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COMPARISONS OF KERNEL FUNCTIONS, BOUNDARY HARNACK PRINCIPLE AND RELATIVE FATOU THEOREM ON LIPSCHITZ DOMAINS

by Jang-Mei G. WU

1. Introduction.

Let $D \subseteq \mathbf{R}^n$, $n \geqslant 2$, be a Lipschitz domain with a point O in D fixed. It is proved by Hunt and Wheeden [8] that corresponding to each positive harmonic function u in D, there is a unique Borel measure U on ∂D, such that when $P \in D$,

$$(1.1) \qquad u(P) = \int_{\partial D} K(P,Q)\, dU(Q)\,,$$

where $K(P,Q)$ is the kernel function defined by

$$(1.2) \qquad K(P,Q) = \frac{d\omega^P}{d\omega^0}(Q)$$

in the Radon-Nikodým sense and $\omega^P(E)$ is the harmonic measure of $E \subseteq \partial D$ at P; moreover u has nontangential limit $dU/d\omega^0$ at ω^0-almost every point of ∂D.

The kernel function defined in (1.2) does not have an explicit form as it does in spheres. In this paper, we shall prove a kernel comparison theorem (Theorem 2) on $K(P,Q_1)$ and $K(P,Q_2)$ with different poles Q_1 and Q_2 on ∂D. The spherical version of Theorem 2 follows easily from the explicit form of Poisson kernel and can be stated as follows :

Let $K(P,Q)$ be the Poisson kernel on the unit ball Σ in $\mathbf{R}^n$. Then there are positive constants c and C so that

$$\text{(i)} \qquad CK(P,Q_1) \leqslant K(P,Q_2) \leqslant \frac{1}{C} K(P,Q_1)$$

whenever Q_1, Q_2 and Q are on $\partial\Sigma$, P is on the radius through Q and $\frac{1}{2} |Q_1 - Q| < |Q_2 - Q| < 2|Q_1 - Q| < c$; and

$$\text{(ii)} \qquad K(P,Q_1) \leqslant K(P,Q_2)$$

whenever Q_1, Q_2 and Q are on $\partial\Sigma$, P is on the radius through Q and $|Q_2 - Q| < |Q_1 - Q| < c$.

We also have the following boundary Harnack principle on positive harmonic functions.

THEOREM 1. — *Suppose* D *is a Lipschitz domain,* P_0 *is a point in* D, E *is a relatively open set on* ∂D *and* S *is a sub-domain of* D *satisfying* $\partial S \cap \partial D \subseteq E$. *Then there is a constant* C, *so that whenever* u_1 *and* u_2 *are two positive harmonic functions in* D *vanishing on* E *and* $u_1(P_0) = u_2(P_0)$, *then* $u_1(P) \leqslant Cu_2(P)$ *for all* $P \in S$.

It is pointed out to me by Professors A. Ancona and M. Brelot that Theorem 1 is stated in [10], but the proof there is not complete.

As an application of Theorems 1 and 2, we prove the following relative Fatou theorem.

THÉOREM 3. — *Let* $D \subseteq \mathbf{R}^n$ *be a Lipschitz domain and* u, h *be two positive harmonic functions on* D. *With respect to a fixed point* O *in* D, *let* U *and* H *be the Borel measures on* ∂D *corresponding to* u *and* h. *Then* u/h *has a finite non-tangential limit at* H-*almost every point of* ∂D. *This limit is* H-*almost everywhere the Radon-Nikodým derivative of the absolutely continuous component of* U *with respect to* H.

The choice of the fixed point O in D is not important in Theorem 3. In fact, with respect to a fixed point O_i in D, $i = 1, 2$, let H_i be the measure on ∂D corresponding to h, then, a simple deduction from (1.1) and (1.2), H_1

and H_2 are absolutely continuous with respect to each other.

In [5], Dahlberg proved that in the Lipschitz domain D, ω^0 and the $n-1$ dimensional Hausdorff measure σ of ∂D are absolutely continuous with respect to each other. Roughly the bad points on ∂D where $d\omega^0/d\sigma = \infty$ or $d\sigma/d\omega^0 = \infty$ are those points at which the norm of the gradient of the Green's function $G(0,\cdot)$ of D tends to ∞ or 0. We note that the set of exceptional points in Theorem 3 need not contain those bad points on ∂D.

Theorem 3 is false when u is positive superharmonic, even if only the radial limits on spheres are considered [6]. But there are positive results about boundary limits along curves for positive superharmonic functions on D [11].

Theorem 3 is first studied by Doob for open solid spheres [6]. The idea of the proof of Theorem 3 is partly from that in [6].

Professors Ancona and Brelot have kindly pointed out to me that a fine limit version of Theorem 3 was proved by Gowrisankaran [7; Theorem 8] and Theorem 3 can be derived from the cited result in [7] by applying the method, introduced by Brelot and Doob, (see [4; Theorem 3] or [9; Theorem 5.5]), of showing the existence of nontangential limit of a positive harmonic function at a boundary point from the existence of the fine limit. However, the method used here is more direct and indicates the exceptional set (see the concluding remark).

For literature of fine limits of quotients of positive superharmonic functions see [3], [4], [6], and [7]. For fine topology approach for boundary Harnack principle and relative Fatou Theorem about elliptic operators on Lipschitz domains see [1].

The author wishes to thank the referee for many helpful comments.

2. Definitions and Preliminaries.

We denote by D a Lipschitz domain in $\mathbf{R}^n$, that is, D is a bounded domain and to each $Q \in \partial D$ there corresponds a local coordinate system (x,y), $(x \in \mathbf{R}^{n-1}$ and $y \in \mathbf{R})$ and a function f of class Lip 1 from $\mathbf{R}^{n-1}$ to $\mathbf{R}$ such that

$$N \cap D = \{(x,y): \ y > f(x)\} \cap D$$

for some neighborhood N of Q. We denote by 0 a fixed point in D.

We call S a cone at $Q \in \partial D$ if S is a subdomain of D and S is the union of a family of rays of equal length starting at Q. We say S is a nontangential cone at Q if $S \cup \partial S \setminus \{Q\}$ is contained in another cone at Q. We say a function F in D has nontangential limit β at $Q \in \partial D$ if for each nontangential cone S at Q, lim $F(P) = \beta$ as $P \longrightarrow Q$, $P \in S$.

Choosing and fixing local coordinate systems on ∂D properly, we denote by $M > 1$ an upper bound of the Lipschitz constants of the functions f and note that there is a similar circular cone at each $Q \in \partial D$ with axis on the local y-axis which lies completely in D, and whose reflection about the local x-hyperplane lies completely outside $\overline{D}$. For geometrical properties of Lipschitz domains and their relation to Harnack's inequality, the reader is referred to [8] and [9].

To each $Q \in \partial D$, we assume the origin of the local coordinate system is at Q and denote by $L(Q,r)$ the region $D \cap \{(x,y): |x| < r, \ y - f(x) < 10\,Mr\}$ and by $\Delta(Q,r)$ the base $\{(x,y) \in \partial D : |x| < r\}$ for small r. We note that there is a constant $C_0 > 0$ such that the fixed point $0 \notin L(Q,4C_0) \subseteq D$ for each $Q \in \partial D$ and we shall only consider those r, $0 < r < C_0$.

We shall use $C_1, C_2, \ldots, C, c$ to denote strictly positive constants which depend at most on n, 0, C_0 and M. The actual values of C and c may vary from line to line but the values of $C_1, C_2, \ldots$ are fixed. If the constants will also depend on some other variables, we indicate this by parentheses.

Here is a sequence of lemmas on estimates of harmonic measures. Lemmas 1 and 2 are due to L. Carleson; detailed proofs can be found in [9].

LEMMA 1. — *If* $Q \in \partial D$ *and* $P \in L(Q,r/2)$, *then* $\omega^P(\Delta(Q,r)) \geqslant C$.

LEMMA 2. — *Suppose* $Q \in \partial D$ *and* A *is the center of the top of* $L(Q,r)$, *that is,* $A = (0,10\,Mr)$. *Then for any* $\Delta' = \Delta(Q, r')$, $0 < r' < r$, *we have*

$$\omega^P(\Delta') \leqslant C\omega^A(\Delta')$$

for $P \in D \setminus L(Q,r)$.

We say D is starlike Lipschitz about O if it is starlike about O and the y-axis of a local coordinate system at any $Q \in \partial D$ can be chosen to contain the segment OQ, moreover the Lipschitz constants of the corresponding Lip 1 functions are bounded above by a constant M, $M > 1$. We note that « Lipschitz and starlike » need not imply « starlike Lipschitz ». In this case, we choose these local coordinate systems as the fixed ones on ∂D and also assume $0 \notin L(Q,4C_0) \subseteq D$ for each $Q \in \partial D$.

LEMMA 3. — *Suppose* $Q \in \partial D$, $\Delta = \Delta(Q,r)$ *and* $A = (0,10\,M\,r)$. *Then for every measurable set* $E \subseteq \Delta(Q,r/4)$,

$$\omega^O(E) \leqslant C\omega^A(E)\omega^O(\Delta);$$

moreover, if D *is starlike Lipschitz about* O, *we also have*

$$\omega^O(E) \geqslant C\omega^A(E)\omega^O(\Delta).$$

Lemma 3 is a special case of a result of Hunt and Wheeden, see (2.4) and (3.1) in [9].

LEMMA 4. — *Suppose* D *is starlike Lipschitz about* O. *Then if* $Q \in \partial D$, $0 < r < C_0$, $P = (0,10\,Mr)$ *and* $\Delta = \Delta(Q,r)$, *we have*

$$cr^{n-2}G(0,P) \leqslant \omega^O(\Delta) \leqslant Cr^{n-2}G(0,P),$$

where G *is the Green's function on* D.

Proof. — The validity of the second inequality is due to Dahlberg [5], only it is stated for slightly more general domains here. In order to prove the first inequality, we let $0 < r < C_0$ and D_1 be $\{(x,y) : |x| < r,\ 5\,M\,r < y - f(x) < 20\,Mr\}$. Because P is of distance at least Cr from ∂D_1, $G(T,P) \leqslant Cr^{2-n}$ for $T \in \partial D_1$. By Lemma 1 and Harnack's inequality we have $\omega^T(\Delta) \geqslant C$ for $T \in \partial D_1$. Applying the maximum principle to $r^{n-2}G(T,P)$ and $\omega^T(\Delta)$ for $T \in D\setminus D_1$, we have the first inequality.

The author is indebted to Robert Kaufman for a suggestion concerning the proof of Lemma 5.

LEMMA 5. — *Suppose* D *is a Lipschitz domain,* $Q \in \partial D$, A *is the center of the top of* $L(Q,r)$ *and* $a > 0$. *Then there is a constant* $C(a)$ *depending on* a, $0 < C(a) < 1$, *such that* $\omega^A(E) < a$ *for* $0 < t \leqslant C(a)$ *and*

$$E = \Delta(Q,r(1+t))\setminus\Delta(Q,r(1-t)).$$

Proof. — Let $\gamma = \gamma(Q,r) = \{(x,y) : |x| = r, \ y = f(x)\}$, l be the $n-2$ dimensional Hausdorff measure on γ and

$$\varphi(P) = \int_\gamma \frac{1}{|P-X|^{n-2}}\, dl(X)$$

for $P \in \mathbf{R}^n$. It is known that $\varphi(P)$ is a positive harmonic function on $\mathbf{R}^n\setminus\gamma$. Let x and p denote the x-coordinates of X and P respectively. We write $\alpha \sim \beta$ if $c < \alpha/\beta < C$. For $P \in E$, write $|p|$ as $(1-s)r$ or $(1+s)r$, $0 \leqslant s \leqslant t$; thus we have

$$\begin{aligned}
\varphi(P) &\sim \int_{|x|=r} \frac{1}{|(1 \pm s)r - x|^{n-2}}\, dx \\
&= \int_{|x|=1} \frac{1}{|1 \pm s - x|^{n-2}}\, dx \\
&= 2\int_0^\pi \frac{(\sin\theta)^{n-3}}{[(1-\cos\theta \pm s)^2 + (\sin\theta)^2]^{(n-2)/2}}\, d\theta \\
&\geqslant c\int_s^{\sqrt{s}} \frac{\theta^{n-3}}{[(c\theta^2 \pm s)^2 + \theta^2]^{(n-2)/2}}\, d\theta \\
&\sim \log\frac{1}{s} \geqslant \log\frac{1}{t}.
\end{aligned}$$

By the maximum principle, for $P \in D$,

$$\varphi(P) \geqslant \log\frac{1}{t}\,\omega^P(E). \tag{2.1}$$

We note that $\varphi(A) \leqslant C$. Letting $P = A$ in (2.1), we see the choice of $C(a)$.

3. Proof of Theorem 1.

Suppose D is starlike Lipschitz about O. We shall introduce two new Lipschitz domains $\overline{D}$ and $\underline{D}$, $\underline{D} \subseteq D \subseteq \overline{D}$, and compare certain harmonic measures on these three regions. Notations are retained from last section.

For $0 < t < 1$, let $g(x)$ be the Lipschitz function defined on $\mathbf{R}^{n-1}$ by

$$g(x) = \begin{cases} 0 , & |x| < 1 - t \text{ or } |x| > 1 + t , \\ t - |1 - |x|| , & 1 - t \leqslant |x| \leqslant 1 + t . \end{cases}$$

For $Q \in \partial D$, f the Lipschitz function associated with Q and $0 < r < C_0$, we define

$$\overline{f}(x) = f(x) - rg(x/r) \qquad \text{and} \qquad \underline{f}(x) = f(x) + rg(x/r)$$

for $x \in \mathbf{R}^{n-1}$. Let $\overline{D} = \overline{D}(Q,r,t)$ be the region

$D \cup \{(x,y): \ (1 - t)r < |x| < (1 + t)r , \ \overline{f}(x) < y \leqslant f(x)\}$

and $\underline{D} = \underline{D}(Q,r,t)$ be the region

$D \setminus \{(x,y): \quad (1 - t)r < |x| < (1 + t)r, \quad f(x) < y \leqslant \underline{f}(x)\}$.

We assume C_0 was chosen small enough so that $\overline{D} = \overline{D}(Q,r,t)$ and $\underline{D} = \underline{D}(Q,r,t)$ are starlike Lipschitz about O for $0 < r < C_0$; and let C_1M be an upper bound for the Lipschitz constants corresponding to $\partial\overline{D}$ and $\partial\underline{D}$. Choose C_2, less than C_0, depending on C_0 and C_1M but not on r and t so that if $0 < r < C_2$ then $0 \notin \overline{L}(Q,4C_2) \subseteq \overline{D}$ or $0 \notin \underline{L}(Q,4C_2) \subseteq \underline{D}$ for $Q \in \partial\overline{D}$ or $\partial\underline{D}$, where $\overline{L}$ and $\underline{L}$ are related to $\overline{D}$ and $\underline{D}$ as L is related to D. Thus Lemmas 1, 2, 3 and 4 are applicable to $\overline{D}$ and $\underline{D}$. We observe that all the points in

$$\Delta(Q,r(1 + t/2) \setminus \Delta(Q,r(1 - t/2))$$

are of distance at least Ctr from $\partial\overline{D}$ and from $\partial\underline{D}$. We denote by $\overline{\omega}^P(E)$ the harmonic measure of $E \subseteq \partial\overline{D}$ at $P \in \overline{D}$ with respect to the region $\overline{D}$. For the region $\underline{D}$ we define $\underline{\omega}^P(E)$ similarly.

LEMMA 6. — *Suppose* D *is starlike Lipschitz about* O, $Q \in \partial D$, $0 < r < C_2$, $A = (0, 10\,Mr)$, $\overline{D} = \overline{D}(Q,r,t)$, $\underline{D} = \underline{D}(Q,r,t)$ *and* $\Delta' = \Delta(Q',r') \subseteq \Delta(Q,r/2)$. *Then there is a constant* $C_3 < 1/4$ *such that*

$$c\overline{\omega}^A(\Delta') \leqslant \omega^A(\Delta') \leqslant C\underline{\omega}^A(\Delta')$$

if $0 < t \leqslant C_3$.

Proof. — It is enough to prove the first inequality because the second one follows from the first one by replacing $\overline{D}$ by D and D by $\underline{D}$. Assume $0 < t < 1/4$. It follows from Lemma 2 for $\overline{D}$ and Harnack's inequality that

$$\overline{\omega}^P(\Delta') \leqslant C_4\overline{\omega}^A(\Delta') \tag{3.1}$$

for $P \in \overline{D}\setminus L(Q, 5r/8)$ and some constant C_4. From (3.1) and the maximum principle, it follows that

$$\overline{\omega}^P(\Delta') \leqslant \omega^P(\Delta') + C_4\overline{\omega}^A(\Delta')\omega^P(E) \tag{3.2}$$

for $P \in D$, where $E = \Delta(Q, r(1+t))\setminus\Delta(Q, r(1-t))$. In the statement of Lemma 5, let a be $1/(2C_4)$. We may find a constant $C_3 < 1/4$ such that $\omega^A(E) < 1/(2C_4)$ for $0 < t \leqslant C_3$. Letting $P = A$ in (3.2), we conclude that $\overline{\omega}^A(\Delta') \leqslant 2\omega^A(\Delta')$. This completes the proof of Lemma 6.

The following lemma compares two harmonic functions near a piece of boundary where they both vanish.

LEMMA 7. — *Suppose* D *is starlike Lipschitz about* O, $0 < r < C_2$, $Q \in \partial D$, $L = L(Q,r)$, $\Gamma_1 = \partial L \setminus \partial D$,

$$\Gamma_2 = \Gamma_1 \setminus \{(x,y) : \ |x| = r \quad \textit{and} \quad y < f(x) + C_3 r\}.$$

Let v_1 *and* v_2 *be two positive harmonic functions on* L; v_1 *has boundary value* 1 *on* Γ_1, 0 *on* $\partial L\setminus\Gamma_1$ *and* v_2 *has boundary value* 1 *on* Γ_2 *and* 0 *on* $\partial L \setminus \Gamma_2$. *Then*

$$v_1(P) \leqslant C\, v_2(P) \tag{3.3}$$

for $P \in L(Q, r/4)$.

Proof. — For $0 < r < C_2$, let $\overline{D} = \overline{D}(Q,r,C_3)$,

$$\underline{D} = \underline{D}(Q,r,C_3) \qquad \text{and} \qquad \overline{G}, \underline{G}$$

be the Green's functions on $\overline{D}$ and $\underline{D}$ respectively.

Suppose $P \in L(Q,r/4)$. With the aid of Harnack's inequality, we may assume P is on OQ' and dist $(P,Q') \leqslant 5\,M\,r/2$ for some $Q' \in \Delta(Q,r/4)$. It follows from lemma 4 that

$$(3.4) \qquad c(r')^{n-2}\overline{G}(O,P) \leqslant \overline{\omega}^{o}(\Delta')$$

and

$$(3.5) \qquad \underline{\omega}^{o}(\Delta') \leqslant C(r')^{n-2}\overline{G}(0,P)\,,$$

where $r' = \text{dist}(P,Q')/(10\ M)$ and $\Delta' = \Delta(Q',r')$. By Harnack's inequality and the observation before Lemma 6, we have

$$\overline{G}(0,X) \geqslant C\overline{G}(0,A) \qquad \text{for} \qquad X \in \Gamma_1$$

and

$$\underline{G}(0,X) \leqslant G(0,X) \leqslant CG(0,A) \qquad \text{for} \qquad X \in \Gamma_2$$

where A is the center of the top of $L(Q,r)$. By the maximum principle,

$$(3.6) \quad v_1(X)\overline{G}(0,A) \leqslant C\overline{G}(0,X) \qquad \text{for} \qquad X \in L$$

and

$$(3.7) \quad \underline{G}(0,X) \leqslant Cv_2(X)G(0,A) \qquad \text{for} \qquad X \in L \cap \underline{D}\,.$$

It follows from (3.6) and (3.7) that to prove (3.3) for $P \in L(Q,r/4)$, it is enough to show

$$\overline{G}(0,P)G(0,A) < C\underline{G}(0,P)\overline{G}(0,A)\,.$$

With the aid of (3.4) and (3.5) it suffices to show

$$(3.8) \qquad \overline{\omega}^{o}(\Delta') \leqslant C\underline{\omega}^{o}(\Delta')\,.$$

From Lemma 6 we have $\overline{\omega}^{A}(\Delta') \leqslant C\underline{\omega}^{A}(\Delta')$. Applying Lemma 3 to $\overline{D}$ and $\underline{D}$, we obtain

$$(3.9) \qquad \overline{\omega}^{o}(\Delta')/\overline{\omega}^{o}(\overline{\Delta}) \leqslant C\underline{\omega}^{o}(\Delta')/\underline{\omega}^{o}(\underline{\Delta})\,,$$

where $\overline{\Delta} = \partial\overline{D}\diagdown(\partial D\diagdown\Delta(Q,2r))$ and $\underline{\Delta} = \partial\underline{D}\diagdown(\partial D\diagdown\Delta(Q,2r))$. We note that $\overline{\omega}^{o}(\overline{\Delta}) < \underline{\omega}^{o}(\underline{\Delta})$. Thus (3.8) follows from (3.9) and the proof is complete.

Proof of Theorem 1. — We choose $\varepsilon > 0$ and a finite family of cylinders $\{L(Q_i\,,2\,\varepsilon)\}_{1\leqslant i\leqslant k}$ in D depending on

D, E and S so that $\bigcup_1^k \bar{L}(Q_i, \varepsilon/8) \supseteq \partial S \cap \partial D$ and $\bigcup_1^k \bar{L}(Q_i, 2\varepsilon) \cap \partial D \subseteq E$. By Harnack's inequality, we need only show that $u_1(P) \leqslant Cu_2(P)$ for $P \in L(Q_i, \varepsilon/4)$ and $1 \leqslant i \leqslant k$ where C is a constant depending on D, E, S and P_0.

Fix i, $1 \leqslant i \leqslant k$ and denote by $L(\delta) = L(Q_i, \delta)$, by A the center top of $L(\varepsilon)$. We claim that $u_1(P) \leqslant Cu_1(A)$ on $\partial L(\varepsilon) \cap D$ for some constant C depending on D, E and S. Let K be the kernel function for the domain $L(2\varepsilon)$, normalized by $K(A,Q) = 1$. Since u_1 vanishes on $\partial L(2\varepsilon) \cap \partial D$, it suffices to show that for $P \in \partial L(\varepsilon) \cap D$ and $Q \in \partial L(2\varepsilon) \cap D$, $K(P,Q) \leqslant C$ for some constant C, or $\omega^P(\Delta(Q,r)) \leqslant C\omega^A(\Delta(Q,r))$ whenever r is sufficiently small; where ω and Δ are defined relative to $L(2\varepsilon)$. By Lemma 2 and Harnack's inequality, the last inequality holds for some constant C depending on $L(2\varepsilon)$ and A, hence depending only on D, E and S.

Let C_3' be a constant depending on $L(2\varepsilon)$, which plays the same role as C_3 in Lemma 7. By Harnack's inequality, $u_2(P) \geqslant cu_2(A)$ on $\partial L(\varepsilon) \setminus \{(x,y) : |x| = \varepsilon$ and $y < f(x) + C_3'\varepsilon\}$. By Lemma 7, $\dfrac{u_1(P)}{u_1(A)} \leqslant C \dfrac{u_2(P)}{u_2(A)}$ on $L(Q_i, \varepsilon/4)$.

We recall that $u_1(P_0) = u_2(P_0)$. Thus by Harnack's inequality $u_1(P) \leqslant C\, u_2(P)$ on $L(Q_i, \varepsilon/4)$, where C depends on D, E, S and P_0 only. This completes the proof of Theorem 1.

4. Inequalities on Kernel Functions.

In this section we shall prove a theorem concerning inequalities of the kernel functions on Lipschitz domains. The analogue for Poisson kernels on solid spheres can be derived easily from the explicit form of the kernels. Notations are retained from before.

The following is a variant of Besicovitch's theorem on the differentiation of Borel measures: *Suppose* μ *and* ν *are* *finite Borel measures on the boundary of the Lipschitz domain* D , *then for* ν*-almost every point* $Q \in \partial D$, $\mu(\Delta(Q,r))/\nu(\Delta(Q,r))$ *tends to* $d\mu/d\nu(Q)$, *the Radon-Nikodým derivative of* μ *with respect to* ν , *as* $r \longrightarrow 0$. (See [2] or [9]).

Applying this result to (1.2), we see that for each $P \in D$

$$K(P,Q) = \lim_{r\to 0} \frac{\omega^P(\Delta(Q,r))}{\omega^O(\Delta(Q,r))} \tag{4.1}$$

for ω^O-almost every $Q \in \partial D$. In fact, Hunt and Wheeden [9] have proved that (4.1) holds for each $Q \in \partial D$ and that for fixed $P \in D$, $K(P,Q)$ is a continuous function of Q on ∂D . Because of (4.1) we may use the lemmas in the previous sections to show the following results.

LEMMA 8. — *Suppose* D *is starlike Lipschitz about* O , $Q \in \partial D$, $0 < r < C_2$ *and* Q_1 , Q_2 *are in* $\Delta(Q,r)\setminus\Delta(Q,r/2)$. *Then there are constants* c *and* C *such that*

$$cK(P,Q_1) \leqslant K(P,Q_2) \leqslant CK(P,Q_1)$$

for P *on* $L(Q,r/32)$ *or on the segment* OQ .

Proof. — It is enough to prove that

$$K(P,Q_1) \leqslant CK(P,Q_2) \tag{4.2}$$

for some constant C . Given Q_1 and Q_2 on $\Delta(Q,r)\setminus\Delta(Q,r/2)$, we denote by Δ_i , $i = 1$ or 2 , the region $\Delta(Q_i,a_i)$ for small $a_i > 0$. From (4.1),

$$\frac{K(P,Q_1)}{K(P,Q_2)} \sim \frac{\omega^P(\Delta_1)}{\omega^O(\Delta_1)} \frac{\omega^O(\Delta_2)}{\omega^P(\Delta_2)}$$

for sufficiently small Δ_1 and Δ_2 . (By $\alpha \sim \beta$ we mean $c < \dfrac{\alpha}{\beta} < C$.) Let $A = (0,10Mr)$. From Lemma 3 we have

$$\frac{K(P,Q_1)}{K(P,Q_2)} \sim \frac{\omega^P(\Delta_1)}{\omega^A(\Delta_1)} \cdot \frac{\omega^A(\Delta_2)}{\omega^P(\Delta_2)} \tag{4.3}$$

for sufficiently small Δ_1 and Δ_2 . Let $L = L(Q,r/8)$, $\Gamma_1 = \partial L\setminus\partial D$, $\Gamma_2 = \Gamma_1\setminus\{(x,y) : |x| = r/8$ and $y < f(x) + C_3 r/8\}$.

From Harnack's inequality we have

$$C\,\frac{\omega^{P}(\Delta_2)}{\omega^{A}(\Delta_2)} > 1 \tag{4.4}$$

for $P \in \Gamma_2$ and small Δ_2. We may choose $C_5 > 0$ so that $L(Q_1, C_5 r)$ does not meet Γ_1 and if A_1 is the center of the top of $L(Q_1, C_5 r)$ then $\operatorname{dist}(A, A_1) \leqslant Cr$ for some constant C depending on C_5. Therefore by Lemma 2 and Harnack's inequality we have

$$\omega^{P}(\Delta_1) \leqslant C\omega^{A_1}(\Delta_1) \leqslant C\omega^{A}(\Delta_1) \tag{4.5}$$

for $P \in L_1$ and small Δ_1. From (4.4), (4.5), Lemma 7 and the maximum principle we conclude that

$$\frac{\omega^{P}(\Delta_1)}{\omega^{A}(\Delta_1)} < C\,\frac{\omega^{P}(\Delta_2)}{\omega^{A}(\Delta_2)}$$

for P on $L(Q, r/32)$ and thus from (4.3),

$$K(P, Q_1) < CK(P, Q_2)\,. \tag{4.6}$$

Let $B = (0, 5Mr/16)$ be the center of the top of $L(Q, r/32)$. We want to show (4.2) for P on the segment OB. Let $S = \{(x,y) : |x| = 2r,\ y = f(x)\}$, $\partial_1 = \{P \in D$ and P is on OQ' for some $Q' \in S$ and $\operatorname{dist}(P, Q') \leqslant 5Mr/16\}$ and $\partial_2 = \{P \in D$, P is on OQ' for some $Q' \in \Delta(Q, 2r)$ and $\operatorname{dist}(P, Q') = 5Mr/16\}$. Because of (4.6) and Harnack's inequality,

$$K(P, Q_1) \leqslant CK(P, Q_2) \tag{4.7}$$

for $P \in \partial_2$. Using the cone properties for D we may find constants C, c such that for each $Q' \in S$, with respect to the local coordinate system at Q', the quotient of the moduli of the x-coordinates of Q_1 and Q_2 is bounded above and below by C and c respectively. Thus repeating the argument in the last paragraph with Q replaced by points $Q' \in S$ and using Harnack's inequality, we obtain

$$K(P, Q_1) \leqslant C\,K(P, Q_2) \tag{4.8}$$

for $P \in \partial_1$. Let D' be the subdomain of D containing O and bounded by ∂D, ∂_1 and ∂_2. By (4.7), (4.8) and maximum principle on D', we conclude that for P on the segment $OB \subseteq D'$,

$$K(P,Q_1) \leqslant C\,K(P,Q_2)\,.$$

Hence the proof is complete.

LEMMA 9. — *Suppose* D *is starlike Lipschitz about* O, $0 < r < C_2$. $Q_2 \in \Delta(Q,r/2)$ *and* Q_1 *is on the* $n-2$ *dimensional boundary of* $\Delta(Q,r)$. *Then there is a constant* C *such that*

$$(4.9) \qquad K(P,Q_1) \leqslant CK(P,Q_2)$$

for P *on the ray* OQ.

Proof. — The idea of the proof is very similar to that of Lemma 8; we shall not put down too much detail.

First we shall show (4.9) for P on the segment QB where B is the center of the top of $L = L(Q,3r/4)$. Let $A = (0,20\,M\,r)$ and $\Delta_i = \Delta(Q_i,a_i)$, $a_i > 0$ and $i = 1,2$. To show (4.9) it is enough to prove

$$(4.10) \qquad \omega^P(\Delta_1)/\omega^A(\Delta_1) \leqslant C\omega^P(\Delta_2)/\omega^A(\Delta_2)$$

for small Δ_1 and Δ_2. Let $\Gamma_1 = \partial L \setminus \partial D$ and

$$\Gamma_2 = \Gamma_1 \setminus \{(x,y): \ |x| = 3r/4 \quad \text{and} \quad y < f(x) + 3C_3r/4\}\,.$$

We have

$$c\omega^P(\Delta_2)/\omega^A(\Delta_2) \geqslant 1 \ \text{for} \ P \in \Gamma_2 \ \text{and} \ c\omega^P(\Delta_1)/\omega^A(\Delta_1) \leqslant 1$$

for $P \in \Gamma_1$ and small Δ_1, Δ_2. By Lemma 7 and the maximum principle, (4.10) holds for P on QB.

Choose a point Q' on $\Delta(Q,r)$ so that with respect to the local coordinate system at Q', the quotient of the moduli of the x-coordinates for Q_1 and Q_2 is bounded above and below by 2 and 1/2 respectively. If P is a point on the segment OB, we let P' be the point on OQ' so that dist $(P',Q) =$ dist (P,Q). By Lemma 8, $K(P',Q_1) \leqslant K(P',Q_2)$. By Harnack's inequality, (4.2) follows for P on OB. This completes the proof.

For a general Lipschitz domain D, we choose and fix a finite covering of ∂D, $\mathscr{V} = \{\bar{L}(\tilde{Q}_i, C_0/4)\}_{1 \leqslant i \leqslant k}$, with $\tilde{Q}_i \in \partial D$

and $\bigcup_{1}^{k} L(\tilde{Q}_i,C_0/4) \supseteq \{P \in D: \text{dist}\ (P,\partial D) < \varepsilon\}$ for some $\varepsilon > 0$. Then we have the following theorem.

THEOREM 2. — *Suppose* D *is a Lipschitz domain. There exist constants* C_6, $\bar{c}$, $\bar{C}$ *depending on* $\mathcal{V}$ *such that*

(1) if $0 < r < C_6$, $Q \in \partial D$ *and* Q_1, Q_2 *are in* $\Delta(Q,r)\setminus\Delta(Q,r/2)$,

then

$$\bar{c}K(P,Q_1) \leqslant K(P,Q_2) \leqslant \bar{C}K(P,Q_1)$$

for P *on the local y-axis at* Q *with* dist $(P,Q) < C_0$ *or on* $L(Q,r/32)$;

(2) *if* $0 < r < C_6$, $Q \in \partial D$, $Q_2 \in \Delta(Q,r/2)$ *and* Q_1 *is on the* $n-2$ *dimensional boundary of* $\Delta(Q,r)$, *then*

$$K(P,Q_1) \leqslant \bar{C}K(P,Q_2)$$

for P *on the local y-axis at* Q *with* dist $(P,Q) < C_0$.

Proof. — We first prove (1). Choose r_0 depending on $\mathcal{V}$ so that for each $Q \in \partial D$, $L(Q,3r_0)$ is in some $L(\tilde{Q}_i,C_0/4) \in \mathcal{V}$. For each $Q \in \partial D$, we choose one $L(\tilde{Q}_i,C_0/4)$ from $\mathcal{V}$ which contains $L(Q,3r_0)$ and denote by $\tilde{D}$ the cylinder $L(\tilde{Q}_i,C_0)$ and let $\tilde{O}$ be the point $(0,5\ MC_0)$ with respect to the local coordinate system at $\tilde{Q}_i$. Then $\tilde{D}$ is starlike Lipschitz about $\tilde{O}$ and with C_7M as an upper bound for the Lipschitz constants of functions defining $\tilde{D}$ (C_7 is independent of $\mathcal{V}$). We denote by $\tilde{L}(Q,r)$ the cylinder corresponding to Q,r with respect to the starlike Lipschitz domain $\tilde{D}$. Let $\tilde{K}$ be the kernel function on $\tilde{D}$ relative to the fixed point $\tilde{O}$, that is, for $X \in \tilde{D}$ and $Y\ \omega\ \partial\tilde{D}$

$$\tilde{K}(X,Y) = d\tilde{\omega}^X/d\tilde{\omega}^{\tilde{O}}(Y),$$

where $\tilde{\omega}^X(E)$ is the harmonic measure of $E \subseteq \partial\tilde{D}$ at X with respect to the region $\tilde{D}$. By Lemma 8 and the choice of C_2, we may find C_6 less than r_0, depending on C_7M such that if $0 < r < C_6$, Q_1, Q_2 are in $\Delta(Q,r)\setminus\Delta(Q,r/2)$

and P is on the segment $\tilde{O}Q$ or on $\tilde{L}(Q,r/32)$ then

$$\tilde{K}(P,Q_1) \leqslant C\tilde{K}(P,Q_2);$$

hence

$$\tilde{\omega}^P(\Delta_1)/\tilde{\omega}^{\tilde{O}}(\Delta_1) \leqslant C\tilde{\omega}^P(\Delta_2)/\tilde{\omega}^{\tilde{O}}(\Delta_2) \tag{4.11}$$

for small $\Delta_i = \Delta(Q_i,a_i)$, $a_i > 0$ and $i = 1,2$.

Suppose $0 < r < C_6$, $Q \in \partial D$ and Q_1 , Q_2 are in $\Delta(Q,r)\diagdown\Delta(Q,r/2)$. We need to show that there is a constant $\overline{C}$ depending on $\mathscr{V}$, so that

$$K(P,Q_1) \leqslant \overline{C}K(P,Q_2) \tag{4.12}$$

for those P described in (1). With the aid of Harnack's inequality, we may assume P is on $\tilde{O}Q$ with dist $(P,Q) \leqslant C_0$ or on $\tilde{L}(Q,r/32)$ and need only to show (4.12) for these P's . Thus it is enough to show

$$\omega^P(\Delta_1)/\omega^O(\Delta_1) \leqslant \overline{C}\omega^P(\Delta_2)/\omega^O(\Delta_2) \tag{4.13}$$

for small Δ_1 and Δ_2 . Because both quotients in (4.13) have limits as the diameters of Δ_1 and Δ_2 tend to zero, and $\tilde{\omega}^{\tilde{O}}$ and $n-1$ dimensional Hausdorff measure on $\partial\tilde{D}$ are absolutely continuous with respect to each other, we need only to show (4.13) for those Δ_1 and Δ_2 satisfying

$$\tilde{\omega}^{\tilde{O}}(\Delta_1) = \tilde{\omega}^{\tilde{O}}(\Delta_2) . \tag{4.14}$$

First, we want to show that there is a constant $\overline{C}$ depending on $\mathscr{V}$,

$$\omega^O(\Delta_2) \leqslant \overline{C}\omega^O(\Delta_1) . \tag{4.15}$$

Let $\Gamma_1 = \partial\tilde{D}\diagdown\partial D$, we have

$$\omega^{\tilde{O}}(\Delta_2) = \tilde{\omega}^{\tilde{O}}(\Delta_2) + \int_{\Gamma_1} \tilde{K}(\tilde{O},X)\omega^X(\Delta_2)d\tilde{\omega}^{\tilde{O}}(X) .$$

Therefore

$$\begin{aligned}&\tilde{\omega}^{\tilde{O}}(\Delta_2)\\ &= \int_{\Delta_2} K(\tilde{O},Y)d\omega^O(Y) - \int_{\Gamma_1} \tilde{K}(\tilde{O},X)\int_{\Delta_2} K(X,Y)d\omega^O(Y)d\tilde{\omega}^{\tilde{O}}(X)\\ &= \int_{\Delta_2}\Big[K(\tilde{O},Y) - \int_{\Gamma_1} \tilde{K}(\tilde{O},X)K(X,Y)d\tilde{\omega}^{\tilde{O}}(X)\Big]d\omega^O(Y) .\end{aligned}$$

We observe that for each $Y \in \partial\tilde{D} \cap \partial D$, the term enclosed by brackets in the above equality is positive. By (4.1), Lemma 2 and Harnack's inequality we observe that

$$\max \{K(X,Y) : X \in \Gamma_1 , \quad Y \in \Delta(\tilde{Q}_i , C_0 - r_0)\} < C$$

for some constant C. Because for each X, $K(X,Y)$ is a continuous function of $Y \in \partial D$, there is a constant $\bar{C} > 0$ depending on $\mathscr{V}$ so that

$$K(\tilde{O},Y) - \int_{\Gamma_1} \tilde{K}(\tilde{O},X)K(X,Y)d\tilde{\omega}^{\tilde{O}}(X) > \bar{C}$$

for each $Y \in \Delta(\tilde{Q}_i ,C_0 - r_0)$. Hence for sufficiently small Δ_2,

$$\tilde{\omega}^{\tilde{O}}(\Delta_2) \geqslant \bar{C}\omega^{O}(\Delta_2) . \tag{4.16}$$

By Harnack's inequality,

$$\omega^{\tilde{O}}(\Delta_1) \leqslant C\omega^{O}(\Delta_1) . \tag{4.17}$$

From (4.14), (4.16), (4.17) and monotonicity, we have $\omega^{O}(\Delta_2) \leqslant \bar{C}\omega^{O}(\Delta_1)$, which is (4.15). Switching the roles of Q_1 and Q_2, we also have

$$\omega^{O}(\Delta_1) \leqslant \bar{C}\omega^{O}(\Delta_2) . \tag{4.18}$$

Next we want to show that for P on $\tilde{O}Q$ with dist $(P,Q) < C_0$ or on $\tilde{L}(Q,r/32)$,

$$\omega^{P}(\Delta_1) \leqslant \bar{C}\omega^{P}(\Delta_2) . \tag{4.19}$$

Let B be the center of the top of $\tilde{D}$, and

$$\Gamma_2 = \Gamma_1 \setminus \{(x,y) : \quad |x| = C_0, y < f(x) + C_3C_0\}$$

where (x,y) are the local coordinates and $y = f(x)$ is the Lipschitz function corresponding to $\tilde{Q}_i$ in defining the Lipschitz domain D. From Harnack's inequality, it follows that for $i = 1, 2$,

$$c\omega^{B}(\Delta_i) \leqslant \omega^{O}(\Delta_i) \leqslant C\omega^{B}(\Delta_i) .$$

Combining the above inequalities with (4.15) and (4.18) we obtain

$$(4.20) \qquad \bar{c}\omega^{B}(\Delta_1) \leqslant \omega^{B}(\Delta_2) \leqslant \bar{C}\omega^{B}(\Delta_1)\,.$$

Applying Lemma 2 and Harnack's inequality properly we have

$$(4.21) \qquad C\omega^{X}(\Delta_1)/\omega^{B}(\Delta_1) \leqslant 1$$

for $X \in \Gamma_1$, and

$$(4.22) \qquad C\omega^{X}(\Delta_2)/\omega^{B}(\Delta_2) \geqslant 1$$

for $X \in \Gamma_2$. Let $v_j(X) = \omega^{X}(\Delta_j) - \tilde{\omega}^{X}(\Delta_j)$ for $X \in \tilde{D} \cup \partial\tilde{D}$ and $j = 1$ or 2. Thus $v_j(X)$ has the same boundary values as $\omega^{X}(\Delta_j)$ on Γ_1 and vanishes on $\partial\tilde{D} \cap \partial D$. Embedding $\tilde{D}$ in a sufficiently large domain D^* which is starlike Lipschitz about 0^* and has $2M$ as an upper bound for the Lipschitz constants, we may conclude from (4.20), (4.21), (4.22) and Lemma 7 that $v_1(P) \leqslant \bar{C}v_2(P)$ for P on $\tilde{L}(Q_i,C_0/4)$. From (4.11) and (4.14), we have $\tilde{\omega}^{P}(\Delta_1) \leqslant C\tilde{\omega}^{P}(\Delta_2)$ for P on $\tilde{O}Q$ or on $\tilde{L}(Q,r/32)$. Combining the last two inequalities we have (4.19). Thus (4.13) follows. This concludes the proof for (1).

The proof of (2) is the same as (1) except that we use Lemma 9 instead of Lemma 8. This completes the proof of Theorem 2.

5. Proof of Theorem 3.

Before we prove Theorem 3, we need the following lemma.

LEMMA 10. — *Let h be a positive harmonic function on D, corresponding to a Borel measure H on ∂D. Then for H-almost every $Q \in \partial D$*

$$(5.1) \qquad \liminf h(P) > 0$$

as $P \longrightarrow Q$ along the local y-axis at Q.

Proof. — Let E be the set of $Q \in \partial D$, where $\liminf h(P) = 0$ as $P \longrightarrow Q$ along the local y-axis. For each $Q \in E$, choose $\{P_n\}$ on the local y-axis, satisfying $P_n \longrightarrow Q$ and $h(P_n) < \dfrac{1}{n}\cdot$

Let $\Delta_n = \Delta(Q,|P_n - Q|)$ and Δ_X be any $\Delta(X,a)$, $X \in \partial D$, $a > 0$. From Harnack's inequality and Lemma 3, we have $\omega^0(\Delta_n)\omega^{P_n}(\Delta_X) \geqslant C\omega^0(\Delta_X)$ for $X \in \Delta_n$ and sufficiently small Δ_X. Therefore

$$
\begin{aligned}
h(P_n) &\geqslant \int_{\Delta_n} K(P_n,X)dH(X) \\
&= \int_{\Delta_n} \lim_{a\to 0} \frac{\omega^{P_n}(\Delta(x,a))}{\omega^0(\Delta(x,a))}\, dH(x) \\
&\geqslant C \int_{\Delta_n} \frac{1}{\omega^0(\Delta_n)}\, dH(X) = C\,\frac{H(\Delta_n)}{\omega^0(\Delta_n)}.
\end{aligned}
$$

Thus the symmetric derivative $\lim_{n\to\infty} \frac{H(\Delta_n(Q))}{\omega^0(\Delta_n(Q))} = 0$ for each $Q \in E$. It follows from a theorem by Besicovitch [2; Theorem 5] that $H(E) = 0$.

Now we shall prove Theorem 3. Let S be a nontangential cone at Q. We say S is of class N (N: positive integer) if every point $P \in S$ can be connected by an arc γ in D to the point P' on the local y-axis satisfying $\operatorname{dist}(P',Q) = \operatorname{dist}(P,Q)$, such that the length of γ is less than $N \operatorname{dist}(\gamma,\partial D)$. For each $Q \in \partial D$, let $S(Q,N)$ be the union of cones of class N at Q.

To prove the theorem it is enough to show that, for each N, $\lim \frac{u(P)}{h(p)} = \frac{dU}{dH}(Q)$ as $P \longrightarrow Q$, $P \in S(Q,N)$, for H-almost every $Q \in \partial D$. A slight variant of an argument in [6] says that it is enough to show that there is a constant $C(Q,N)$ for each Q and N, independent of u and h, such that

$$
\limsup \frac{u(P)}{h(P)} \leqslant C(Q,N) \lim_{r\to 0} \frac{U(\Delta(Q,r))}{H(\Delta(Q,r))} \tag{5.2}
$$

as $P \longrightarrow Q$, $P \in S(Q, N)$ whenever the symmetric derivative exists and (5.1) is satisfied. By Harnack's inequality, it is enough to prove that

$$
\limsup \frac{u(P)}{h(P)} \leqslant C(Q) \lim_{r\to 0} \frac{U(\Delta(Q,r))}{H(\Delta(Q,r))} \tag{5.3}
$$

as $P \longrightarrow Q$ along local y-axis whenever the symmetric derivative exists and (5.1) is satisfied.

Since U and H are Borel measures, for H-almost every $Q \in \partial D$,

$$\lim_{a \to 0} \frac{U(\Delta(Q,a))}{H(\Delta(Q,a))} = \frac{dU}{dH}(Q) .$$

Let Q be a point on ∂D satisfying (5.1) and

$$\lim_{n \to \infty} U(\Delta_n)/H(\Delta_n) = \alpha < \infty ,$$

where $\Delta_n = \Delta(Q, 2^{-n})$. Choose and fix an integer k, such that $2^{-k} < C_6$ and $U(\Delta_n) \leqslant 2\alpha H(\Delta_n)$ for $n \geqslant k$. Suppose P is on the local y-axis at Q and satisfies dist $(P,Q) < C_0$. We let $I_n(P)$ be $\sup \{K(P,X) : X \in \Delta_n \setminus \Delta_{n+1}\}$ and $J_n(P)$ be $\inf \{I_m : m \geqslant n\}$. It follows from Theorem 2 that if $n \geqslant k$ then

$$J_n(P) \leqslant I_n(P) \leqslant \overline{C} J_n(P) \tag{5.4}$$

for some constant $\overline{C}$ depending on a preassigned finite cylindrical covering of ∂D. Therefore

$$\begin{aligned} \int_{\Delta_k} K(P,X)dU(X) &\leqslant \sum_{n=k}^{\infty} I_n(P)U(\Delta_n \setminus \Delta_{n+1}) + K(P,Q)U(\{Q\}) \\ &\leqslant \overline{C} \sum_{n=k}^{\infty} J_n(P)U(\Delta_n \setminus \Delta_{n+1}) + K(P,Q)U(\{Q\}) . \end{aligned}$$

By Abel summation formula, the observation

$$\lim_{N \to \infty} J_N(P)U(\Delta_{N+1}) = K(P,Q)U(\{Q\}) ,$$

the choice of k, (5.4) and Theorem 2, we have

$$\begin{aligned} &\int_{\Delta_k} K(P,X)dU(X) \\ &\qquad \leqslant \overline{C}\left[\sum_{n=k}^{\infty} (J_n - J_{n-1})(P)U(\Delta_n) + J_{k-1}(P)U(\Delta_k)\right] \\ &\qquad \leqslant 2\alpha\overline{C}\left[\sum_{n=k}^{\infty} (J_n - J_{n-1})(P)H(\Delta_n) + J_{k-1}(P)H(\Delta_k)\right] \\ &\qquad = 2\alpha\overline{C}\left[\sum_{n=k}^{\infty} J_n(P)H(\Delta_n \setminus \Delta_{n+1}) + K(P,Q)H(\{Q\})\right] \\ &\qquad \leqslant 2\alpha\overline{C}\left[\sum_{n=k}^{\infty} I_n(P)H(\Delta_n \setminus \Delta_{n+1}) + K(P,Q)H(\{Q\})\right] \\ &\qquad \leqslant 2\alpha\overline{C} \int_{\Delta_k} K(P,X)dH(X) \leqslant 2\alpha\overline{C}\, h(P) . \end{aligned}$$

Therefore

$$u(\mathrm{P})/h(\mathrm{P}) = \left[\int_{\Delta_k} \mathrm{K}(\mathrm{P},\mathrm{X})d\mathrm{U}(\mathrm{X}) + \int_{\partial\mathrm{D}\setminus\Delta_k} \mathrm{K}(\mathrm{P},\mathrm{X})d\mathrm{U}(\mathrm{X})\right]/h(\mathrm{P})$$
$$\leqslant 2\alpha\overline{\mathrm{C}} + \left[\int_{\partial\mathrm{D}\setminus\Delta_k} \mathrm{K}(\mathrm{P},\mathrm{X})d\mathrm{U}(\mathrm{X})\right]/h(\mathrm{P}).$$

For the same reason as in the proof of Theorem 2, $\mathrm{K}(\mathrm{P},\mathrm{X})$ is bounded for $\mathrm{X} \in \partial\mathrm{D}\setminus\Delta_k$ and P near Q. Because (5.1) holds at Q, $\limsup u(\mathrm{P})/h(\mathrm{P}) \leqslant 2\alpha\overline{\mathrm{C}}$ as $\mathrm{P} \longrightarrow \mathrm{Q}$ along local y-axis at Q. Hence (5.3) is proved. This completes the proof of the theorem.

Remark. — Professor Ancona suggests that it is interesting to indicate the exceptional set in Theorem 3. We write $\mathrm{U} = \frac{d\mathrm{U}}{d\mathrm{H}}\mathrm{H} + \mathrm{U}_s$ where U_s is singular with respect to H. At a point $\mathrm{Q}_0 \in \partial\mathrm{D}$ where condition (5.1) is satisfied and at which the symmetric derivative of $\left|\frac{d\mathrm{U}}{d\mathrm{H}} - \frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}_0)\right|\mathrm{H} + \mathrm{U}_s$ with respect to H is zero, u/h has nontangential limit $\frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}_0)$. In fact,

$$\left|u(\mathrm{P}) - \frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}_0)h(\mathrm{P})\right|$$
$$\leqslant \int_{\partial\mathrm{D}} \mathrm{K}(\mathrm{P},\mathrm{Q})\left[\left|\frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}) - \frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}_0)\right|d\mathrm{H}(\mathrm{Q}) + d\mathrm{U}_s(\mathrm{Q})\right].$$

dividing both sides of the inequality by $h(\mathrm{P})$ and applying (5.2) to the right side, we obtain

$$\limsup \left|\frac{u(\mathrm{P})}{h(\mathrm{P})} - \frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}_0)\right| = 0$$

as $\mathrm{P} \longrightarrow \mathrm{Q}_0$ nontangentially, hence,

$$\lim \frac{u(\mathrm{P})}{h(\mathrm{P})} = \frac{d\mathrm{U}}{d\mathrm{H}}(\mathrm{Q}_0)$$

as $\mathrm{P} \longrightarrow \mathrm{Q}_0$ nontangentially. The set of Q_0 on $\partial\mathrm{D}$ with properties described above has full H-measure (a classical argument).

We notice that the condition (5.1) and the existence of symmetric derivative of U with respect to H at Q_0 need not give the nontangential limit of u/h at Q_0, for example $D = \{z \in C : |z| < 1, Im\, z > 0\}$, $u(z) = \arg z (0 < \arg z < \pi)$, $h \equiv 0$ and Q_0 is the origin.

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