alpha(x_\alpha) \geqslant \|x_\alpha\|^2.$$

Since $\| \ \|$ is w^*-lower semi-continuous, we get that $\|x_\alpha\| \longrightarrow \|x\| = 1$. P^* is w^*-w^* continuous so

$$P^* x_\alpha \longrightarrow P^* x \quad \text{and} \quad x_\alpha - P^* x_\alpha \longrightarrow x - P^* x \, (w^*).$$

Hence $\|x_\alpha\|^2 \longrightarrow 1$ and

$$1 = \|x\| \leqslant \|P^* x\| + \|x - P^* x\| \leqslant \underline{\lim} \|P^* x_\alpha\| + \underline{\lim} \|x_\alpha - P^* x_\alpha\| \leqslant 1$$

so

$$1 = \|x\| = \|P^* x\| + \|x - P^* x\|.$$

The proof is complete.

5. Nice operators and the E.P.I.P.

We will now consider $A(X^*, Y)$ = the operators from X^* to Y which are w^*-norm continuous on X_1^*.

THEOREM 5.1. – *Let* F *be a maximal proper face of the unit ball of* $A(X^*, Y)$. *Then there exist* $x \in \partial_e X_1^*$ *and a maximal proper face* G *of* Y_1 *such that*

$$F = \{T \in A(X^*, Y) : \|T\| \leqslant 1 \text{ and } Tx \in G\}.$$

Proof. – Consider the ordering $<$ on F where $T < S$ if and only if $T \in \text{face}(S)$. Then $(F, <)$ is a directed set. Since each $T \in F$ is w^*-norm continuous on X_1^* we get that $K_T = \{x \in X_1^* : \|Tx\| = 1\}$ is a non-empty, w^*-compact union of faces of X_1^*. If $T < S$, then $S = \alpha T + (1 - \alpha) U$ for some $\alpha \in \langle 0,1]$ and some $U \in F$. It follows that $T < S$ implies $K_S \subseteq K_T$. Since $S < \frac{1}{2}(S + T)$ and $T < \frac{1}{2}(S + T)$, $\{K_T\}_{T \in F}$ is directed by inclusion. Hence $K = \bigcap_{T \in F} K_T \neq \emptyset$ and K is a w^*-compact union of faces. From [25] we get $K \cap \partial_e X_1^* \neq \emptyset$. Choose $x \in K \cap \partial_e X_1^*$. Then it follows that if $T < S$, then $Tx \in \text{face}(Sx)$. Hence we get that $\bigcup_{S \in F} \text{face}(Sx)$ is a proper face of Y_1. Let G be a maximal proper face of Y_1 such that $\bigcup_{S \in F} \text{face}(Sx) \subseteq G$. Then clearly

$$F \subseteq \{T : \|T\| \leqslant 1 \text{ and } Tx \in G\}$$

and since the latter set is a face, we get

$$F = \{T : \|T\| \leqslant 1 \text{ and } Tx \in G\}.$$

The proof is complete.

THEOREM 5.2. – *Assume* Y *has the E.P.I.P.. If every* $T \in \partial_e A(X^*, Y)_1$ *satisfies* $T(\partial_e X_1^*) \subseteq \partial_e Y_1$, *then* $A(X^*, Y)$ *has the E.P.I.P.*

Proof. – Let $T \in \partial_e A(X^*, Y)_1$ and let $S \in A(X^*, Y)$. Choose $x \in \partial_e X_1^*$ such that $\|S\| = \|Sx\|$. Let G be a maximal proper face of Y_1 such that $\frac{S(x)}{\|S\|} \in G$. Since $Tx \in \partial_e Y_1$ and Y has the E.P.I.P., we get by Theorem 2.2 that $\theta Tx \in G$ for some $\theta \in \Gamma$. Hence

$$\|S\| + \|T\| = \|Sx\| + \|\theta Tx\| = \|(S + \theta T)(x)\| = \|S + \theta T\|$$

and $A(X^*, Y)$ has the E.P.I.P. by Theorem 2.2.

The next lemma is an easy consequence of Theorem 2.1. We omit the proof.

LEMMA 5.3. – *Let* $x \in A$ *with* $\|x\| = 1$. *If* $x \in \Gamma \cdot G$ *for every maximal proper face* G *of* A_1, *then* $x \in \partial_e A_1$.

LEMMA 5.4. – *Let* $x \in \partial_e X_1^*$ *and assume* span(x) *is a semi L-summand. Let* G *be a maximal proper face of* Y_1. *Then* $F = \{T \in A(X^*, Y) : \|T\| \leqslant 1, Tx \in G\}$ *is a maximal proper face of* $A(X^*, Y)_1$.

Proof. – Clearly F is a face. By Theorem 5.1 there exists $z \in \partial_e X_1^*$ and a maximal proper face H of Y_1 such that

$$F \subseteq \{T \in A(X^*, Y) : \|T\| \leqslant 1 , \ Tz \in H\} .$$

Let $K = \{u \in X : \|u\| = 1 = x(u)\}$. By Theorem 3.5 we have $X_1 = \overline{co}(\Gamma \cdot K)$. For each $u \in K$, define $T_u \in A(X^*, Y)$ by

$$T_u(v) = v(u) \cdot y$$

where $y \in G$ is a fixed element. Then $\|T_u\| \leqslant 1$ and $T_u(x) = y \in G$. Hence we have $T_u(z) = z(u) \cdot y \in H$, i.e. $|z(u)| = 1$. If $u_1, u_2 \in K$, then

$$1 = \left| z\left(\frac{u_1 + u_2}{2}\right) \right| = \frac{1}{2} \ |z(u_1)| + \frac{1}{2} \ |z(u_2)| .$$

Hence by rotating z if necessary, we may assume $z = 1$ on K. But then $z = x$ sinc $X_1 = \overline{co}(\Gamma \cdot K)$. From the argument above it also follows that $G \subseteq H$, so $G = H$. The proof is complete.

THEOREM 5.5. – *Assume* span(e) *is a semi L-summand for every* $e \in \partial_e X_1^*$. *If* $A(X^*, Y)$ *has the E.P.I.P., then every* $T \in \partial_e A(X^*, Y)_1$ *satisfies* $T(\partial_e X_1^*) \subseteq \partial_e Y_1$.

Proof. – Let $T \in \partial_e A(X^*, Y)_1$ and let $x \in \partial_e X_1^*$. Let G be a maximal proper face of Y_1 and define

$$F = \{S \in A(X^*, Y) : \|S\| \leqslant 1 \text{ and } Sx \in G\} .$$

F is a maximal proper face of $A(X^*,Y)_1$ by Lemma 5.4. Hence by Theorem 2.2, $\theta T \in F$ for some $\theta \in \Gamma$, so $\theta Tx \in G$. By Lemma 5.3 $Tx \in \partial_e Y_1$. The proof is complete.

COROLLARY 5.6. – *Let X and Y be real or complex spaces such that X^* and Y (or Y^*) are L_1-spaces. Then the maximal proper faces of $A(X^*,Y)_1$ are exactly the sets*

$$\{T \in A(X^*,Y) : \|T\| \leqslant 1 \, , \; Tx \in G\}$$

where $x \in \partial_e X_1^*$ and G is a maximal proper face of Y_1. Moreover $A(X^*,Y)$ has the E.P.I.P. if and only if $T(\partial_e X_1^*) \subseteq \partial_e Y_1$ for all $T \in \partial_e A(X^*,Y)_1$.

Remark. – If X and Y are real spaces, then Corollary 5.6. remains true if we only assume that Y and X^* have the E.P.I.P.

Remark. – Since $C(X,Y)$ is isometric to $A(Y^*,X^*)$ by the map $T \longrightarrow T^*$ (the adjoint operator), Corollary 5.6. could also have been formulated for $C(X,Y)$.

Let $T \in C(X,Y)_1$. If both X^* and Y^* are CL-spaces, then the statements below are related as follows :

$$\text{(i)} \iff \text{(ii)} \iff \text{(iii)} \Longrightarrow \text{(iv)}$$

(i) For every maximal proper face G of Y_1, there exists a maximal proper face F of X_1 such that $T(F) \subseteq G$.

(ii) For every $y \in \partial_e Y_1^*$, there exists a maximal proper face F of X_1 such that

$$1 = T^*y(x) = y(Tx) \quad \text{for all} \quad x \in F \, .$$

(iii) $T^*(\partial_e Y_1^*) \subseteq \partial_e X_1^*$

(iv) $T \in \partial_e C(X,Y)_1$.

(The proof is an easy application of Theorem 3.5).

We will now give a partial extension of the results above to $L(X,Y)$.

THEOREM 5.7. – *Assume X^* has the E.P.I.P. If every $T \in \partial_e L(X,Y)_1$ satisfies $T(\partial_e Y_1^*) \subseteq \partial_e X_1^*$, then $L(X,Y)$ has the E.P.I.P.*

We will need a theorem of T. Johannesen. Since the proof is published in Norwegian, we will indicate the proof.

THEOREM 5.8 (T. Johannesen). – *Let* K *be a compact convex set and let* F *be a subset of* K *such that* i) F *is a union of faces of* K *and* ii) $K \setminus F$ *is a countable union of compact convex sets. Then* $F \cap \partial_e K \neq \emptyset$.

Proof. – We may write $K \setminus F = \bigcup_{n=1}^{\infty} C_n$ where $C_n \subseteq C_{n+1}$ for all n and every C_n is compact and convex. Let f_n be the characteristic function to C_n . Since F is a union of faces it follows that if $x \in F$ and μ is a probability measure on K representing x , then $\mu(F) = 1$. Hence $\hat{f}_n = 0$ on F . (See [29 ; p. 27].) If $F \cap \partial_e K = \emptyset$, then $\liminf \hat{f}_n \geqslant 1$ on K by [29 ; Th. I.4.10.]. This contradict $\liminf \hat{f}_n = 0$ on F , so we get $F \cap \partial_e K \neq \emptyset$.

Proof of theorem 5.7. – Let $T \in \partial_e L(X,Y)_1$.

Clearly if suffices to show that (iv) in Theorem 2.2 is satisfied for a dense subset of $L(X,Y)$. The operators in $L(X,Y)$ such that the adjoint operator attains its norm on Y_1^* are dense [28]. Hence it suffices to show that $\|T + \theta S\| = 1 + \|S\|$ for some $\theta \in \Gamma$ when $S \in L(X,Y)$ satisfies $\|S\| = \|S^* y\|$ for some $y \in Y_1^*$. Let $F = \{y \in Y_1^* : \|S\| = \|S^* y\|\}$. By Theorem 5.8 $F \cap \partial_e Y_1^* \neq \emptyset$, so $\|S\| = \|S^* y\|$ for some $y \in \partial_e Y_1^*$. Since $T^* y \in \partial_e X_1^*$ and X^* has the E.P.I.P. we get $\|S + \theta T\| = \|S^* y + \theta T^* y\| = 1 + \|S\|$ for some $\theta \in \Gamma$. The proof is complete.

Remark. – In this section we have tried to generalize results of Blumenthal, Lindenstrauss and Phelps [22] and of Sharir [21]. Other results in this direction is in [10], [20], [23], and [24].

6. The 3.2.I.P. for $C(X,Y)$. Sufficient conditions.

We use the following notation.

$L(X,Y)$ = the Banach space of all bounded operators from X to Y .

$C(X,Y)$ = the Banach space of all compact operators from X to Y.

$A(X^*,Y)$ = the Banach space of operators from X^* to Y which are w^*-norm continuous on X_1^*.

It is well known that $C(X,Y) \cong A(Y^*,X^*)$.

PROPOSITION 6.1. – *Let* Q *be an* M*-projection in* Y. *Then* $T \longrightarrow Q \cdot T$ *is an* M*-projection in* $C(X,Y)$, *in* $L(X,Y)$, *and in* $A(X^*,Y)$.

Proof. – Clearly

$$\max(\|QT\|, \|T - QT\|) \leqslant \|T\|$$

for all bounded operators. Fix $T \in L(X,Y)$, let $\epsilon > 0$ and choose $x \in X$ with $\|x\| = 1$ such that $\|Tx\| > \|T\| - \epsilon$. Then

$$\|T\| - \epsilon < \|Tx\| = \max(\|QTx\|, \|Tx - QTx\|)$$
$$\leqslant \max(\|QT\|, \|T - QT\|) \leqslant \|T\|.$$

Hence $\|T\| = \max(\|QT\|, \|T - QT\|)$.

The proof is complete.

COROLLARY 6.2.

$$L(X, 1_\infty^n) \cong (X^* \oplus \cdots \oplus X^*)_{1_\infty^n}$$
$$A(X^*, 1_\infty^n) \cong (X \oplus \cdots \oplus X)_{1_\infty^n}.$$

PROPOSITION 6.3. – *Let* P *be an* L*-projection in* X. *Then* $T \longrightarrow T \cdot P$ *is an* M*-projection in* $L(X,Y)$ *and in* $C(X,Y)$. *If* P *is an* M*-projection in* X, *then* $T \longrightarrow T \cdot P^*$ *is an* M*-projection in* $A(X^*,Y)$.

Proof. – We have for $T \in L(X,Y)$,

$$\|T\| = \|T^*\| = \max(\|P^*T^*\|, \|T^* - P^*T^*\|) = \max(\|TP\|, \|T - TP\|)$$

by Proposition 6.1. From this the conclusion easily follows.

COROLLARY 6.4. – $L(1_1^n, Y) \cong (Y \oplus \cdots \oplus Y)_{1_\infty^n}$.

Remark. – Corollary 6.2 and Corollary 6.4 are well known and are used in [10] and [17].

THEOREM 6.5 (Real case). – *Let* $n = 3$ *or* 4. *Assume* X *has the n.2.I.P. and that* Y *has the* 4.2.I.P. *Then* $A(X^*,Y)$ *has the* n.2.I.P.

Proof. – Let $T_1, \dots, T_n \in A(X^*,Y)$ and let $r_1, \dots, r_n > 0$ such that $\|T_i - T_j\| \leq r_i + r_j$ for all i and j. Then $S = \bigcup_{i=1}^{n} T_i(X_1^*)$ is norm-compact in Y. Let $\epsilon > 0$. By Theorem 3.1 in [11], there exists a subspace Z of Y such that $Z \cong l_\infty^m$ for some m and $d(x, Z) \leq \epsilon \|x\|$ for all $x \in S$. Let Q be a projection of norm 1 in Y such that $Q(Y) = Z$. Then $\|T_i - QT_i\| \leq \epsilon$ for all i and $QT_i \in A(X^*,Z) \cong (X \oplus \dots \oplus X)_{l_\infty^m}$. This last space has the n.2.I.P. by Theorem 4.6 in [15]. Hence $\bigcap_{i=1}^{n} B(QT_i, r_i + \epsilon) \neq \emptyset$ in $A(X^*,Y)$. But then $\bigcap_{i=1}^{n} B(T_i, r_i + 2\epsilon) \neq \emptyset$ in $A(X^*,Y)$. By Lemma 4.2 in [15], $A(X^*,Y)$ has the n.2.I.P. The proof is complete.

COROLLARY 6.6 (Real case). – *Assume that* Y *is an* L_1*-space and that* X *has the n.2.I.P.* ($n = 3$ or 4). *Then* $C(Y,X)$ *has the n.2.I.P.*

THEOREM 6.7 (Complex case). – *If* X *and* Y *are preduals of* L_1*-spaces, then* $A(X^*,Y)$ *is a predual of an* L_1*-space. If* Y *is an* L_1*-space and* X *is a predual of an* L_1*-space, then* $C(Y,X)$ *is a predual of an* L_1*-space.*

Proof. – Proceed as in the proof of Theorem 6.5 and replace [11 ; Theorem 3.1] by [18 ; Theorem 1.3] and [15 ; Lemma 4.2] by [8 ; Theorem 4.8].

THEOREM 6.8 (Real case). – *Assume* Y *has the 3.2.I.P. and that* X *has the 4.2.I.P. Then* $C(Y,X)$ *has the* 3.2.I.P.

Proof. – Similar to the proof of Theorem 6.5.

THEOREM 6.9 (Real case). – *Assume* X^* *is an* L_1*-space and that* Y *has the 3.2.I.P. Then* $A(X^*,Y)$ *has the 3.2.I.P.*

Proof. – $A(X^*,Y) \subseteq A(X^*,Y^{**})$. $A(X^*,Y^{**}) \cong C(Y^*,X)$ has the 3.2.I.P. by Theorem 6.8. and the operators with finite rank are dense. Assume $\{B(T_i, r_i)\}_{i=1}^3$ are balls in $A(X^*,Y)$ such that $\| T_i - T_j \| \leqslant r_i + r_j$ for all i and j. Let $\epsilon > 0$. Then there exists $S \in A(X^*,Y^{**})$ such that $\| T_i - S \| \leqslant r_i + \epsilon$ $(i = 1,2,3)$ and dim range $S < \infty$. Choose $x_1, \ldots, x_p \in X_1^*$ such that

$$T_i(X_1^*) \subseteq \bigcup_{j=1}^{p} B(T_i(x_j), \epsilon) \text{ for all } i .$$

Let $E = \text{span}\,\{T_i(x_j) : i = 1,2,3 \ ; j = 1, \ldots, p\} + \text{range } S \subseteq Y^{**}$. By the "principle of local reflexivity" [16] there exists an operator $U : E \longrightarrow Y$ such that $U = I$ on $Y \cap E$ and

$$(1 - \epsilon)\, \| y \| \leqslant \| Uy \| \leqslant (1 + \epsilon)\, \| y \| \quad \text{for all } y \in E .$$

Then $U \cdot S \in A(X^*,Y)$ and if $x \in X_1^*$ and $T_i(x) \in B(T_i(x_j), \epsilon)$, then

$$\begin{aligned} \| T_i x - U \cdot Sx \| &\leqslant \| T_i x - T_i x_j \| + \| T_i x_j - U \cdot Sx \| \\ &\leqslant \epsilon + \| UT_i x_j - USx \| \\ &\leqslant \epsilon + (1 + \epsilon)\, \| T_i x_j - Sx \| \\ &\leqslant \epsilon + (1 + \epsilon)\, [\| T_i x - T_i x_j \| + \| T_i x - Sx \|] \\ &\leqslant \epsilon + (1 + \epsilon)\, [\epsilon + r_i + \epsilon] \\ &= r_i + \epsilon(r_i + 3 + 2\epsilon) . \end{aligned}$$

By Lemma 4.2 in [15], $\bigcap_{i=1}^{3} B(T_i, r_i) \neq \emptyset$ in $A(X^*,Y)$. The proof is complete.

Remark. – 1) The proof of Theorem 6.9 uses the principle of local reflexivity in the same way as Lindenstrauss and Tzafriri use it in the proof of Theorem 1.e.5 in [17].

2) Theorem 6.5 was proved by Lazar [10] in the case that X is a simplex space. Fakhoury [24] has proved Corollary 6.6 in the case $n = 4$. He also considered spaces of weakly compact operators.

Using Zorn's lemma, we can prove the following algebraic selection theorem.

THEOREM 6.10. – *Let* E *and* F *be Hausdorff locally convex vector spaces. Let* $K \subseteq E$ *be a convex set with the Riesz decomposition property and let* F^c *be the family of all compact convex non-empty subsets of* F. *If* $\varphi : K \longrightarrow F^c$ *is a convex map i.e.*

$$\lambda\varphi(x) + (1 - \lambda)\varphi(y) \subseteq \varphi(\lambda x + (1 - \lambda) y)$$

for all $x, y \in K$ *and all* $\lambda \in [0,1]$, *then there exists an affine function* $\psi : K \longrightarrow F$ *such that* $\psi(x) \in \varphi(x)$ *for all* $x \in K$.

We omit the proof. We only note that we must prove that there exists a minimal affine map $\eta : K \longrightarrow F^c$ such that $\eta(x) \subseteq \varphi(x)$ for all $x \in K$. That η is affine means that

$$\lambda\eta(x) + (1 - \lambda)\eta(y) = \eta(\lambda x + (1 - \lambda) y)$$

for all $x, y \in K$ and all $\lambda \in [0,1]$.

THEOREM 6.11 (Real case). – *Let* $n = 3$ or 4. *Let* X *be an* L_1*-space and assume* Y *is a dual space with the n.2.I.P. Then* L(X,Y) *has the n.2.I.P.*

Proof. – Let $\{B(T_i, r_i)\}_{i=1}^n$ be n balls in L(X,Y) such that $\|T_i - T_j\| \leqslant r_i + r_j$ for all i, j. Let F be a maximal proper face of X_1. Define $\varphi : F \longrightarrow 2^Y$ by

$$\varphi(x) = \bigcap_{i=1}^{n} B(T_i(x), r_i).$$

Then $\varphi(x) \neq \emptyset$ and $\varphi(x)$ is a w^*-compact set for each x. φ is convex, so φ has an affine selection ψ by Theorem 6.10. Extend ψ to a linear map $T : X \longrightarrow Y$. Then T is bounded and $\|T - T_i\| \leqslant r_i$ for all i. The proof is complete.

COROLLARY 6.12. – *Assume* X *and* Y *are* L_1*-spaces. Then* L(X,Y) *has the 3.2.I.P. and* L(X,Y*) *is a* P_1*-space.*

Proof. – It follows by the proof of Theorem 6.11 that L(X,Y*) is a P_1-space. L(X,Y**) has the 3.2.I.P. by Theorem 6.11 and since Y is range of a projection of norm 1 in Y**, we get that L(X,Y) has the 3.2.I.P. The proof is complete.

COROLLARY 6.13. – *Assume* X *has the* 3.2.I.P. *and that* Y *is a dual space with the 4.2.I.P. Then* L(X,Y) *has the 3.2.I.P.*

Proof. – L(Y*,X*) has the 3.2.I.P. by Theorem 6.11. Let $\{B(T_i, r_i)\}_{i=1}^3$ be balls in L(X,Y) such that $\|T_i - T_j\| \leq r_i + r_j$ for all i, j. Then there exists $T \in L(Y^*,X^*)$ such that $\|T - T_i^*\| \leq r_i$ for all i. Hence also $\|T^* - T_i^{**}\| \leq r_i$ for all i. Since Y is a dual space, there exists a projection P in Y^{**} with $\|P\| = 1$ and $P(Y^{**}) = Y$. Then $P \cdot T^*|_X \in L(X,Y)$ and $P \cdot T^*|_X \in \bigcap_{i=1}^{3} B(T_i, r_i)$.
Hence L(X,Y) has the 3.2.I.P.

In the complex case we can prove.

THEOREM 6.14. – *Let* X *and* Y *be complex* L_1*-spaces. Then* L(X,Y*) *is a complex* P_1*-space.*

THEOREM 6.15. – *Let* X *be a Banach space. Then* $C(X, c_0)$ *is an* M*-ideal in* $L(X, c_0)$ *and* $C(X, c_0)$ *is isometric to* $(X^* \oplus \cdots \oplus X^* \oplus \cdots)\, c_0$.

Proof. – Let P_n be the M-projection

$$P_n((x_m)) = (x_1, \ldots, x_n, 0, \ldots).$$

Then clearly

$$C(X, c_0) = \overline{\bigcup_{n=1}^{\infty} L(X, P_n(c_0))}.$$

$L(X, P_n(c_0))$ is an M-ideal in $L(X, c_0)$ by Proposition 6.1. Hence $C(X, c_0)$ is an M-ideal in $L(X, c_0)$. (See [1] or [12]).

Let Q_n be the projection $Q_n((x_m)) = (0, \ldots, 0, x_n, 0, \ldots)$. Then by Proposition 6.1 $L(X, P_n(c_0)) \cong (X^* \oplus \cdots \oplus X^*)_{1_\infty^n}$ by the map $S \longrightarrow (Q_1 S, Q_2 S, \ldots, Q_n S)$. The map $S \longrightarrow (Q_1 S, \ldots, Q_n S, \ldots)$ is an isometry of $C(X, c_0)$ onto $(X^* \oplus \ldots \oplus X^* \oplus \ldots)_{c_0}$. The verification is easy and we leave it to the reader.

Remark. – In [6] Hennefeld showed that $C(c_0, c_0)$ is an M-ideal in $L(c_0, c_0)$.

7. The 3.2.I.P. for C(X,Y). Necessary conditions.

THEOREM 7.1 (Real case). – *Let* $n = 3$ *or* 4. *If* C(X,Y) *or* L(X,Y) *has the n.2.I.P., then* X* *and* Y *has the n.2.I.P.*

Proof. – We show the theorem in the case that C(X,Y) has the n.2.I.P. Let Z be a subspace of Y with $\dim Z = 1$. Then X^* is isometric to $C(X,Z) \subseteq C(X,Y)$. Let P be a projection in Y with $\|P\| = 1$ and $P(Y) = Z$. Then the map $T \longrightarrow P \cdot T$ is a projection in C(X,Y) onto C(X,Z) with norm 1, so C(X,Z) and also X* has the n.2.I.P.

We have that C(X,Y) is isometric to $A(Y^*,X^*)$. Let now Z be a subspace of X* with $\dim Z = 1$ and let $\epsilon > 0$. Then there exists a w^*-continuous projection P in X* such that $P(X^*) = Z$ and $\|P\| \leqslant 1 + \epsilon$. Now Y is isometric to $A(Y^*,Z) \subseteq A(Y^*,X^*)$ and $T \longrightarrow P \cdot T$ is a projection in $A(Y^*,X^*)$ onto $A(Y^*,Z)$ with norm $\leqslant 1 + \epsilon$. Since $\epsilon > 0$ is arbitrary, we get that Y has the n.2.I.P. The proof is complete.

COROLLARY 7.2. – C(X,Y) *has the 4.2.I.P. if and only if* X *is an* L_1*-space and* Y *has the 4.2.I.P.*

Proof. – Use Corollary 6.6 and Theorem 7.1.

COROLLARY 7.3. – *Let* Y *be a dual space. Then* L(X,Y) *has the 4.2.I.P. if and only if* X *is an* L_1*-space and* Y *has the 4.2.I.P.*

Proof. – Use Theorem 6.11 and Theorem 7.1.

Remark. – Theorem 7.1 and the corollaries above can also be generalized to the complex case.

PROPOSITION 7.4. – $L(l_\infty^3, l_1^3)$ *does not have the 3.2.I.P.*

Proof. – Let $x_1 = (1, -1, -1)$, $x_2 = (1, 1, -1)$, $x_3 = (1,1,1)$ and $x_4 = (1, -1, 1)$ in l_∞^3. Define $G_2 = co((0,1,0), (0,0,-1), (-1,0,0))$ and $G_4 = co((0,-1,0), (0,0,-1), (-1,0,0))$. Define disjoint faces F_1 and F_2 of $L(l_\infty^3, l_1^3)_1$ by

$$F_1 = \{T : \|T\| \leqslant 1 \text{ and } Tx_1 = (1,0,0)\}$$

and

$$F_2 = \{T : \|T\| \leqslant 1 , Tx_3 = (0,0,-1) \text{ and } Tx_i \in G_i \text{ for } i = 2,4\}.$$

Note that a $T \in F_1 \cap F_2$ would have $T(x_1 + x_3) = (1,0,-1)$ while $T(x_2 + x_4)$ has a negative first component. Hence $F_1 \cap F_2 = \phi$ since $T(x_1 + x_3) = T(x_2 + x_4)$. Assume F is a maximal proper face of $L(1^3_\infty , 1^3_1)_1$ such that $F_1 \subseteq F$. Then by Theorem 5.1

$$F = \{T : \|T\| \leqslant 1 \text{ and } Tx_0 \in G\}$$

for some $x_0 \in \{x_1, \ldots, x_4\}$ and some proper maximal face G of $(1^3_1)_1$. Since $F_1 \subseteq F$ it follows that $x_0 = x_1$ and $(1,0,0) \in G$. We have to consider four cases.

(i) $G = co((1,0,0), (0,1,0), (0,0,1))$. Define $T \in L(1^3_\infty , 1^3_1)$ by $Tx_1 = (0,0,1)$, $Tx_2 = (0,1,0)$, $Tx_3 = (0,0,-1)$ and $Tx_4 = (0,-1,0)$. Then $T \in F \cap F_2$ so $F_2 \not\subseteq -F$.

(ii) $G = co((1,0,0), (0,-1,0), (0,01))$. The operator T in (i) shows that $F_2 \not\subseteq -F$.

(iii) $G = co(1,0,0), (0,1,0), (0,0,-1))$. Define $T \in L(1^3_\infty , 1^3_1)_1$ by $Tx_1 = Tx_2 = Tx_3 = Tx_4 = (0,0,-1)$. Then $T \in F \cap F_2$ so $F_2 \not\subseteq -F$.

(iv) $G = co((1,0,0), (0,-1,0), (0,0,-1))$. The operator T in (iii) shows that $F_2 \not\subseteq -F$.

By Theorem 1.2 we get that $L(1^3_\infty , 1^3_1)$ does not have the 3.2.I.P. The proof is complete.

THEOREM 7.5. – *Assume* $C(X,Y)$ *or* $L(X,Y)$ *has the 3.2.I.P. Then either* X *is an* L_1*-space or* Y *has the 4.2.I.P.*

Proof. – The two cases are similar so we will assume $C(X,Y)$ has the 3.2.I.P. Assume for contradiction that X is not an L_1-space and that Y does not have the 4.2.I.P. By Theorem 7.1 X^* and Y have the 3.2.I.P. By Proposition 7.4 we can choose balls $\{B(a_i , 1)\}_{i=1}^3$ in $L(1^3_\infty , 1^3_1)$ such that $\|a_i - a_j\| \leqslant 2$ for all i and j and $\bigcap_{i=1}^{3} B(a_i , 1) = \phi$. Choose $\epsilon > 0$ such that

$$\bigcap_{i=1}^{3} B\left(a_i , \frac{(1+\epsilon)^6}{(1-\epsilon)^2}\right) = \phi .$$

Since X^* has the 3.2.I.P. and not the 4.2.I.P, , there exists by [12 ; Theorem 5.14] an isometry $T : l_\infty^3 \longrightarrow X^{**}$. By the principle of local reflexivity [16], we can imbed $T(l_\infty^3)$ almost isometrically into X. Hence we can find an operator $S : l_\infty^3 \longrightarrow X$ and a projection P in X such that

$$(1-\epsilon)\,\|x\| \leqslant \|S(x)\| \leqslant (1+\epsilon)\,\|x\|$$

for all $x \in l_\infty^3$, $\|P\| \leqslant 1+\epsilon$ and $P(X) = S(l_\infty^3)$. Since Y has the 3.2.I.P. and not the 4.2.I.P. , we can find [12 ; Corollary 4.5] an operator $U : l_1^3 \longrightarrow Y$ and a projection Q in Y such that

$$(1-\epsilon)\,\|x\| \leqslant \|U(x)\| \leqslant (1+\epsilon)\,\|x\|$$

for all $x \in l_1^3$, $\|Q\| \leqslant 1+\epsilon$ and $Q(Y) = U(l_1^3)$. Define a projection R in $C(X,Y)$ by $R(V) = Q \,.\, V \,.\, P$. Then $\|R\| \leqslant (1+\epsilon)^2$. Let $T_i = U \cdot a_i \cdot S^{-1} \cdot P \in C(X,Y)$. Then

$$\|T_i - T_j\| \leqslant 2(1+\epsilon)^2\,(1-\epsilon)^{-1} .$$

Since $C(X,Y)$ has the 3.2.I.P. we can find $W \in C(X,Y)$ such that $\|T_i - W\| \leqslant (1+\epsilon)^2\,(1-\epsilon)^{-1}$ for all i. Then we have

$$\|T_i - R(W)\| = \|Q \cdot T_i \cdot P - Q \cdot W \cdot P\| \leqslant (1+\epsilon)^4\,(1-\epsilon)^{-1}.$$

Hence

$$\|(T_i - R(W))|_{P(X)}\| \leqslant (1+\epsilon)^4\,(1-\epsilon)^{-1}$$

and

$$\begin{aligned}\|a_i - U^{-1} \cdot R(W)|_{P(X)} \cdot S\| &= \|U^{-1} \cdot (U \cdot a_i \cdot S^{-1} - R(W)|_{P(X)}) \cdot S\| \\ &\leqslant \|U^{-1}\|\,\|(T_i - R(W))|_{P(X)}\| \cdot \|S\| \\ &\leqslant (1+\epsilon)^6\,(1-\epsilon)^{-2}.\end{aligned}$$

This contradicts that $\bigcap_{i=1}^{3} B(a_i , (1+\epsilon)^6\,(1-\epsilon)^{-2}) \neq \phi$. The proof is complete.

COROLLARY 7.6. – $C(X,Y)$ *has the 3.2.I.P. if and only if* X *and* Y *has the 3.2.I.P. and either* X *is an* L_1*-space or* Y *has the 4.2.I.P.*

COROLLARY 7.7. – *Let* Y *be a dual space.* L(X,Y) *has the 3.2.I.P. if and only if* X *and* Y *has the 3.2.I.P. and either* X *is an* L_1*-space or* Y *has the 4.2.I.P.*

Remark. – By the method of proof used in Theorem 7.1 we can also prove that if C(X,Y) is a real L_1-space, then $\dim X = 1$ or $\dim Y = 1$. In fact, by Theorem 3.9 and Theorem 3.10 in [12] X* and Y are L_1-spaces, and by Theorem 7.5 X or Y* is an L_1-space. But then $\dim X \leqslant 2$ or $\dim Y \leqslant 2$. Now Proposition 6.3 shows that $\dim Y = 1$ or $\dim Y = 1$.

Remark. – We have made a detailed study of $L(l_\infty^3, l_1^3)$. Some of the results we obtained are the following:

1) If $T \in \partial_e L(l_\infty^3, l_1^3)_1$, then $T(\partial_e(l_\infty^3)_1) \subseteq \partial_e(l_1^3)_1$.

From Theorem 5.2 and Corollary 3.6 we get :

2) $L(l_\infty^3, l_1^3)$ is a CL-space.

From Theorem 5.1 we get that the unit ball of $L(l_\infty^3, l_1^3)$ contains 32 maximal proper faces. Hence the unit ball of the dual space contains 32 extreme points. From this we get :

3) $L(l_\infty^3, l_1^3)$ does not contain any non-trivial L-summand. Counting the extreme points of the unit ball of $L(l_\infty^3, l_1^3)$ we get 90. Since 90 is not divisible by 4 we get.

4) $L(l_\infty^3, l_1^3)$ does not contain any non-trivial M-summand.

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