ANNALES DE L'INSTITUT FOURIER

ALAIN BERNARD

EDDY A. CAMPBELL

A. M. DAVIE

Brownian motion and generalized analytic and inner functions

BROWNIAN MOTION AND GENERALIZED ANALYTIC
AND INNER FUNCTIONS

by A. BERNARD, E.A. CAMPBELL and A.M. DAVIE

Dédié à Monsieur Claude Chabauty.

Introduction.

Let f be an entire function on the complex plane $\mathbf{C}$, let $z_0 \in \mathbf{C}$ and let $b(t)\,(t > 0)$ denote plane Brownian motion started at z_0. According to a theorem of P. Levy, $f(b(t))$ is, in a sense which will be made precise below, Brownian motion started at $f(z_0)$, with a different time-scale. The purpose of this paper is to investigate which functions $f: \mathbf{R}^n \longrightarrow \mathbf{R}^p\,(n \geqslant p)$ between Euclidean spaces preserve Brownian motion in this manner. A necessary and sufficient condition is that f satisfy a certain system of partial differential equations. If $p = 1$ this system reduces to the equation $\nabla^2 f = 0$, i.e. f must be harmonic. If $p = 2$, then treating f as a complex-valued function, the requirement is that f and f^2 both be harmonic. An alternative characterization is that $\varphi \circ f$ is harmonic for any harmonic function φ on $\mathbf{R}^p$. Functions between Riemannian manifolds possessing this property have been studied by Fuglede [6] but he did not consider the case of $\mathbf{R}^n \longrightarrow \mathbf{R}^p$ in detail. We concentrate on the case $f: \mathbf{R}^3 \longrightarrow \mathbf{R}^2$, for which we obtain a reasonable description based on the fact that the level sets of f must be straight lines. The general case, for which we have less information, is considered briefly.

We also consider a notion of inner function which includes the classical one. Roughly speaking, if U and V are domains in $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively, then a mapping $f: \text{U} \longrightarrow \text{V}$ which maps Brownian motion started at $x_0 \in \text{U}$ and stopped when it reaches the boundary of U to Brownian motion started at $f(x_0)$ and stopped at the boundary of V, we call a stochastic inner function. When U

and V are both the open unit disc in $R^2 = C$ the stochastic inner functions are just the classical (non-constant) inner functions and their complex conjugates. We conjecture that if $U \subseteq R^n$ is bounded ($n > 2$) and $V \subseteq R^2$ there are no stochastic inner functions, and prove this when $n = 3$. A proof of this conjecture for $n = 2m$ would prove the non-existence of inner functions (in the usual sense) on the unit ball of C^m ($m > 1$). We exhibit some non-trivial stochastic inner functions with $p > 2$.

It was hoped that the Brownian path preserving property would enable probabilistic methods to be brought to bear on problems such as the existence of inner functions on balls. Unfortunately we have not realized this hope, and the arguments presented here are based mainly on the differential equations.

1. Brownian path preserving functions and their differential equations.

Let us first recall that a Brownian motion in R^n is a family of R^n-valued random variable $\{b(t): t \geqslant 0\}$ on a probability space (Ω, F, P) satisfying:

   (i) for each $t > 0$ the random variable $b(t) - b(0)$ has a Gaussian distribution with density $(2\pi t)^{-n/2} \exp(-|x|^2/2t)$ where $|x|^2 = \displaystyle\sum_1^n x_k^2$;

   (ii) if $0 \leqslant t_1 < t_2 < \ldots < t_n$ then the random variables $b(t_n) - b(t_{n-1}), \ldots, b(t_2) - b(t_1)$ are independent;

   (iii) for each $\omega \in \Omega$ the mapping $t \longrightarrow b(t)(\omega)$ from $\{t \geqslant 0\}$ to R^n is continuous.

Let $x_0 \in R$. If moreover $b(0) = x_0$ almost surely, we say $b(t)$ is a Brownian motion issued from x_0.

We next make precise the notion of a function which preserves Brownian motion.

DEFINITION 1.1. – *Let* U *be a domain in* R^n *and let* $f: U \longrightarrow R^p$ *be continuous. We say that* f *is Brownian path preserving* (BPP) *if for each* $x_0 \in U$ *and for each Brownian motion* $b(t)$ *defined on* (Ω, F, P), *issued from* x_0, *there exist:*

1) *a mapping $\omega \longrightarrow \sigma_\omega$ on Ω such that for each $\omega, \sigma = \sigma_\omega$ is a continuous strictly increasing function on $[0, \tau[$, where $\tau = \sup \{t : b(s) \in U$ for $0 \leqslant s \leqslant t\}$ is the exit time of b from U, and such that for each $t > 0$, the mapping $\omega \longrightarrow \sigma_\omega(t)$ is measurable on the set $\{\tau > t\} \subseteq \Omega$. We also require that for each s the random variable $\sigma(s)$ be independent of the family of random variables $\{b(t) - b(s) : t > s\}$* ;

2) *a Brownian motion $b'(t)$ defined on a probability space (Ω', F', p') in $\mathbf{R}^p$, issued from 0, such that*

3) *on the probability space product of (Ω, F, P) and (Ω', F', P') the random function $a(s) = a_\omega(s)$ defined for $s \geqslant 0$ by*

$$a(s) = \begin{cases} f(b(\sigma^{-1}(s))), \ s < \sigma(\tau) = \lim\limits_{t \to \tau} \sigma(t) \\ f(b(\tau)) + b'(s - \sigma(\tau)), \ s \geqslant \sigma(\tau) \end{cases}$$

is a Brownian motion issued from $f(x_0)$.

The following characterization of BPP functions is due essentially to P. Levy.

THEOREM 1.2. — *Let U be a domain in $\mathbf{R}^n$ and let $f = (f_1, \ldots, f_p)$ be a non-constant continuous mapping of U into $\mathbf{R}^p$. For f to be BPP it is necessary and sufficient that the following two conditions be satisfied:*

(i) *for each $j = 1, \ldots, p, f_j$ is harmonic on U*

(ii) *for each $x \in U$ the gradient vectors $\nabla f_j(x) \ (j = 1, \ldots, p)$ have equal length and are mutually orthogonal.*

Before giving the proof, we remark that (i) plus (ii) is equivalent to the statement that $\varphi \circ f$ is harmonic for every harmonic function φ on $\mathbf{R}^p$.

Proof. — We first prove necessity of (i) and (ii). We suppose f is BPP and let φ be a harmonic function on $\mathbf{R}^p$; it suffices to show that $\varphi \circ f$ is harmonic.

Let W be an open ball in $\mathbf{R}^p$, let $V = f^{-1}(W)$, and let $x_0 \in V$. Let $U_m = \{x \in U : |x| < m, d(x, \mathbf{R}^n \backslash U) > m^{-1}\}$ and let $V_m = V \cap U_m$. For m large enough, $x_0 \in V_m$. Let τ, τ_m denote the exit times of b from U and U_m respectively and let σ and

a be as in Definition 1.1. Let ψ be the exit time of a from W and let $\theta = \min(\psi, \sigma(\tau))$, $\theta_m = \min(\psi, \sigma(\tau_m))$. ψ and $\sigma(\tau)$ are stopping times for a, hence so is θ; using the fact that φ is harmonic, and bounded convergence, we obtain

$$\varphi \circ f(x_0) = \varphi(a(0))$$

$$= E\,\varphi(a(\theta))$$

$$= \lim_m E\,\varphi \circ f(b(\sigma^{-1}(\theta_m)))$$

$$= \lim_m E\,\varphi \circ f(b(\min(\sigma^{-1}(\psi), \tau_m)))$$

$$= \lim_m g_m(x_0)$$

where g_m is the harmonic extension to V_m of $\varphi \circ f | \partial V_m$, and E denotes expectation (note that $\min(\sigma^{-1}(\psi), \tau_m)$ is the exit time of b from V_m). Then $g_m \longrightarrow \varphi \circ f$ pointwise an V; also $|g_m(x)| \leqslant \sup |\varphi|$ for $x \in V_m$, so f is harmonic on V. Since W is any open ball in $\mathbf{R}^p$, $\varphi \circ f$ is harmonic on U.

The proof of sufficiency is based on the theory of stochastic integrals, for which we refer to [9]. Given f satisfying (i) and (ii) we define σ by

$$\sigma(t) = \int_0^t |\nabla f_i(b(s))|^2 ds, \quad 0 \leqslant t < \tau;$$

by (ii) this is independent of i. This σ satisfies condition (1) of Definition 1.1. Define U_m, τ_m as above and let

$$\sigma_m(t) = \begin{cases} \sigma(t): t \leqslant \tau_m \\ \sigma(\tau_m) + t - \tau_m, t > \tau_m . \end{cases}$$

With b' as in 1.1(2) define a random function $a^{(m)}$, indexed by $\Omega \times \Omega'$, by

$$a^{(m)}(s) = \begin{cases} a(s), & s < \sigma(\tau_m) \\ f(b(\tau_m)) + b'(s - \sigma(\tau_m)), & s \geqslant \sigma(\tau_m), \end{cases}$$

where $a(s)$ is as in 1.1(3). Then a.s. $a^{(m)}$ is continuous on $\{s : s \geqslant 0\}$, and for each s, $a^{(m)}(s) \longrightarrow a(s)$ a.s. If we can show that $a^{(m)}$ is a Brownian motion for each m, it will follow that $a(s)$ is a Brownian motion.

Fix m, and define

$$\widetilde{b}(t) = \begin{cases} b(t), & t < \tau_m \\ b(\tau_m) + b'(t - \tau_m), & t \geqslant \tau_m . \end{cases}$$

Then $\widetilde{b}$ is a Brownian motion by [9, p. 10]. Now let $\gamma_1, \ldots, \gamma_n$ be real numbers with $\Sigma \gamma_j^2 = 1$. Then using Ito's formula [9, section 2.9]

$$\sum_j \gamma_j a_j^{(m)}(\sigma(t)) = \sum_j \gamma_j f_j(b(0)) + \int_0^t \langle e, d\widetilde{b} \rangle$$

where the n-dimensional non-anticipating functional e of $\widetilde{b}$ is defined by

$$e_j(t) = \begin{cases} \sum_i \gamma_i \, \partial f_i / \partial x_j(b(t)), & t < \tau_m \\ \gamma_j & t \geqslant \tau_m . \end{cases}$$

Since $|e(b(t))|^2 = |\nabla f_i(b(t))|^2$ it follows from [9, Section 2.9, problem 1] that $\Sigma \gamma_i a_i^{(m)}$ is a one-dimensional Brownian motion. This is true whatever the choice of γ_i, hence $a_i^{(m)}$ is a p-dimensional Brownian motion, as required.

Remark 1.3. — We point out some special cases. If $p = 1$, the BPP functions are just the harmonic functions. If $p = 2$, we can identify $\mathbf{R}^2$ with $\mathbf{C}$ and regard f as complex-valued; then f is BPP if and only if f is harmonic and $\displaystyle\sum_{i=1}^n (\partial f / \partial x_i)^2 = 0$, or equivalently if f and f^2 are both harmonic (and then all powers of f are harmonic). If $n = p = 2$ the BPP functions are the (complex-) analytic or conjugate-analytic functions. Identifying $\mathbf{C}^m$ with $\mathbf{R}^{2m}$, a complex-analytic function on $U \subseteq \mathbf{C}^m$ defines a BPP function from U to $\mathbf{R}^2$.

The only affine BPP functions from $\mathbf{R}^n$ to $\mathbf{R}^p$ are of the form $f(x) = \lambda A x + b$, where A is a real $p \times n$ matrix with $A A^t = I_p$, $\lambda \geqslant 0$, and $b \in \mathbf{R}^p$. Note also that a composition of BPP functions is BPP; in particular if $f: U \longrightarrow \mathbf{C}$ is BPP and g is analytic on a domain containing $f(U)$ then $g \circ f$ is BPP. However, linear combinations of BPP functions are not in general BPP.

Remark 1.4. – In the case where $U = \mathbf{R}^n$, we have $\tau = \infty$ and one may ask whether $\sigma(\infty)$, which is the explosion time of the image motion, is necessarily infinite a.s. If $n = 2$, the fact that plane Brownian motion returns infinitely often to any neighborhood of the origin implies that $\sigma(\infty) = \infty$ a.s. In general it is not hard to show that either $\sigma(\infty) = \infty$ a.s. or $\sigma(\infty) < \infty$ a.s. But for $n \geqslant 3$ we can construct a harmonic (and hence BPP) function $f : \mathbf{R}^n \longrightarrow \mathbf{R}$ such that $\sigma(\infty) < \infty$ a.s., as follows:

Let $\epsilon(t)$ be a positive continuous function $(t > 0)$ with $\epsilon(t) \longrightarrow 0$ very fast as $t \longrightarrow \infty$. Let $S = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 > 0, |x_k| < \epsilon(x_1), k = 2, \ldots, n\}$. Provided $\epsilon(t) \longrightarrow 0$ fast enough, Brownian motion in $\mathbf{R}^n$ (issued from any given point) $b(t)$ eventually avoids S, in the sense that a.s. there exists T such that $t > T \Longrightarrow b(t) \notin S$. We assert the existence of a harmonic real-valued non-constant function f on $\mathbf{R}^n$ with $|f| < 1$ on $\mathbf{R}^n \backslash S$. Given this, a.s. $\limsup_{t \to \infty} |f(b(t))| \leqslant 1$ so that if $a(s)$ denotes the image Brownian motion then a.s. $\limsup_{s \to \sigma(\infty)} |a(s)| \leqslant 1$, whence $\sigma(\infty) < \infty$ a.s.

It remains to construct f. Choose a sequence $\{y_k\}$ of points in S, with $|y_k| \longrightarrow \infty$, so that for $k \geqslant 2$ there is an open ball B_k centered at y_k, contained in S, and containing y_{k-1}. Let f_1 be harmonic on $\mathbf{R}^n \backslash \{y_1\}$, vanishing at ∞, with $|f_1| < 1$ outside S, but not identically zero. Define f_k inductively so that f_k is harmonic on $\mathbf{R}^n \backslash \{y_k\}$ and $|f_k - f_{k-1}| < \delta \, 2^{-k}$ outside B_k (for example by truncating the expansion of f_{k-1} in spherical harmonics about y_k). Then provided δ is small enough f_k converges to a limit f with the desired properties.

2. BPP functions from domains in $\mathbf{R}^3$ to $\mathbf{C}$.

We have already remarked that a complex-valued function f on a domain in $\mathbf{R}^n$ is BPP if and only if f and f^2 are both harmonic. In general it seems to be hard to describe such functions, but for $n = 3$ a good deal can be said. The reason for this is that BPP functions in this case have the property that their level sets are straight lines. Before proving this we give two examples.

Example 2.1. – Let V be a domain in **C**, let g be an analytic function on V and define f on $V \times \mathbf{R} \subseteq \mathbf{R}^3$ by $f(x, y, z) = g(x + iy)$. Then f is BPP. The level sets of f are lines parallel to the z-axis.

Example 2.2. – The function $f(x, y, z) = (x + iy)/(r - z)$, where $r = (x^2 + y^2 + z^2)^{1/2}$, is BPP on $\mathbf{R}^3 \backslash \{(0, 0, z) : z \geqslant 0\}$. The level sets of f are rays through 0.

THEOREM 2.3. – *Let* U *be a domain in* $\mathbf{R}^3$ *and let* $f : U \longrightarrow \mathbf{C}$ *be a BPP function. Then for each* $p \in U$ *there is a unique straight line* l *through* p *such that* f *is constant on the component of* $l \cap U$ *containing* p.

Proof. – We first assume that p is not a critical point of f (a critical point of f is a point where all first derivatives of f vanish). Using the notation $f_x = \partial f / \partial x$ etc., we have

$$f_x^2 + f_y^2 + f_z^2 = 0 . \tag{1}$$

Differentiating (1) with respect to y and z, multiplying respectively by f_z and f_y, and adding:

$$f_x(f_y f_{xz} + f_z f_{xy}) + f_z f_y(f_{yy} + f_{zz}) + (f_y^2 + f_z^2)f_{yz} = 0 . \tag{2}$$

In view of (1) and the fact that f is harmonic,

$$f_y f_z f_{xx} = f_x(f_x f_{xz} + f_z f_{xy} - f_x f_{yz}) . \tag{3}$$

Now since p is not a critical point we may by rotating and translating the coordinate frame, assume that $p = (0, 0, 0)$ and, at p, $f_x = 0, f_y \neq 0, f_z \neq 0$. Then from (3) we have, in some neighborhood of p, $|f_{xx}| \leqslant M|f_x|$ for some constant $M > 0$. Let $g(t) = f_x(t, 0, 0)$. Then $g(0) = 0$ and for $|t|$ small enough, $|g'(t)| \leqslant M|g(t)|$. Hence $g(t) = 0$ in a neighborhood of 0, so f is constant on a neighborhood of p on the x-axis, and hence, since f is real-analytic, on the component of $U \cap (x$-axis) containing p.

This proves existence of l for a non-critical point p ; in view of (1) the (real) Jacobian of f at p has rank 2 so l is unique, and we denote it by $l(p)$.

The set of all straight lines through the origin can be identified with the real projective plane $\mathbf{RP}^2$; by translation the set of lines

through any given point can also be identified with RP^2. If p is a non-critical point we denote by $\delta(p)$ the point of RP^2 associated in this way with $l(p)$.

We now prove existence of a line of constancy for a critical point p; we denote by K the set of critical points of f in U. If $p \in K$ we can find a sequence $p_n \in U \backslash K$ with $p_n \longrightarrow p$. Since RP^2 is compact, by taking a subsequence we may assume that $\delta(p_n)$ converges to a limit δ. δ corresponds to a line l through p and the continuity of f implies that f is constant on the component of $l \cap U$ containing p.

To prove uniqueness for a critical point we need two lemmas.

LEMMA 2.4. — *If f is BPP on an open ball B and K is the set of critical points of f in B then $B \backslash K$ is connected.*

Proof. — Suppose $B \backslash K$ is disconnected. Then [11, V. 14.3 and V.2] there is a component D of K such that $B \backslash D$ is disconnected. Since D is a connected set of critical points of the smooth function f, f is constant on D [4, p. 38]. Let $x_0 \in B \backslash D$ and consider Brownian motion $b(t)$ issued from x_0. As $B \backslash D$ is disconnected, with positive probability $b(t)$ hits D before leaving B. But then $f(b(t))$ hits the singleton $f(D)$ with positive probability, in contradiction to the BPP property.

LEMMA 2.5. — *Let B be an open ball centered at p and let $\pi : B \backslash \{0\} \longrightarrow RP^2$ be the mapping which sends each line through p to the corresponding point of RP^2. Let E be a closed connected set in RP^2 with more than one point. Then $\pi^{-1}(E)$ has positive capacity (i.e. Brownian motion has positive probability of hitting it).*

Proof. — In R^2 any compact connected set with more than one point has positive capacity. The lemma follows from this using the BPP function of example 2.2.

We now complete the proof of the theorem.

Suppose $p \in K$ and let E be the (closed) set of $\delta \in RP^2$ such that there is a sequence $p_n \in U \backslash K$ with $p_n \longrightarrow p$ and $\delta(p_n) \longrightarrow \delta$. If B is any small open ball centred at p then $q \longrightarrow \delta(q)$ is con-

tinuous on $B\backslash K$, and by lemma 2.4 $B\backslash K$ is connected, so $\delta(B\backslash K)$ is connected. Hence E is connected. As f is constant on the union of the lines through p corresponding to points of E, lemma 2.5 ensures that E is a single point, which we denote by $\delta(p)$, and the corresponding line $l(p)$. This defines δ as a continuous mapping from U into RP^2.

Again let $p \in K$ and suppose there is a line of constancy l through p, corresponding to $\delta_0 \neq \delta(p)$. Then there is an open segment J of l containing p such that $\overline{J} \subseteq U$ and if $q \in J$ then $\delta(q) \neq \delta$. Let F be the set in $\mathbf{R}^3$ formed by the lines $l(q), q \in J$, intersected with a small cylindrical domain with axis J, contained in U. Let G be the projection of F onto a plane normal to l. Since $\{\delta(q): q \in J\}$ is connected, G is either a line segment or contains an open subset of this plane. In the former case F contains an open subset of a plane, and so has positive capacity. In the latter G has positive capacity, as a subset of $\mathbf{R}^3$, hence so does F, by [8, Theorem 2.9]. Since f is constant on F we obtain a contradiction, so $l(p)$ is the only line of constancy through p.

The following partial converse to Theorem 2.3 is useful.

PROPOSITION 2.6. – *Let* U *be a domain in* $\mathbf{R}^3$ *and* f *a non constant twice continuously differentiable complex valued function on* U. *Suppose* f *satisfies*

    (a) *through each point of* U *there is a straight line segment on which* f *is constant, and*

    (b) $f_x^2 + f_y^2 + f_z^2 = 0$ *on* U.

Then f *is harmonic (and hence BPP) on* U.

Proof. – Let p be a non-critical point of f. Rotate the coordinates so that the line of constancy through p is parallel to the x-axis. Then $f_y \neq 0$, $f_z \neq 0$. From (b) we deduce the identity (2) obtained in the proof of 2.3. Since $f_x(p) = 0$ and $f_{xx}(p) = 0$ we deduce that $\nabla^2 f(p) = 0$. Thus $\nabla^2 f = 0$ at the non-critical points of f, and hence, since $\nabla^2 f$ is continuous, throughout U.

We now show how the differential equations for a BPP function can locally be reduced to a system of two-dimensional equations.

Suppose f is BPP on a domain $U \subseteq \mathbf{R}^3$. Fix $p \in U$ and choose orthogonal coordinates so that p is the origin and $l(p)$ is not parallel to the plane $z = 0$. Choose a ball B, centered at p, contained in U, such that, for $q \in B$, $l(q)$ is not parallel to the plane $z = 0$. Let $W = B \cap \{z = 0\}$, which we can regard as a domain in $\mathbf{R}^2$. Define g on W by $g(u,v) = f(u,v,0)$. The line of constancy through $(u,v,0)$ can be parametrised as

$$\{(u + \varphi(u,v)\lambda,\; v + \psi(u,v)\lambda, \lambda) : \lambda \in \mathbf{R}\}\;.$$

Then φ and ψ are continuous real functions on W. For λ small enough

$$f(u + \varphi(u,v)\lambda,\; v + \psi(u,v)\lambda, \lambda) = g(u,v)\;. \tag{1}$$

THEOREM 2.7. – *With the above notations, the real functions φ and ψ and the complex function g on W are real analytic and satisfy the equations*

$$\frac{\varphi_v}{1 + \varphi^2} = -\frac{\psi_u}{1 + \psi^2}\;,\quad \psi_v - \varphi_u = \frac{2\varphi\psi}{1 + \varphi^2}\,\varphi_v \tag{2}$$

and

$$(1 + \varphi^2)g_u^2 + 2\varphi\psi\, g_u g_v + (1 + \psi^2)g_v^2 = 0\,. \tag{3}$$

Conversely, suppose φ and ψ are twice continuously differentiable real functions satisfying (2) on a domain $W \subseteq \mathbf{R}^2$. Then φ and ψ are real analytic, and there exists a solution g of (3) on W. For any such g, (1) defines a BPP function f in an open set containing $W \times \{0\}$ in $\mathbf{R}^3$.

Proof. – Assume first that W contains no critical points of f. Then the real analyticity of φ, ψ and g follows from that of f. Writing $x = u + \varphi(u,v)\lambda$, $y = v + \psi(u,v)\lambda$, $z = \lambda$, differentiating (1) yields

$$\begin{aligned} g_u &= (1 + \lambda\varphi_u)f_x + \lambda\psi_u f_y \\ g_v &= \lambda\varphi_v f_x + (1 + \lambda\psi_v)f_y \end{aligned} \tag{4}$$

$$0 = \varphi f_x + \psi f_y + f_z \tag{5}$$

(5) together with $f_x^2 + f_y^2 + f_z^2 = 0$ gives

$$(1 + \varphi^2)f_x^2 + 2\varphi\psi\, f_x f_y + (1 + \psi^2)f_y^2 = 0$$

and solving (4),

$$\begin{aligned} \Delta f_x &= g_u + \lambda(\psi_v g_u - \psi_u g_v) \\ \Delta f_y &= g_v + \lambda(\varphi_u g_v - \varphi_v g_u) \end{aligned}$$

where

$$\Delta = 1 + \lambda(\varphi_u + \psi_v) + \lambda^2(\varphi_u \psi_v - \varphi_v \psi_u) \neq 0$$

for λ small enough. Also f_x and f_y never vanish on W, so f_x/f_y is a root of the quadratic

$$(1 + \varphi^2)\alpha^2 + 2\varphi\psi\,\alpha + (1 + \psi^2) = 0 \tag{6}$$

for all $(u, v) \in W$ and all small λ. So if we fix u and v we find that

$$\frac{g_u + \lambda(\psi_v g_u - \psi_u g_v)}{g_v + \lambda(\varphi_u g_v - \varphi_v g_u)}$$

is a root of (6) for all small λ, and hence is independent of λ. We deduce (3) and the equation

$$\varphi_v g_u^2 + (\psi_v - \varphi_u)g_u g_v - \psi_u g_v^2 = 0.$$

For this latter equation to be compatible with (3) requires (2). So (2) and (3) hold.

Conversely, given twice continuously differentiable φ and ψ satisfying (2), by the theory of the Beltrami equation [1, Chapter 5], (3) possesses solutions and all solutions are twice continuously differentiable. The above argument can then be reversed to show that f defined by (1) is twice continuously differentiable and satisfies $f_x^2 + f_y^2 + f_z^2 = 0$, so is BPP by Proposition 2.6.

To complete the proof we must remove from the first half of the proof the assumption that f has no critical points in W. Let K be the set of all points of W which are critical points of f. Then all points of K are isolated (this can be seen as follows: first, K is totally disconnected, since if J is a component of K with more than one point, the f is constant on J [4, p. 38] and hence on the union of the lines of constancy through J, which is impossible by an argument similar to the last part of the proof of Theorem 2.3. Then since K is a real analytic set the argument of corollary 3 of [10, p. 100] shows that each point of k is isolated). Let $q \in k$; we wish to prove that the function $\delta : W \longrightarrow RP^2$ is smooth at q. To do this we re-select the coordinate system so that φ and ψ as defined above vanish at q. Let B be an open disc centered at q, contained in W, so that $B \cap K = \{q\}$. Let h be a solution in B of the Beltrami equation

$$(1 + \varphi^2) h_u^2 + 2\varphi\psi\, h_u h_v + (1 + \psi^2) h_v^2 = 0$$

as constructed in [1, Ch. 5, Theorem 1]. By taking B small we can make the suprema of $|\varphi|$ and $|\psi|$ over B arbitrarily small, so that in the notation of [1, Ch. 5] we can arrange that $p > 4$. Then by equation (13) of [1, p. 96]

$$|h(r) - h(q)| \geqslant M\,|r - q|^\alpha \, , \; r \in B, \qquad (8)$$

for some $M > 0$, $\alpha > \dfrac{1}{2}$. Moreover h is homeomorphic on B. Now define $F(u + \varphi\lambda, \, v + \psi\lambda, \lambda) = h(u, v)$. Then since $g = G \circ h$ on $B \backslash \{q\}$, for some G, analytic or conjugate analytic on $h(B \backslash \{q\})$, it follows that F is harmonic on $\tilde{B} \backslash l(q)$ for some open set $\tilde{B}$ in $\mathbf{R}^3$ containing B. Moreover F is continuous on $\tilde{B}$, so F is harmonic on $\tilde{B}$. Finally by (8) q is not a critical point of F, so the desired result follows by applying the first part of the proof to the BPP function F.

Remarks 2.8. — (A) Equation (3) can be interpreted as saying that W admits a conformal structure with respect to which g must be analytic or conjugate-analytic. To state this fact globally, if f is BPP on a domain U in $\mathbf{R}^n$, then the family of lines of constancy of f can be given the structure of a Riemann surface so that f, or any other BPP function with the same level sets, is analytic or conjugate analytic with respect to this structure.

(B) Theorem 2.7 implies that any twice continuously differentiable solutions of (2) are in fact real analytic; this also follows from general results about elliptic equations. If in (2) we write $\varphi = \tan G$, $\psi = -\tan H$ then $G_v = H_u$ so there exists F with $F_u = G$, $F_v = H$ and then F satisfies

$$\cos^2(f_v)\, F_{uu} + \cos^2(F_u)\, F_{vv} - \frac{1}{2}\sin(2F_u)\sin(2F_v)\, F_{uv} = 0,$$

with the constraint that F_u and F_v lie in the interval $\left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. This is a quasi-linear elliptic equation for which the Dirichlet problem can be solved in any smooth strictly convex domain, given sufficiently small smooth boundary values [7, Ch. 6, Th. 42]. Thus there exist many solutions of (2).

(C) Another way to reduce the equations for a BPP function to two variables is to note that any BPP function satisfies

$f_y^2 f_{xx} + f_x^2 f_{yy} - 2f_x f_y f_{xy} = 0$ (this is obtained by differentiating $f_x^2 + f_y^2 + f_z^2 = 0$ with respect to x, y and z and then eliminating all derivatives involving z). Thus the restriction of any BPP function to the plane $z = 0$ satisfies this equation. Conversely, given a real-analytic complex function g on a simply connected domain W in $\mathbf{R}^2$ satisfying $g_y^2 g_{xx} + g_x^2 g_{yy} - 2g_x g_y g_{xy} = 0$, and such that $g_x^2 + g_y^2$ does not vanish, one can use the Cauchy-Kowalewski theorem to extend g to a BPP function on an open set in $\mathbf{R}^3$ containing W (see Prop. 3.1 below). A further application of the Cauchy-Kowalewski theorem shows that, given real-analytic complex functions g and h on an open interval I on the x-axis, with g' and $g'^2 + h^2$ non-vanishing, there exists a BPP function f on an open set in $\mathbf{R}^3$ containing I, with $f = g$ and $f_y = h$ on I. This says essentially that one can find a BPP function with given values on a given line, and given lines of constancy through the points of this line, provided these data are real analytic.

(D) The construction of global solutions raises interesting problems. In particular, we do not know of any BPP functions defined on the whole of $\mathbf{R}^3$, apart from functions whose lines of constancy are all parallel, i.e. functions of the form $f(x, y, z) = g(x + iy)$, where g is an entire function, or obtained from such a function by an orthogonal transformation. One way to try to construct a BPP function on $\mathbf{R}^3$ is to seek a solution of (2) on $\mathbf{R}^2$, such that the associated lines fill up $\mathbf{R}^3$ without meeting each other. It is not hard to show that, given φ and ψ satisfying (2) in a convex domain $W \subseteq \mathbf{R}^2$, the necessary and sufficient condition for the lines never to meet is that φ_v have constant sign. If $W \subseteq \mathbf{R}^2$ and φ_v does have constant sign, then the lines fill $\mathbf{R}^3$ if and only if $[u^2 + v^2 + (u\psi - v\varphi)^2]/(\varphi^2 + \psi^2) \longrightarrow \infty$ as $u^2 + v^2 \longrightarrow \infty$. This is equivalent to saying that $r(\theta + \rho^{-1}) \longrightarrow \infty$ as $r \longrightarrow \infty$, where $r = (u^2 + v^2)^{1/2}$, $\rho = (\varphi^2 + \psi^2)^{1/2}$ and θ is the acute angle between the vector (u, v) and (φ, ψ). Thus, to construct a non-trivial BPP function on $\mathbf{R}^3$, it suffices to construct φ and ψ satisfying (2) on $\mathbf{R}^2$ and in addition $\varphi_v > 0$ on $\mathbf{R}^2$ and $r(\theta + \rho^{-1}) \longrightarrow \infty$ as $r \longrightarrow \infty$. Example 2.9 below shows that φ and ψ can be constructed satisfying all these conditions except the last.

We can easily show that a polynomial in x, y, z which is a BPP function must be of the trivial type. Write $f = \sum\limits_{k=0}^{m} f_k$ where f_k is homogeneous of degree k and f_m is not identically zero. Now suppose u is a vector parallel to some line of constancy of f. Then for some $x \in \mathbf{R}^3$, the function $g(\lambda) = f(x + \lambda u)$ is constant on $\mathbf{R}$, which implies $f_m(u) = 0$. Unless all the lines of constancy are parallel, the set of such vectors u is open, so $f_m = 0$, a contradiction. So all the lines are parallel.

Example 2.9. — A solution of equations (2) on the disc $u^2 + v^2 < 1$ is given by $\varphi(u, v) = -v/s$, $\psi(u, v) = -u/s$ where $s = (1 - u^2 - v^2)^{1/2}$. The equation (3) for g becomes $(1 - u^2)g_u^2 - 2uv\, g_u g_v + (1 - v^2)g_v^2 = 0$, and has one solution $g(u, v) = (u + iv)/(1 + s)$. The associated straight line through $(u, v, 0)$ has equation $x = u - vz/s$, $y = v + uz/s$. These lines are disjoint and fill up the domain

$$U = \mathbf{R}^3 \backslash \{(x, y, 0): x^2 + y^2 \geqslant 1\}.$$

So $f(x, y, z) = g(u, v)$ defines a BPP function on U. Explicitly,

$$f(x, y, z) = (x + iy)/[(1 + s)(1 + iz/s)]$$

where

$$s = [A + (A^2 + z^2)^{1/2}]^{1/2}, A = (1 - x^2 - y^2 - z^2)/2.$$

Note that if $0 < r < 1$ then the union of the lines of constancy passing through the points of the circle $u^2 + v^2 = r^2$ is the hyperboloid of revolution $x^2 + y^2 = r^2[1 + z^2/(1 - r^2)]$.

Note also that by choosing $\lambda \neq 0$ and considering the lines of constancy of f through the points of the plane $z = \lambda$, we obtain a solution of equations (2) on $\mathbf{R}^2$ such that the associated lines never meet, yet fail to fill up $\mathbf{R}^3$.

The idea of the above example was suggested by B. Øksendal and A. Stray.

3. BPP functions from higher dimensions to $\mathbf{C}$.

We have less information about BPP functions on domains in $\mathbf{R}^n$ for $n > 3$. We describe some properties of the level sets and characterize restrictions of BPP functions to affine hypersurfaces.

Let f be a complex-valued BPP function on a domain U in $\mathbf{R}^n$, let $z \in f(U)$, let $E = f^{-1}(z)$ and let $p \in E$. Suppose p is not a critical point of f. Then we can rotate the coordinates so that, near p, $E = \{x : x_1 = g_1(x_3 \ldots x_n), x_2 = g_2(x_3 \ldots x_n)\}$, where g_1 and g_2 are smooth real functions on $\mathbf{R}^{n-2}$ and grad g_1 and grad g_2 vanish at p. Then we have the following result.

THEOREM 3.1. − *With the above assumptions, $\nabla^2 g_1$ and $\nabla^2 g_2$ vanish at p.*

Proof. − We write f_i for $\partial f / \partial x_i$, etc. The choice of coordinates ensures that $f_j(p) = 0$, $j = 3, \ldots, n$. From the equation $\sum_{j=1}^{n} f_j^2 = 0$ we find that, at p, $f_1 f_{11} + f_2 f_{12} = 0$ and $f_1 f_{12} + f_2 f_{22} = 0$, and also $f_1 = \pm i f_2 \neq 0$, whence $f_{11} + f_{22} = 0$, so $\sum_{j=3}^{n} f_{jj} = 0$. Now let $h(x_1 \ldots x_n) = f(g_1, g_2, x_3, \ldots, x_n)$. Then h has the constant value z, so

$$0 = \nabla^2 h(p) = f_1(p) \nabla^2 g_1(p) + f_2(p) \nabla^2 g_2(p) + \sum_{j=3}^{n} f_{jj}(p).$$

The last term is zero and g_1 and g_2 are real so we conclude that $\nabla^2 g_1(p) = 0$, $\nabla^2 g_2(p) = 0$.

The following corollary will be used in section 5.

COROLLARY 3.2. − *A non-empty level set of a BPP function $f : U \longrightarrow \mathbf{C}$ cannot be compact.*

Proof. − Suppose $E = f^{-1}(z)$ is non-empty and compact. First suppose that E contains no critical point of f. Let B be a closed ball such that $E \subseteq B$ and $E \cap \partial B \neq \emptyset$. Then if $p \in E \cap \partial B$, the conclusion of Theorem 3.1 is false at p, a contradiction.

If E contains critical points, let V be an open set containing E so that $\overline{V}$ is a compact subset of U. By Sard's theorem [4, p. 38] the image under f of the critical points of f has Lebesgue measure 0, so we can find $z_1 \in f(V) \backslash f(\partial V)$ so that $f^{-1}(z_1)$ contains no critical point. Then $V \cap f^{-1}(z_1)$ is compact, giving a contradiction.

Theorem 3.1 says that the level sets of f have zero mean curvature vector at each point, and so are minimal surfaces (of codimension 2). See e.g. [5, Section 52].

We next describe a property of the second derivatives of a BPP function. Consider a smooth complex-valued function f on a domain in $\mathbf{R}^n$, satisfying $\sum_{j=1}^{n} f_j^2 = 0$ (1). Let A denote the matrix of second derivatives (f_{ik}). Differentiating (1) twice yields the matrix relation

$$\sum_j f_j \, \partial A/\partial x_j = -A^2 .$$

Hence $\sum_j f_j \, \partial A^k/\partial x_j = -kA^{k+1} \ (k = 1, 2, \ldots)$ and so

$$\sum_j f_j \, \partial (tr \, A^k)/\partial x_j = -k \, tr \, A^{k+1} . \qquad (2)$$

If now f is BPP then $tr(A) = 0$, so by successive applications of (2), $tr(A^k) = 0$ for $k = 1, 2, \ldots$, so A is nilpotent. Thus the matrix of second derivatives of a BPP function at any point is nilpotent.

We apply this to a boundary value problem: given a complex-valued function g, defined on a domain W in $\mathbf{R}^m$ (regarded as a subspace of $\mathbf{R}^{m+1}$), when can one extend g to a BPP function in a domain in $\mathbf{R}^{m+1}$ containing W? Clearly g must be real-analytic; if $m = 1$ this is sufficient but we show that if $m > 1$ g must satisfy a system of partial differential equations. We use $x_1, \ldots, x_m$ for coordinates in $\mathbf{R}^m$ and x_0 for the extra dimension.

If f satisfies (1) in a domain in $\mathbf{R}^{m+1}$ then, writing $D = -\sum_{j=1}^{m} f_j^2$ we have $D^{1/2} f_{0k} = -\sum_{l=1}^{m} f_l f_{kl} \ (i = 1, \ldots, m)$ and $D f_{00} = \sum_{k,l=1}^{m} f_k f_l f_{kl}$. Consequently a necessary condition for g to have a BPP extension is the nilpotency of the $(m + 1) \times (m + 1)$ matrix B given by

$$B = \begin{bmatrix} \alpha & \beta \\ \beta^t & c \end{bmatrix}$$

where $\alpha = \sum_{k,l} g_k g_l g_{kl}$, $\beta = -D^{1/2} \sum_l g_l g_{kl}$ and $c_{kl} = D g_{kl}$

$\left(D = - \sum_k g_k^2 \right)$. Note that this condition is independant of the choice of sign of $D^{1/2}$.

We now prove a converse.

PROPOSITION 3.1. – *Let* g *be a real-analytic complex function on a simply – connected domain* W *in* $\mathbf{R}^m$, *and suppose* $\sum g_k^2$ *is never zero. Suppose the matrix* B *defined above is nilpotent at each point of* W. *Then there is a BPP function* f *on a domain* U *in* $\mathbf{R}^{m+1}$ *containing* W, *with* $f = g$ *on* W.

Proof. – Since W is simply connected, $-\sum g_k^2$ has a real-analytic square root on W, so by the Cauchy-Kowalewski theorem [12, Section 1.2] we can find f, real-analytic on a domain U in $\mathbf{R}^{m+1}$ containing w, with $f = g$ on W and $\sum_{k=0}^{m} f_k^2 = 0$ on U. Let A be the $(m + 1) \times (m + 1)$ matrix-valued function $[f_{jk}]$ on U. Then at each point of W, A is nilpotent, so $A^{m+1} = 0$.

The matrix A is singular since $\sum_{k=0}^{m} f_{jk} f_k = 0$ so its character-istic function can be written $\det(\lambda I - A) = \lambda^{m+1} + \sum_{r=1}^{m} \psi_r \lambda^r$ where $\psi_1, \ldots, \psi_m$ are real-analytic functions on U. Then

$$A^{m+1} + \sum \psi_r A^r = 0$$

and using equation (2) above we find that the functions $tr(A^k)$, $k = 1, \ldots, m$, satisfy the equations:

$$f_0 \frac{\partial}{\partial x_0} (tr\ A^k) = - \sum_{r=1}^{m} f_r \frac{\partial}{\partial x_r} (tr\ A^k) - k\ tr(A^{k+1}),$$
$$k = 1, \ldots, m - 1$$

$$f_0 \frac{\partial}{\partial x_0} (tr\ A^m) = - \sum_{r=1}^{m} f_r \frac{\partial}{\partial x_r} (tr\ A^m) + k \sum_{r=1}^{m} \psi_r\ tr\ (A^r) .$$

Moreover $tr\ A^k = 0$ on W so by the uniqueness part of the Cauchy-Kowalewski theorem it follows that $tr(A^k) = 0$, $k = 1, \ldots, m$. In particular $tr(A) = 0$ so f is harmonic, and hence BPP, on U.

We remark that for a given g there are exactly two possible extensions f, corresponding to the two choices of $(-\sum g_k^2)^{1/2}$. One extension is a reflection of the other in the subspace $\mathbf{R}^n$.

In order that B be nilpotent it suffices that $tr(B^k) = 0$ for $k = 1, \ldots, m - 1$. The equation $tr(B) = 0$ can be written

$$(\Sigma \, g_k^2) \, \nabla^2 g = \Sigma \, g_k g_l g_{kl} \, . \tag{3}$$

If $m = 2$, (3) suffices for nilpotency of B but if $m > 2$ the other, highly non-linear equations must be satisfied. The investigation of these equations poses formidable problems. We note merely that if g is constant on an affine subspace of codimension *one* through each point then they are satisfied, and that if g is real-valued this condition is also necessary.

We remark also that if g is real-valued then (3) is equivalent to the statement that its level sets are minimal surfaces. Equation (3) for real g arises from the problem of minimising $\int_U |\mathrm{grad}\, g|$ with given boundary values on ∂U. See e.g. [2].

4. BPP functions with range dimension greater than 2.

We have little information about BPP functions in general. Fuglede [5, Section 10] proved, in a more general context, that BPP functions are open mappings. We show that between spaces of the same dimension $(n > 2)$ the only BPP functions are the affine functions of the form $f(x) = \lambda Ax + b$ where $\lambda > 0$, A is a $p \times n$ matrix with $AA^t = I$, and $b \in \mathbf{R}^p$. This result can be deduced from a theorem of Liouville [3, 20.9 Problem 11] stating that the only conformal mappings of a domain in $\mathbf{R}^n$ into $\mathbf{R}^n (n > 2)$ are compositions of translations, rotations, dilations and inversions. A more general result for "BPP" functions on Riemannian manifolds is proved by Fuglede [6, Section 8]. We give an elementary proof avoiding the machinery of Riemannian geometry.

THEOREM 4.1. – *Let* U *be a domain in* $\mathbf{R}^n (n > 2)$ *and let* $f : U \longrightarrow \mathbf{R}^n$ *be BPP. Then* f *is affine.*

Proof. – Since f is BPP there is a function h on U so that

$$\sum_{i=1}^n \frac{\partial f_j}{\partial x_i} \frac{\partial f_k}{\partial x_i} = h \, \delta_{jk}$$

(here $f_1, \ldots, f_n$ are the components of f).

Hence

$$\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k} = h \, \delta_{jk} \cdot \tag{1}$$

Thus, for each j,

$$\sum_{i=1}^{n} \frac{\partial f_i^2}{\partial x_j} = h$$

so

$$2 \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial^2 f_i}{\partial x_j^2} = \frac{\partial h}{\partial x_j} \tag{2}$$

and

$$\frac{\partial h}{\partial x_j} = 2 \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_k} \frac{\partial^2 f_i}{\partial x_j \partial x_k} \cdot \tag{3}$$

Also if $j \neq k$, then by (1) $\sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial f_i}{\partial x_k} = 0$ so

$$2 \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \frac{\partial^2 f_i}{\partial x_k^2} = - 2 \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_k} \frac{\partial^2 f_i}{\partial x_j \partial x_k} = - \frac{\partial h}{\partial x_j},$$

using (3).

Combining this with (2) yields

$$2 \sum_{i=1}^{n} \frac{\partial f_i}{\partial x_j} \nabla^2 f_i = (2 - n) \frac{\partial h}{\partial x_j} \cdot$$

Since $\nabla^2 f_i = 0$, $\partial h / \partial x_j = 0$ for $j = 1, \ldots, n$, so h is constant. Now, for each j, $\sum_{i=1}^{n} (\partial f_j / \partial x_i)^2 = h$. The proof is completed by the following lemma.

LEMMA 4.2. — *Suppose $g_1, \ldots, g_n$ are harmonic real valued functions on a domain U and satisfy $\Sigma \, g_i^2 = h$, a constant. Then $g_1, \ldots, g_n$ are each constant on U.*

Proof. — Fix $x_0 \in U$ and let $G(x) = \Sigma \, g_i(x_0) g_i(x)$, $x \in U$. Then G is harmonic on U and by the Cauchy-Schwarz inequality $G(x) \leqslant h$. But $G(x_0) = h$, so by the maximum principle G is constant, so $G(x) = h$, $x \in U$. But then equality is attained in the Cauchy-Schwarz inequality, so

$$g_i(x) = g_i(x_0), \quad x \in U, \quad i = 1, \ldots, n.$$

An interesting class of BPP functions can be constructed as follows. Define $f\colon \mathbf{C} \times \mathbf{C} \longrightarrow \mathbf{R} \times \mathbf{C}$ by $f(z, w) = (|z|^2 - |w|^2, 2zw)$. Identifying $\mathbf{C}$ with $\mathbf{R}^2$, f is a BPP function from $\mathbf{R}^4$ to $\mathbf{R}^3$. Replacing $\mathbf{C}$ by the quaternions and Cayley numbers we obtain BPP functions from $\mathbf{R}^8 \longrightarrow \mathbf{R}^5$ and $\mathbf{R}^{16} \longrightarrow \mathbf{R}^9$ respectively. These functions satisfy $|f(x)| = |x|^2$ so they map the unit sphere to the unit sphere. Moreover their level sets are compact, in contrast to corollary 3.2. One may ask whether any other pairs (n, p) with $n > p > 2$ exist for which BPP functions from $\mathbf{R}^n$ to $\mathbf{R}^p$ can have compact level sets.

5. Stochastic inner functions.

Let U and V be domains in $\mathbf{R}^n$ and $\mathbf{R}^p$ respectively, and let $f\colon \mathrm{U} \longrightarrow \mathrm{V}$ be BPP. Let $x_0 \in \mathrm{U}$ and let $b(t)$ be a Brownian motion issued from x_0 and stopped at its exit time τ from U. We say that f is a stochastic inner function from U to V if, for every compact $\mathrm{K} \subseteq \mathrm{V}$, almost surely these exists σ, with $0 < \sigma < \tau$ such that, for $\sigma < t < \tau$, $f(b(t)) \notin \mathrm{K}$. It is not hard to show that this definition is independent of the choice of x_0.

If U and V are bounded domains with smooth boundary, this is equivalent to requiring that the boundary values of f (which exists a.e. on $\partial \mathrm{U}$) are a.e. in $\partial \mathrm{V}$. Thus if U and V each coincide with the unit disc in the plane, then the stochastic inner functions are just the (non-constant) inner functions in the usual sense, together with their complex conjugates.

A special case of interest arises when U and V are respectively the open unit balls in $\mathbf{R}^n$ and $\mathbf{R}^p$. Then one may ask for which values of n and p stochastic inner functions exist. The examples at the end of section 4 show that they exist when $(n, p) = (4,3)$, $(8,5)$ or $(16,9)$. It is possible that these are the only pairs with $n > p \geqslant 2$ for which stochastic inner functions exist. Any non-constant inner function, in the usual sense, on the unit ball in $\mathbf{C}^m$, is a stochastic inner function from the ball in $\mathbf{R}^{2m}$ to the disc. Thus a proof of the non-existence of stochastic inner functions from the ball in $\mathbf{R}^n (n > 2)$ to the disc would solve the well-known problem on the existence of non-constant inner functions on the unit ball in $\mathbf{C}^m (m > 1)$. However, the only case we are able to solve is $n = 3$.

THEOREM 5.1. — *Let* U *be a bounded domain in* $\mathbf{R}^3$ *and* V *a domain in* $\mathbf{C}$. *Then there are no stochastic inner functions* $f: \mathrm{U} \longrightarrow \mathrm{V}$.

Proof. — Suppose such an f exists. Choose a non-critical point $p \in \mathrm{U}$ of f. Choose coordinates and define φ and ψ as in Theorem 2.7. We may suppose $\varphi_v(p) \neq 0$, except in the case where all the lines are parallel or with a common point. Choose a small open disc W about p in the (u, v) plane, such that $\varphi_v \neq 0$ in W. Define

$\mathrm{T}: \mathrm{W} \times \mathbf{R} \longrightarrow \mathbf{R}^3$ by $\mathrm{T}(u, v, \lambda) = (u + \varphi(u, v)\lambda, v + \psi(u, v)\lambda, \lambda)$.

Then T has Jacobian $1 + \lambda(\varphi_u + \psi_v) + \lambda^2(\varphi_u \psi_v - \varphi_v \psi_u)$, which since $\varphi_v \neq 0$ and by equations 3.7(2), does not vanish on $\mathrm{W} \times \mathbf{R}$. This is true also in the case when φ, ψ are constant. T is moreover $(1 - 1)$ on $\mathrm{W} \times \mathbf{R}$. Let $\mathrm{Q} = \mathrm{T}(\mathrm{W} \times \mathbf{R})$, an open set, and let Q_0 be the component of $\mathrm{U} \cap \mathrm{Q}$ containing p. The function h defined on Q by $h(\mathrm{T}(u, v, \lambda)) = f(u, v, 0)$ is real-analytic on Q and coincides with f near p, so $f = h$ on Q_0. Hence $f(\mathrm{Q}_0) = f(\mathrm{W})$.

Now let $b(t)$ be Brownian motion issued from x_0, and let τ be the exit time of b from U. Then with positive probability $b(t) \in \mathrm{Q}$ for $0 < t < \tau$, which is a contradiction since $\overline{f(\mathrm{W})}$ is a compact subset of V.

If we assume continuity of boundary values we can prove non-existence in higher dimensions.

PROPOSITION 5.2. — *Let* U *be a bounded domain in* $\mathbf{R}^n (n > 2)$ *and* V *a domain in* $\mathbf{C}$. *Then there is no continuous* $f: \overline{\mathrm{U}} \longrightarrow \overline{\mathrm{V}}$ *such that* $f(\mathrm{U}) \subseteq \mathrm{V}$, $f(\partial \mathrm{U}) \subseteq \partial \mathrm{V}$, *and* f *is BPP on* U.

Proof. — Suppose such an f exists. Let $z \in f(\mathrm{U})$. Then $f^{-1}(z)$ is a compact subset of U, in contradiction to Corollary 3.2.

BIBLIOGRAPHY

[1] L.V. AHLFORS, Lectures on Quasi-Conformal Mappings, Van Norstrand, 1966.

[2] E. BOMBIERI, E. DE GIORGI and E. GIUSTI, Minimal cones and the Bernstein problem, *Inventiones Math.*, 7 (1969), 243-268.

[3] J. DIEUDONNE, Eléments d'Analyse, Gauthiers-Villars, 1971, Vol. 4. (English translation: Treatise on Analysis, Academic Press, 1974).

[4] J. EELLS, Singularities of Smooth Maps, Nelson, 1967.

[5] L.P. EISENHART, Riemannian Geometry, Princeton, 1949.

[6] B. FUGLEDE, Harmonic morphisms between Riemannian manifolds, Preprint, Copenhagen University, 1976.

[7] O.A. LADYZHENSKAYA and N.N. URAL'TSEVA, Linear and Quasilinear Elliptic Equations, Nauka Press, Moscow 1964, English translation Academic Press, 1968.

[8] N.S. LANDKOF, Foundations of Modern Potential Theory, Springer-Verlag, 1972.

[9] H.P. McKEAN, Stochastic Integrals, Academic Press, 1969.

[10] R. NARASIMHAN, Introduction to the Theory of analytic Spaces, *Lecture Notes in Mathematics*, No. 25, Springer-Verlag, 1966.

[11] M.H.A. NEWMAN, Topology of Plane Sets of Points, Cambridge University Press, 2nd. Edition, 1952.

[12] I.G. PETROVSKY, Lectures on Partial Differential Equations, Interscience, 1954.

A. BERNARD,
Université de Grenoble I
Laboratoire de Mathématiques Pures
associé au CNRS n° 188
Institut Fourier
B.P. 116 38402 Saint-Martin d'Hères.

E.A. CAMPBELL,
Department of Mathematics
University of Edinburgh.

A.M. DAVIE,
Department of Mathematics
University of California at Los Angeles
California 90024.