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THE GENERIC DIMENSION OF THE FIRST DERIVED SYSTEM

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Introduction.

In this paper we study the so-called first derived system of a subbundle of the cotangent bundle of a manifold, which we will call here a Pfaffian system. The first derived system will be explained in full detail in Section 1, but for now suffice it to say that it is a set of subspaces of the Pfaffian system under consideration, one for each point of the manifold, which measures the complete integrability or lack of same of the system. For instance, in the case of a completely integrable Pfaffian system, the first derived system equals the Pfaffian system at each point of the manifold. Now, in general, unlike the completely integrable case, the definition of the first derived system allows the subspaces (giving the first derived system) to vary in dimension from point to point. A natural generalization of complete integrability would be to require these subspaces to have a constant dimension. This generalization of complete integrability is not at all new. Such Pfaffian systems have been studied by E. Cartan (2) in several papers in connection with the equivalence problem for Differential Systems and more recently by myself (1) in a paper investigating the characteristic classes of such systems. One of the first questions which arises in connection with the first derived systems is: What is the generic dimension of the first derived system? More precisely, given any r-dimensional Pfaffian system on a manifold M, we will have a partition of the manifold into subsets $M_0, \dots, M_m$, where m is the minimum of r and $C(n-r, 2)$, with M_i giving the set on which the first derived system has codimension i. The precise question, which we answer in this paper, is: What is the generic dimension of the M_i's? As an example of the type of result which we obtain our work will show that generically a 2-dimensional Pfaffian system on a manifold of dimension $n \geqslant 7$ is completely non-integrable, i.e. the first derived system is the zero subspace at each point of the manifold.

1. The First Derived System.

In this section, given a Pfaffian system I, we define the first derived system, $I^{(1)}$, of I.

For any vector bundle E on a manifold M we let $\Gamma(E)$ denote the module of sections of E.

The exterior derivative d gives a map

$$d\colon \Gamma(I) \longrightarrow \Gamma(\Lambda^2 T^*(M))$$

which is not a module map. Let

$$\pi\colon T^*(M) \longrightarrow T^*(M)/I$$

be the natural projection and let

$$\Lambda^2\pi\colon \Gamma(\Lambda^2 T^*(M)) \longrightarrow \Gamma(\Lambda^2(T^*(M)/I))$$

be the second exterior power of the module map induced by π. Let $D = \Lambda^2\pi \circ d$ be the composition, so that we have:

$$D\colon \Gamma(I) \longrightarrow \Gamma(\Lambda^2(T^*(M)/I))\,.$$

Then D is a module map. Now there is a one-one correspondence between bundle homomorphisms and module maps between the sections of bundles, so that we have a bundle homomorphism, which we also denote by D,

$$D\colon I \longrightarrow \Lambda^2(T^*(M)/I)\,.$$

At each point $p \in M$, D is a linear map from the fibre at p of I to the fibre at p of $\Lambda^2(T^*(M)/I)$, so we will have a kernel at each point p, which we denote by $I_p^{(1)}$. We define the first derived system $I^{(1)}$ of I by

$$I^{(1)} \equiv \bigcup_{p \in M} I_p^{(1)}\,.$$

By definition, the Pfaffian system I is completely integrable if and only if $I^{(1)} = I$. Also we observe that, since the dimension of $\Lambda^2(T^*(M)/I) = C(n-r, 2)$, the codimension of $I_p^{(1)}$ in I_p can be at most equal to the minimum of r and $C(n-r, 2)$. Thus given an r-dimensional Pfaffian system on a manifold M, the existence of $I^{(1)}$ partitions M into subsets $M_0, \ldots, M_m$, where m is the minimum of r and $C(n-r, 2)$ and $M_i = \{p \in M \mid \text{codimension of } I_p^{(1)} = i\}$.

We will say an r-dimensional Pfaffian system I is non-integrable if $M_0 = \emptyset$ and completely non-integrable if $M_m = M$.

2. The Generic Dimension of $I^{(1)}$.

In this section we give an alternate description of the sets M_i, which will enable us to compute their generic dimensions, thus answering the questions posed in the introduction above.

To this end consider the space of r-frames, F_r, in $R_n = (R^n)^*$. Thus $F_r = \{\omega = (\omega_1, \ldots, \omega_r) \in \bigoplus_r R_n \mid \omega_1 \wedge \ldots \wedge \omega_r \neq 0\}$, an open set in $\bigoplus_r R_n$.

Next we define a vector bundle over F_r, $\hat{V}$, as follows: $\hat{V} = \{(\omega, \tau) \mid \omega \in F_r \text{ and } \tau \in R_n/\{\omega\}\}$ where $\{\omega\}$ indicates the space generated by $\omega_1, \ldots, \omega_r$. Obviously $\hat{V}$ is the quotient of the trivial bundle $F_r \times R_n$ by the bundle $E = \{(\omega, \eta) \in F_r \times R_n \mid \eta \in \{\omega\}\}$. The bundle, in which we will be most interested, is

$$V = \bigoplus_r (\Lambda^2 \hat{V}) = \{(\omega, \Omega) \mid \omega \in F_r, \Omega = (\Omega_1, \ldots, \Omega_r) \text{ and } \Omega_i \in \Lambda^2(R_n/\{\omega\}) \ \forall i\}$$

Observe that V is a vector bundle of fibre dimension $r \cdot C(n-r, 2)$.

We will now describe some subbundles of V which will be important to our work. To do this let W be any finite dimensional vector space, dimension p, and consider $\bigoplus_r W$. For $0 \leqslant i \leqslant \min(r, p)$ we define the rank i set, $R_i \subset \bigoplus_r W$, by

$$R_i = \{(v_1, \ldots, v_r) \mid \text{dimension of } \{v_1, \ldots, v_r\} = i\} \cdot$$

Then R_i is easily seen to be a submanifold of $\bigoplus_r W$ of dimension $i \cdot (r + p - i)$ which is also invariant by the standard direct sum action of $Gl(W)$ on $\bigoplus_r W$. Now if E is any vector bundle with frame bundle B and standard fibre W then $\bigoplus_r E$ is an associated bundle of B with standard fibre $\bigoplus_r W$ and group action given by the direct sum action mentioned above. So we may define subbundles of $\bigoplus_r E$, which we denote by $R_i(E)$, as the fibre bundles associated to B with standard fibre R_i. The definition makes sense since the submanifolds R_i are invariant by the $Gl(W)$-action. Of course these subbundles will be submanifolds of $\bigoplus_r E$ of the same codimension as the codimension of R_i in $\bigoplus_r W$, namely $p \cdot r - i \cdot (p + r - i) = (p - i)(r - i)$. In our case we let $E = \Lambda^2 V$ and we get submanifolds of V, which we will still denote by R_i $(= R_i(\Lambda^2 \hat{V}))$ of codimension $(C(n-r, 2) - i)(r - i)$ for $0 \leqslant i \leqslant \min(r, C(n-r, 2))$. Notice that

$$R_i = \{(\omega, \Omega) \in V \mid \Omega = (\Omega_1, \ldots, \Omega_r) \text{ and dimension of } \{\Omega_1, \ldots, \Omega_r\} = i\}.$$

Next we consider $F_r' \equiv J_0^1(R^n, F_r)$, the space of 1-jets at the origin (i.e. the source is $0 \in R^n$) of r-coframes. We let $\rho_r : F_r' \longrightarrow F_r$ be the projection, so that if $j_0^1(\omega) \in F_r'$ then $\rho_r(j_0^1(\omega)) = \omega(0)$. Now it will be useful for us to observe that (F_r', F_r, ρ_r) is a vector bundle and can be easily identified with the direct sum of r copies of the trivial bundle $F_r \times \otimes^2 R_n$. Using this identification we will define a map $\delta : F_r' \longrightarrow V$. To do this we first define

$$\hat{\delta} : F_r \times \otimes^2 R_n \longrightarrow \Lambda^2 \hat{v}$$

via $\hat{\delta}(\omega, T) = (\omega, [\mathrm{Alt}(T)]_{\{\omega\}})$ where $[\ ]_{\{\omega\}} : \Lambda^2 R_n \longrightarrow \Lambda^2 (R_n/\{\omega\})$ is the second exterior power of the projection $R_n \longrightarrow R_n/\{\omega\}$ and $\mathrm{Alt} : \otimes^2 R_n \longrightarrow \Lambda^2 R_n$ is the skew-symetrization map. We now define δ by $\delta = \bigoplus_r \hat{\delta}$. Now obviously $\hat{\delta}$ is a surjective bundle homomorphism and so δ is also a surjective bundle homomorphism. Therefore we have:

PROPOSITION. — *δ is a submersion.*

Now the maps $\pi_i : F_r \longrightarrow F_1$ given by $\pi_i(\omega_1, \ldots, \omega_r) = \omega_i$ induce $\pi_i' : F_r' \longrightarrow F_1'$ given by $\pi_i'(j_0^1(\omega)) = j_0^1(\pi_i \circ \omega)$. This allows us to write down the map δ as follows:

$$\delta(j_0^1(\omega)) = (\rho_r(j_0^1(\omega), [d \circ \pi_1'(j_0^1(\omega)]_{\rho_r(j_0^1(\omega))}, \ldots, [d \circ \pi_r'(j_0^1(\omega))]_{\rho_r(j_0^1(\omega))})$$

This formulation of δ suggests the tie up of the δ map with the notion of first derived system which we will make below.

Since δ is a submersion we can pull back the submanifolds $R_i \subset V$ to submanifolds $S_i = \delta^{-1}(R_i)$, which of course will have the same codimensions as the R_i's.

Next let G_r denote the space of r-planes in R_n. We have $p_r : F_r \longrightarrow G_r$ given by $p_r(\omega) = \{\omega\}$ which is a principal $Gl(r)$-bundle. Now p_r induces $p_r' : F_r' \longrightarrow G_r'$ where $G_r' = J_0^1(R^n, G_r)$, via $p_r'(j_0^1(\omega)) = j_0^1(p_r \circ \omega)$ and the triple (F_r', G_r', p_r') is a principal bundle with structure group given by $Gl'(r) = J_0^1(R^n, Gl(r))$. The sets S_i also possess an important property given by the following:

THEOREM. – *$S_i \subset F_r'$ is invariant by the $Gl'(r)$-action of the principal bundle (F_r', G_r', p_r').*

Proof. – The $Gl'(r)$-action on F_r' is given by

$$j_0^1(\hat{\omega}) \cdot j_0^1(\hat{g}) = j_0^1(\hat{\omega} \cdot \hat{g})$$

where of course $\hat{\omega} \cdot \hat{g}$ indicates matrix multiplication. Now we may identify $Gl'(r)$ with $Gl(r) \times \mathrm{Hom}(R^n, M(r, r))$, where $M(r, r)$ is the space of r by r matrices. Under this identification and the identification of F_r' with $\bigoplus_r (F_r \times \otimes^2 R_n)$, it is not hard to check that the $Gl'(r)$-action on F_r' is given as follows:

$$\begin{array}{ll} \text{if} & j_0^1(\hat{\omega}) \longleftrightarrow (\omega, T_1, \ldots, T_r) \in \bigoplus_r (F_r \times \otimes^2 R_n) \\ \text{and} & j_0^1(g) \longleftrightarrow (g, L) \in Gl(r) \times \mathrm{Hom}(R^n, M(r, r)) \\ \text{then} & j_0^1(\hat{\omega}) \cdot j_0^1(\hat{g}) \longleftrightarrow (\omega \cdot g, T \cdot g + \omega \cdot L) \end{array}$$

where $T = (T_1, \ldots, T_r)$, $T \cdot g$ is matrix multiplication,

$$(\omega \cdot L)_i = \Sigma L_{ji} \otimes \hat{\omega}_j, \quad \hat{\omega}(0) = \omega \text{ and } \hat{g}(0) = g.$$

Now if we apply $\hat{\delta}$ to the i^{th} term of $j_0^1(\hat{\omega}) \cdot j_0^1(\hat{g})$ we obtain

$$\begin{aligned} \hat{\delta}\Big(\omega \cdot g, \sum_j T^j g_{ji} + L_{ji} \otimes \omega_j\Big) & \\ &= \Big(\omega \cdot g, \Big[\sum_j \mathrm{Alt}(T_j)\, g_{ji} + L_{ji} \wedge \omega_j\Big]_{\{\omega \cdot g\}}\Big) \\ &= \Big(\omega \cdot g, \Big[\sum_j \mathrm{Alt}(T_j)\, g_{ji}\Big]_{\{\omega\}}\Big) \end{aligned}$$

where the last equality follows since $\{\omega \cdot g\} = \{\omega\}$. Writing $[\mathrm{Alt(T)}]_{\{\omega\}}$ for $([\mathrm{Alt}(\mathrm{T}_1)]_{\{\omega\}}, \ldots, [\mathrm{Alt}(\mathrm{T}_r)]_{\{\omega\}})$ we have

$$\delta(j_0^1(\hat{\omega}) \cdot j_0^1(\hat{g})) = (\omega \cdot g, [\mathrm{Alt(T)}]_{\{\omega\}} \cdot g)$$

and since $(j_0^1(\hat{\omega})) = (\omega, [\mathrm{Alt(T)}]_{\{\omega\}})$ we see that S_i is invariant by the $Gl'(r)$-action.

Because of the above theorem the projection of S_i to G'_r is a submanifold of G'_r of the same codimension as S_i has in F'_r. We will denote the projection of S_i to G'_r by Σ_i.

Next let $\mathrm{H} = \{j^2_{(0,0)}(f) \in \mathrm{J}^2_{(0,0)}(\mathrm{R}^n, \mathrm{R}^n) | f$ is a diffeomorphism of a neighborhood of 0 in $\mathrm{R}^n\}$. For convenience we will denote $j^2_{(0,0)}(f)$ by $j^2(f)$ when there is no chance of confusion. H is the group of invertible 2-jets of maps carrying the origin in R^n to itself. We will define an H-action on G'_r and show that the Σ_i's are invariant under this action. To do this we first define an H-action on F'_r as follows:

for $\quad j^2(f) \in \mathrm{H}$ and $j_0^1(\omega) \in \mathrm{F}'_r, j_0^1(\omega) \cdot j^2(f) \equiv j_0^1(\omega \circ \mathrm{J}(f))$

where $\qquad (\omega \circ \mathrm{J}(f))_i = \omega_i \circ \mathrm{J}(f)$

and $\qquad (\omega_i \circ \mathrm{J}(f))(x)(v) = \omega_i(f(x))(\mathrm{J}(f)(x)(v))$

for x and v elements of R^n. Of course $\mathrm{J}(f)(x)$ denotes the ordinary jacobian map of f at x. Now trivially

$$p'_r(j_0^1(\omega) \cdot j^2(f)) = p'_r(j_0^1(\tau) \cdot j^2(f))$$

whenever $j_0^1(\tau) = j_0^1(\omega) \cdot j_0^1(g)$ for some $j_0^1(g)$ $Gl'(r)$. Therefore we can define the H-action on G'_r as follows: if $j^2(f) \in \mathrm{H}$ and $j_0^1(\mathrm{Q}) \in \mathrm{G}'_r$, choose any $j_0^1(\omega) \in \mathrm{F}'_r$ such that $p'_r(j_0^1(\omega) = j_0^1(\mathrm{Q})$ and define $j_0^1(\mathrm{Q}) \cdot j^2(f) \equiv p'_r(j_0^1(\omega) \cdot j^2(f))$. Now by definition the actions of H on F'_r and G'_r commute with p'_r, so to show Σ_i is invariant by the H-action on G'_r, we need only show that S_i is invariant by the H-action on F'_r. To see that S_i is invariant by the H-action, we use our identification of F'_r with $\bigoplus_r \mathrm{F}_r \times \otimes^2 \mathrm{R}_n$ and we identify H with $Gl(n) \times \mathrm{Hom}^2_s(\mathrm{R}^n, \mathrm{R}^n)$, where $\mathrm{Hom}^2_s(\mathrm{R}^n, \mathrm{R}^n)$ is the space of symetric bilinear maps from $\mathrm{R}^n \times \mathrm{R}^n$ to R^n, via

$$j^2_{(0,0)}(f) \longleftrightarrow (\mathrm{J}(f)(0), \mathrm{J}^2(f)(0))$$

and $$\mathrm{J}^2(f)(0)(e_i, e_j) = \sum_k \frac{\partial^2 f^k}{\partial x_i \, \partial x_j}(0)\, e_k$$

and $\{e_i\}$, $1 \leqslant i \leqslant n$, denotes the standard basis of R^n. Under the above identifications it is easy to check that the H-action on F'_r is given as follows:

if $\quad j_0^1(\hat{\omega}) \longleftrightarrow (\omega, T)$ where $T = (T_1, \dots, T_r)$, $T_i \in \otimes^2 R_n$

and $\quad j^2(f) \longleftrightarrow (a, b) \in Gl(n) \times \mathrm{Hom}_s^2(R^n, R^n)$

then $\quad j_0^1(\hat{\omega}) \cdot j^2(f) \longleftrightarrow (\omega \circ a, \omega \circ b + T \circ \otimes^2 a)$

where $\qquad (\omega \circ b + T \circ \otimes^2 a)_i = \omega_i \circ b + T_i \circ \otimes^2 a$.

Next we compute δ on $j_0^1(\omega) \cdot j^2(f)$. To do this we compute $\hat{\delta}$ on each component:

$$\hat{\delta}(\omega \circ a, \omega_i \circ b + T_i \circ \otimes^2 a) = (\omega \circ a, [\mathrm{Alt}(\omega_i \circ b + T_i \circ \otimes^2 a]_{\{\omega \circ a\}})$$

but b is symetric so $\mathrm{Alt}(\omega_i \circ b) = 0$ and $\mathrm{Alt}(T_i \circ \otimes^2 a) = \wedge^2 a(\mathrm{Alt}(T_i))$ so $\quad \delta(\omega \circ a, \omega \circ b + T \circ \otimes^2 a) = (\omega \circ a, [\wedge^2 a(\mathrm{Alt}(T))]_{\{\omega \circ a\}})$ and clearly this implies that the S_i's are invariant.

Now let M be an n-dimensional manifold. We let $G_r(M)$ be the associated bundle to the frame bundle of M with standard fibre G_r and let $G'_r(M)$ be the bundle of i-jets of r-dimensional subbundles of the cotangent bundle of M. $G'_r(M)$ is a fibre bundle with standard fibre G'_r and structure group H, associated to the principal H-bundle of l-jets of r-coframes on M. Now since the Σ_i's are invariant submanifolds of the standard fibre of $G'_r(M)$, we can define submanifolds $\Sigma_i(M)$ of $G'_r(M)$ by letting $\Sigma_i(M)$ be the bundle with standard fiber Σ_i and transition functions the same as $G'_r(M)$. These submanifolds will have the same codimension in $G'_r(M)$ as the Σ_i have in G'_r. Furthermore we have $G'_r(M) = \bigcup_i \Sigma_i(M)$. Observe also that if Q is an r-dimensional subbundle of the cotangent bundle of M and $j^1(Q)(p) \in \Sigma_i(M)$ then the dimension of $Q_p^{(1)} = r - i$.

We let D_s^r denote the set of C^s r-dimensional Pfaffian systems on M. Then local triviality of $G_r(M) \longrightarrow M$, and the jet transversality theorem (see for example (3)) gives the following:

THEOREM. – *For* $s \geqslant 2$ *the set* $T(\Sigma)$ *of* $Q \in D_s^r$ *such that* $j^1(Q): M \longrightarrow G'_r(M)$ *is transversal to the stratification* $\Sigma(M) = \Sigma_i(M)$ *is a residual subset of* D_s^r.

Now if $Q \in D_s^r$, $s \geqslant 2$, is a generic r-dimensional Pfaffian system on M, then the theorem above gives a stratification of M associated to Q by

$$M_i = (j^1(Q))^{-1}(\Sigma_i(M)) \qquad 0 \leqslant i \leqslant \min(r, C(n-r, 2))$$

and since $j^1(Q)$ is transversal to $\Sigma_i(M)$ for all i, M_i will have the same codimension in M as $\Sigma_i(M)$ has in $G_r'(M)$. Also by our observation above these M_i's are the same as the M_i's defined in Section 1.

We now list some examples to illustrate our results:

1) $r = 1$. In general we will have M_0 and M_1 with the codimension of $M_0 = C(n-1, 2)$ and the codimension of $M_1 = 0$. Now if $C(n-1, 2) > n$, which happens if $n \geqslant 5$, then $M_0 = \phi$ and $M_1 = M$. So the generic 1-dimensional Pfaffian system on a manifold of dimension $n \geqslant 5$ is completely non-integrable.

2) $r = 2$. For $n \geqslant 7$ a generic 2-dimensional Pfaffian system is completely non-integrable and for $n \geqslant 5$ a generic 2-dimensional system is non-integrable.

3) We note that the polynomial $P(i) = (C(n-r, 2) - i)(r - i)$ is decreasing on $0 \leqslant i \leqslant \min(C(n-r, 2), r) = m$. So $P(i)$ takes on its minimum value at $i = m$ and $P(m) = 0$. Now if $P(m-1) > n$ we will have $M_0 = \ldots = M_{m-1} = \phi$ and $M_m = M$, i.e. the system is completely non-integrable. To see when this happens we look at the three possible cases:

a) $C(n-r, 2) = r$. In this case $P(m-1) = 1$ so $P(m-1) \not> n$.

b) $C(n-r, 2) < r$. In this case $P(m-1) = r - C(n-r, 2) + 1$, and it is easy to check that $P(m-1) > n$ is impossible.

c) $C(n-r, 2) > r$. In this case $P(m-1) = C(n-r, 2) - r + 1$, and this case is possible as examples 1 and 2 above show.

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