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ON CERTAIN BARRELLED NORMED SPACES

by Manuel VALDIVIA

Let $\mathscr{A}$ be a σ-algebra on a set X. If A belongs to $\mathscr{A}$ let $e(A)$ be the function defined on X taking value 1 in every point of A and vanishing in every point of $X \sim A$. Let $\ell_0^\infty(X, \mathscr{A})$ be the linear space over the field K of real or complex numbers generated by $\{e(A) : A \in \mathscr{A}\}$ endowed with the topology of the uniform convergence. We shall prove that if (E_n) is an increasing sequence of subspaces of $\ell_0^\infty(X, \mathscr{A})$ covering $\ell_0^\infty(X, \mathscr{A})$ there is a positive integer p such that E_p is a dense barrelled subspace of $\ell_0^\infty(X, \mathscr{A})$, and we shall deduce some new results in measure theory from this fact.

1. The space $\ell_0^\infty(X, \mathscr{A})$.

If $z \in \ell_0^\infty(X, \mathscr{A})$ and if $z(j)$ denotes its value in the point j of X we define the norm of z in the following way :

$$\|z\| = \sup \{|z(j)| : j \in X\} .$$

Given a member A of $\mathscr{A}$ we denote by $\ell_0^\infty(A, \mathscr{A})$ the subspace of $\ell_0^\infty(X, \mathscr{A})$ generated by $\{e(M) : M \in \mathscr{A}, M \subset A\}$. We write $(\ell_0^\infty(X, \mathscr{A}))'$ to denote the Banach space conjugate of $\ell_0^\infty(X, \mathscr{A})$. If $u \in (\ell_0^\infty(X, \mathscr{A}))'$, $u(A)$ stands for restriction of u to $\ell_0^\infty(A, \mathscr{A})$. The norm of $u(A)$ is denoted by $\|u(A)\|$ and the value of u at the point z is written $\langle u, z \rangle$. If $A_1, A_2, \ldots, A_n$ are disjoint members of $\mathscr{A}$ and contained in A then

$$\sum_{p=1}^{n} \|u(A_p)\| \leqslant \|u(A)\| \tag{1}$$

since if every any $\epsilon > 0$ we take z_p in $\ell_0^\infty(A_p, \mathscr{A})$ with

$$\|z_p\| \leqslant 1, \langle u, z_p\rangle \geqslant \|u(A_p)\| - \frac{\epsilon}{n}, \quad p = 1, 2, \dots, n.$$

Then $z = \sum_{p=1}^{n} z_p$ has norm less than or equal to 1, belongs to $\ell_0^\infty(A, \mathscr{A})$ and

$$\|u(A)\| \geqslant |\langle u, z\rangle| \geqslant \sum_{p=1}^{n} \|u(A_p)\| - \epsilon$$

and (1) follows.

PROPOSITION 1. – *Let* B *be the closed unit ball of real* $\ell_0^\infty(X, \mathscr{A})$. *Then the absolutely convex hull of* $\{e(A) : A \in \mathscr{A}\}$ *contains* $\frac{1}{2}B$.

Proof. – If $z \in \frac{1}{2}B$ and if z takes exactly two non vanishing values, we obtain $A_1, A_2, A_3 \in \mathscr{A}$, $A_i \cap A_j = \emptyset$, $i \neq j$, $i, j = 1, 2, 3$ such that $A_1 \cup A_2 \cup A_3 = X$ and such that

$$z(j) = \alpha,\ j \in A_1 ;\ z(j) = \beta,\ j \in A_2 ;\ z(j) = 0,\ j \in A_3 .$$

Then $|\alpha| \leqslant \frac{1}{2}$, $|\beta| \leqslant \frac{1}{2}$ and

$$z = \alpha e(A_1) + \beta e(A_2)$$

and therefore z belongs to the absolutely convex hull of $\{e(A) : A \in \mathscr{A}\}$.

By recurrence we suppose that for a $p \geqslant 2$ every vector of $\frac{1}{2}B$ taking exactly p non vanishing values belongs to the absolutely convex hull of $\{e(A) : A \in \mathscr{A}\}$. If $z \in \frac{1}{2}B$ taking $p+1$ non vanishing values we descompose X in $A_1, A_2, \dots, A_{p+2}$ members of $\mathscr{A}$ such that z takes the value α_j in A_j, $j = 1, 2, \dots, p+1$ and zero in A_{p+2}. Since p is larger than or equal to 2, z takes two different values of the same sign. We can suppose that $0 < \alpha_1 < \alpha_2$ or $\alpha_2 < \alpha_1 < 0$. If $0 < \alpha_1 < \alpha_2$ we consider the vectors z_1 and z_2 which coincide with z in $A_2 \cup A_3 \cup \dots \cup A_{p+2}$ such that z_1 takes the value α_2 in A_1 and z_2 takes the value zero in A_1. Then z_1 and z_2 take p non zero different values and since $z_1, z_2 \in \frac{1}{2}B$ they belong to the absolutely convex hull M of $\{e(A) : A \in \mathscr{A}\}$. Since $0 < \frac{\alpha_1}{\alpha_2} < 1$ then

$$\frac{\alpha_1}{\alpha_2} z_1 + \left(1 - \frac{\alpha_1}{\alpha_2}\right) z_2 = z$$

belongs to M. If $\alpha_2 < \alpha_1 < 0$ then $0 < -\alpha_1 < -\alpha_2$ and so $-z \in \mathrm{M}$ and therefore $z \in \mathrm{M}$.

q.e.d.

PROPOSITION 2. – *Let* B *be the closed unit ball of complex* $\ell_0^\infty(\mathrm{X}, \mathscr{A})$. *Then the absolutely convex hull of* $\{e(\mathrm{A}) : \mathrm{A} \in \mathscr{A}\}$ *contains* $\frac{1}{4}\mathrm{B}$.

Proof. – If $z \in \frac{1}{4}\mathrm{B}$ we write

$$z = z_1 + i z_2$$

where z_1, z_2 are real vectors of $\frac{1}{4}\mathrm{B}$. According to Proposition 1 the vectors $2z_1$ and $2z_2$ belong to the absolutely convex hull of $\{e(\mathrm{A}) : \mathrm{A} \in \mathscr{A}\}$. Then

$$z = \frac{1}{2}(2z_1) + \frac{i}{2}(2z_2)$$

belongs also to the absolutely convex hull of $\{e(\mathrm{A}) : \mathrm{A} \in \mathscr{A}\}$.

q.e.d.

Note 1. – If $\mathrm{A} \in \mathscr{A}$ let $\mathscr{B} = \{\mathrm{A} \cap \mathrm{B} : \mathrm{B} \in \mathscr{A}\}$. Then $\mathscr{B}$ is a σ-algebra and we can suppose that $\ell_0^\infty(\mathrm{A}, \mathscr{B})$ coincides with $\ell_0^\infty(\mathrm{A}, \mathscr{A})$. Given an absolutely convex set T of $\ell_0^\infty(\mathrm{A}, \mathscr{A})$ which is not a neighbourhood of the origin and given any positive real number λ we can apply Proposition 1 or Proposition 2 to $\ell_0^\infty(\mathrm{A}, \mathscr{A}) = \ell_0^\infty(\mathrm{A}, \mathscr{B})$ to obtain a member A_1 of $\mathscr{A}$ contained in A so that $\lambda e(\mathrm{A}_1) \notin \mathrm{T}$.

Given a closed absolutely convex set U of $\ell_0^\infty(\mathrm{X}, \mathscr{A})$ we say that the member $\mathrm{A} \in \mathscr{A}$ has property U if there is a finite set Q in $\ell_0^\infty(\mathrm{X}, \mathscr{A})$ such that if V is the absolutely convex hull of $\mathrm{U} \cup \mathrm{Q}$ then $\mathrm{V} \cap \ell_0^\infty(\mathrm{A}, \mathscr{A})$ is a neighbourhood of the origin in $\ell_0^\infty(\mathrm{A}, \mathscr{A})$. Obviously, if A has property U, $\mathrm{B} \subset \mathrm{A}$, $\mathrm{B} \in \mathscr{A}$, then B also has property U.

PROPOSITION 3. – *If* $\mathrm{A} \in \mathscr{A}$ *does not possess property* U *and if* $\mathrm{A}_1, \mathrm{A}_2, \ldots, \mathrm{A}_n$ *are elements of* $\mathscr{A}$ *which are a partition of* A *there is an integer* q, $1 \leqslant q \leqslant n$, *such that* A_q *does not have property* U.

Proof. – We suppose that A_p, $p = 1, 2, \ldots, n$, have property U. There is a finite set Q_p in $\ell_0^\infty(X, \mathscr{A})$ such that if U_p is the absolutely convex hull of $U \cup Q_p$ then $V_p = U_p \cap \ell_0^\infty(A_p, \mathscr{A})$ is a neighbourhood of the origin in $\ell_0^\infty(A_p, \mathscr{A})$. Let V be the absolutely convex hull of $U \cup \left(\bigcup_{p=1}^{n} Q_p\right)$. Since A does not have property U, $V \cap \ell_0^\infty(A, \mathscr{A})$ is not a neighbourhood of the origin in $\ell_0^\infty(A, \mathscr{A})$. Since $\ell_0^\infty(A, \mathscr{A})$ is the topological direct sum of $\ell_0^\infty(A_1, \mathscr{A})$, $\ell_0^\infty(A_2, \mathscr{A}), \ldots, \ell_0^\infty(A_n, \mathscr{A})$, the absolutely convex hull W of $\bigcup_{p=1}^{n} V_p$ is a neighbourhood of the origin in $\ell_0^\infty(A, \mathscr{A})$. On the other hand, W is contained in V and we arrive at a contradiction.

q.e.d.

PROPOSITION 4. – *Suppose that $A \in \mathscr{A}$ does not have property* U. *Given a positive integer $p \geqslant 2$, the elements $x_1, x_2, \ldots, x_n$ of $\ell_0^\infty(X, \mathscr{A})$ and a positive real number α, there are p elements $A_1, A_2, \ldots, A_p$ of $\mathscr{A}$, which are a partition of* A, *and p vectors $u_1, u_2, \ldots, u_p$ in $(\ell_0^\infty(X, \mathscr{A}))'$ such that, if $i = 1, 2, \ldots, p$,*

$$|\langle u_i, e(A_i)\rangle| > \alpha, \quad \sum_{j=1}^{n} |\langle u_i, x_j\rangle| \leqslant 1, \quad |\langle u_i, x\rangle| \leqslant 1, \quad \forall x \in U.$$

Proof. – Let Q be the absolutely convex hull of

$$\{e(A), nx_1, nx_2, \ldots, nx_n\}.$$

Since Q is compact, $V = U + Q$ is a closed absolutely convex set of $\ell_0^\infty(X, \mathscr{A})$. Since A does not have property U, $V \cap \ell_0^\infty(A, \mathscr{A})$ is not a neighbourhood of the origin in $\ell_0^\infty(A, \mathscr{A})$ and therefore, according to Note 1, we can choose a subset P_{11} in A, $P_{11} \in \mathscr{A}$, such that

$$\frac{1}{1+\alpha} e(P_{11}) \notin V.$$

If V° denotes the polar set of V in $(\ell_0^\infty(X, \mathscr{A}))'$ we can find an element $u_1 \in V^\circ$ such that

$$\left|\langle u_1, \frac{1}{1+\alpha} e(P_{11})\rangle\right| > 1$$

and therefore

$$|\langle u_1, e(P_{11})\rangle| > 1 + \alpha > \alpha.$$

On the other hand, if $P_{12} = A \sim P_{11}$ we have

$$\langle u_1 e(P_{11})\rangle = \langle u_1, e(A))\rangle - \langle u_1, e(P_{12})\rangle$$

thus

$$|\langle u_1, e(P_{11})\rangle| \leqslant |\langle u_1, e(A)\rangle| + |\langle u_1, e(P_{12})\rangle|$$

and so

$$|\langle u_1, e(P_{12})\rangle| \geqslant |\langle u_1, e(P_{11})\rangle| - |\langle u_1, e(A)\rangle| > 1 + \alpha - 1 = \alpha .$$

According to Proposition 3, P_{11} or P_{12} does not have property U. We suppose that P_{12} does not have property U and we set $A_1 = P_{11}$. We have that

$$|\langle u_1, e(A_1)\rangle| > \alpha \quad , \quad |\langle u_1, x\rangle| \leqslant 1 \ , \quad \forall x \in U ,$$

$$\sum_{j=1}^{n} |\langle u_1, x_j\rangle| = \sum_{j=1}^{n} \frac{1}{n} |\langle u_1, nx_j\rangle| \leqslant \sum_{j=1}^{n} \frac{1}{n} = 1 .$$

(The same result is obtained if P_{11} does not have property U and we set $A_1 = P_{12}$).

We apply the same method substituting P_{12} for A to obtain a division of P_{12} into two subsets A_2 and P_{22} belonging to $\mathscr{A}$ and an element $u_2 \in (\ell_0^\infty(X, \mathscr{A}))'$ so that

$$|\langle u_2, e(A_2)\rangle| > \alpha \quad , \quad |\langle u_2, x\rangle| \leqslant 1 \quad , \quad \forall x \in U ,$$

$$\sum_{j=1}^{n} |\langle u_2, x_j\rangle| \leqslant 1$$

so that P_{22} does not have property U.

Following the same way we obtain a partition A_{p-1}, $P_{(p-1)2}$ of $P_{(p-2)2}$ and an element $u_{p-1} \in (\ell_0^\infty(X, \mathscr{A}))'$ such that

$$|\langle u_{p-1}, e(A_{p-1})\rangle| > \alpha \quad , \quad |\langle u_{p-1}, e(P_{(p-1)2})\rangle| > \alpha ,$$

$$|\langle u_{p-1}, x\rangle| \leqslant 1 \quad , \quad \forall x \in U \quad , \quad \sum_{j=1}^{n} |\langle u_{p-1}, x_j\rangle| \leqslant 1 .$$

Setting $u_{p-1} = u_p$, $P_{(p-1)2} = A_p$ the conclusion follows.

q.e.d.

Now we consider a sequence (U_n) of closed absolutely convex subsets of $\ell_0^\infty(X, \mathscr{A})$ such that the member $A \in \mathscr{A}$ does not have property U_n for $n = n_1, n_2, \ldots, n_p$ and for an infinity of values of n.

PROPOSITION 5. – *Given a positive real number* α *and the vectors* $x_1, x_2, \ldots, x_r$ *in* $\ell_0^\infty(X, \mathscr{A})$ *there are* p *pairwise disjoint subsets* $M_1, M_2, \ldots, M_p$ *in* A, *belonging to* $\mathscr{A}$ *and* p *elements* $u_1, u_2, \ldots, u_p$ *in* $(\ell_0^\infty(X, \mathscr{A}))'$ *so that, for every* $i = 1, 2, \ldots, p$,

$$|\langle u_i, e(M_i)\rangle| > \alpha \quad , \quad \sum_{j=1}^{r} |\langle u_i, x_j\rangle| \leqslant 1 \quad , \quad |\langle u_i, x\rangle| \leqslant 1 \, , \quad \forall x \in U_{n_i}$$

and $A \sim \bigcup_{i=1}^{p} M_i$ *does not have property* U_n *for* $n = n_1, n_2, \ldots, n_p$ *and for an infinity of values of* n.

Proof. – According to Proposition 4 we can find a partition $Q_1, Q_2, \ldots, Q_{p+2} \in \mathscr{A}$ of A and $v_1, v_2, \ldots, v_{p+2}$ in $(\ell_0^\infty(X, \mathscr{A}))'$ such that, for $i = 1, 2, \ldots, p+2$,

$$|\langle v_i, e(Q_i)\rangle| > \alpha \quad , \quad \sum_{j=1}^{r} |\langle v_i, x_j\rangle| \leqslant 1 \quad , \quad |\langle v_i, x\rangle| \leqslant 1 \quad , \quad \forall x \in U_{n_1}.$$

It is obvious that, for an infinity of values of n, some of the sets

$$Q_1, Q_2, \ldots, Q_{p+2} \tag{2}$$

do not have property U_n. We suppose that Q_1 does not have property U_n for an infinity of values of n. On the other hand, given a positive integer q, $1 \leqslant q \leqslant p$, some of the sets (2) do not have property U_{n_q}. Since in (2) are $p+2$ elements we can find an element Q_h, $1 < h \leqslant p+2$, such that $A \sim Q_h$ does not have property U_n for $n = n_1, n_2, \ldots, n_p$. Obviously $A \sim Q_h$ contains Q_1 and therefore does not have property U_n for an infinity of values of n. We set $M_1 = Q_h$, $u_1 = v_h$, and then

$$|\langle u_1, e(M_1)\rangle| > \alpha \quad , \quad \sum_{j=1}^{r} |\langle u_1, x_j\rangle| \leqslant 1 \quad , \quad |\langle u_1, x\rangle| \leqslant 1 \quad , \quad \forall x \in U_{n_1} .$$

By recurrence we suppose that we already obtained elements $u_i \in (\ell_0^\infty(X, \mathscr{A}))'$, $i = 1, 2, \ldots, s < p$, and pairwise disjoint subsets $M_1, M_2, \ldots, M_s \in \mathscr{A}$ such that, for $i = 1, 2, \ldots, s$,

$$|\langle u_i, e(M_i)\rangle| > \alpha \quad , \quad \sum_{j=1}^{r} |\langle u_i, x_j\rangle| \leqslant 1 \quad , \quad |\langle u_i, x\rangle| \leqslant 1 \, , \quad \forall x \in U_{n_i}$$

and $A \sim \bigcup_{j=1}^{s} M_j$ does not have property U_n for $n = n_1, n_2, \ldots, n_p$ and for an infinity of values of n. Since $A \sim \bigcup_{j=1}^{s} M_j$ does not have property $U_{n_{s+1}}$, we apply Proposition 4 to obtain a partition $R_1, R_2, \ldots, R_{p+2}$ of $A \sim \bigcup_{j=1}^{s} M_j$, by members of $\mathscr{A}$, and elements $w_1, w_2, \ldots, w_{p+2}$ in $(\ell_0^\infty(X, \mathscr{A}))'$ so that, for $i = 1, 2, \ldots, p+2$,

$$|\langle w_i, e(R_i)\rangle| > \alpha \;, \quad \sum_{j=1}^{r} |\langle w_i, x_j\rangle| \leqslant 1 \;, \quad |\langle w_i, x\rangle| \leqslant 1 \,, \quad \forall x \in U_{n_{s+1}} \,.$$

Then some of the subsets

$$R_1, R_2, \ldots, R_{p+2} \tag{3}$$

do not have property U_n for an infinity of values of n. We suppose that R_1 does not have property U_n for an infinity of values of n. As we did before we find an element R_k, $1 < k \leqslant p+2$, such that $\left(A \sim \bigcup_{j=1}^{s} M_j\right) \sim R_k$ does not have property U_{n_i}, $i = 1, 2, \ldots, p$. We set $M_{s+1} = R_k$, $u_{s+1} = w_k$. Then, for $i = 1, 2, \ldots, s+1$,

$$|\langle u_i, e(M_i)\rangle| > \alpha \;, \quad \sum_{j=1}^{r} |\langle u_i, x_j\rangle| \leqslant 1 \;, \quad |\langle u_i, x\rangle| \leqslant 1 \,, \quad \forall\, x \in U_{n_i} \;,$$

and $A \sim \bigcup_{j=1}^{s+1} M_j$ does not have property U_n for $n = n_1, n_2, \ldots, n_p$ and for an infinity of values of n.

q.e.d.

Now we consider a sequence (U_n) of closed absolutely convex subsets of $\ell_0^\infty(X, \mathscr{A})$ such that X does not property U_n for $n = 1, 2, \ldots$

PROPOSITION 6. – *There are :* (i) *a family* $\{A_{ij} : i, j = 1, 2, \ldots\}$ *of pairwise disjoint members of* $\mathscr{A}$, (ii) *a strictly increasing sequence* (n_i) *of positive integers and* (iii) *a set* $\{u_{ij} : i, j = 1, 2, \ldots\}$ *in* $(\ell_0^\infty(X, \mathscr{A}))'$ so that, for $i, j = 1, 2, \ldots$

$$\left.\begin{aligned} &|\langle u_{ij}, e(A_{ij})\rangle| > i + j \\ &\sum_{h+k < i+j} |\langle u_{ij}, e(A_{hk})\rangle| \leqslant 1 \\ &|\langle u_{ij}, x\rangle| \leqslant 1 \,, \quad \forall x \in U_{n_i} \end{aligned}\right\} \tag{4}$$

Proof. – We apply the preceding proposition to obtain an element $u_{11} \in (\ell_0^\infty(X, \mathscr{A}))'$ and an element $A_{11} \in \mathscr{A}$ so that

$$|\langle u_{11}, e(A_{11})\rangle| > 2 \;, \quad |\langle u_{11}, x\rangle| \leqslant 1 \,, \quad \forall\, x \in U_1$$

and such that $X \sim A_{11}$ does not have property U_n for $n = 1$ and an infinity of values of n. By recurrence suppose we have obtained q integers

$$1 = n_1 < n_2 < \ldots < n_q ,$$

and a family $\{A_{ij} : i + j \leqslant q + 1\}$ of pairwise disjoint elements of $\mathscr{A}$ and a set $\{u_{ij} : i + j \leqslant q + 1\}$ in $(\ell_0^\infty(X, \mathscr{A}))'$ so that (4) is verified for $i + j \leqslant q + 1$ and such that $X \sim \bigcup_{i+j\leqslant q+1} A_{ij}$ does not have property U_n for $n = n_1, n_2, \ldots, n_q$ and for an infinity of values of n. Let n_{q+1} be smallest integer larger than n_q such that $X \sim \bigcup_{i+j\leqslant q+1} A_{ij}$ does not have property $U_{n_{q+1}}$. We apply now Proposition 5 to $A = X \sim \bigcup_{i+j\leqslant q+1} A_{ij}$, $p = q + 1$, $\alpha = q + 2$ and $\{x_1, x_2, \ldots, x_r\} = \{e(A_{hk}) : h + k \leqslant q + 1\}$. We obtain the pairwise disjoints subsets

$$A_{1(q+1)}, A_{2q}, A_{3(q-1)}, \ldots, A_{(q+1)1}$$

in $X \sim \bigcup_{i+j\leqslant q+1} A_{ij}$ belonging to $\mathscr{A}$, and the elements

$$u_{1(q+1)}, u_{2q}, u_{3(q-1)}, \ldots, u_{(q+1)1}$$

in $(\ell_0^\infty(X, \mathscr{A}))'$ such that, for $i = 1, 2, \ldots, q + 1$

$$|\langle u_{i(q+2-i)}, e(A_{i(q+2-i)})\rangle| > q + 2$$

$$\sum_{h+k<q+2} |\langle u_{i(q+2-i)}, e(A_{hk})\rangle| \leqslant 1$$

$$|\langle u_{i(q+2-i)}, x\rangle| \leqslant 1 , \quad \forall x \in U_{n_i}$$

and $X \sim \bigcup_{i+j\leqslant q+2} A_{ij}$ does not have property U_n for $n = n_1, n_2, \ldots, n_{q+1}$ and for an infinity of values of n. Proceeding this way we arrive at the desired conclusion.

q.e.d.

PROPOSITION 7. – *Let* V *be a closed absolutely convex subset of* $\ell_0^\infty(X, \mathscr{A})$. *If* V *is not a neighbourhood of the origin in its linear hull* L, *then* X *does not have property* V.

Proof. – Suppose first that the codimension of L in $\ell_0^\infty(X, \mathscr{A})$ is finite. Let $\{z_1, z_2, \ldots, z_p\}$ be a cobasis of L in $\ell_0^\infty(X, \mathscr{A})$. Let M be the absolutely convex hull of $\{z_1, z_2, \ldots, z_p\}$. Then $W = V + M$ is a barrel in $\ell_0^\infty(X, \mathscr{A})$ such that $(V + M) \cap L = V$

and thus W is not a neighbourhood of the origin in $\ell_0^\infty(X, \mathscr{A})$. Let B be any finite subset of $\ell_0^\infty(X, \mathscr{A})$ and let Z be the absolutely convex hull of $V \cup B$. We find a positive integer n such that $B \subset nW$. Then

$$Z \subset V + nW \subset (n+1)W$$

and therefore Z is not a neighbourhood of the origin in $\ell_0^\infty(X, \mathscr{A})$, i.e. X does not have property V. If L has infinite codimension in $\ell_0^\infty(X, \mathscr{A})$ and B is any finite subset of $\ell_0^\infty(X, \mathscr{A})$ let Z be the absolutely convex hull of $V \cup B$. Then Z is not absorbing in $\ell_0^\infty(X, \mathscr{A})$ and therefore X does not have property V.

q.e.d.

THEOREM 1. – *Let* (E_n) *be an increasing sequence of subspaces of* $\ell_0^\infty(X, \mathscr{A})$ *covering* $\ell_0^\infty(X, \mathscr{A})$. *Then there is a positive integer* p *such that* E_p *is a barrelled dense subspace of* $\ell_0^\infty(X, \mathscr{A})$.

Proof. – Suppose first that E_n is not barrelled, $n = 1, 2, \ldots$ Then, for every positive integer n we can find a barrel W_n in E_n which is not a neighbourhood of the origin in E_n. Let U_n be the closure of W_n in $\ell_0^\infty(X, \mathscr{A})$. According to the preceding proposition, X does not have property U_n for every n positive integer. We apply Proposition 6 to obtain the pairwise disjoints subsets $\{A_{ij} : i, j = 1, 2, \ldots\}$ of X belonging to $\mathscr{A}$, the strictly increasing sequence of positive integers (n_i) and the set $\{u_{ij} : i, j = 1, 2, \ldots\}$ in $(\ell_0^\infty(X, \mathscr{A}))'$ with conditions (4).

We order the pairs of all the positive integers in the following way: given two of those pairs (p_1, p_2) and (q_1, q_2) we set $(p_1, p_2) < (q_1, q_2)$ if either $p_1 + p_2 < q_1 + q_2$ or $p_1 + p_2 = q_1 + q_2$ and $p_1 < q_1$. Setting $G = \cup \{A_{ij} : i, j = 1, 2, \ldots\}$ we find a positive integer m such that $\|u_{11}(G)\| < m$. We make a partition of the set of pairs of positive integers $\{(i, j) : i + j > 2\}$ in m parts $\mathscr{P}_1^{(11)}, \mathscr{P}_2^{(11)}, \ldots, \mathscr{P}_m^{(11)}$, so that, in each one, given any positive integer i there are infinitely many elements whose first component is i. According to (1)

$$\sum_{h=1}^{m} \|u_{11}(\cup \{A_{ij} : (i, j) \in \mathscr{P}_h^{(11)}\})\| \leqslant \|u_{11}(G)\|$$

and thus there is an integer k, $1 \leqslant k \leqslant m$, such that

$$\|u_{11}(\cup \{A_{ij} : (i, j) \in \mathscr{P}_k^{(11)}\})\| < 1 .$$

Setting $\mathscr{P}_k^{(11)} = \mathscr{P}^{(11)}$ and using recurrence suppose $\mathscr{P}^{(11)}, \ldots, \mathscr{P}^{(wt)}$ have already been constructed. If (r, s) is the pair following (w, t) we take in $\mathscr{P}^{(wt)}$ an element of the form (r, r_s) with $r_s > s + 2$. We find a positive integer q such that $\|u_{rr_s}(\mathrm{G})\| < q$. We make a partition of the set $\{(i, j) \in \mathscr{P}^{(wt)} : i + j > r + r_s\}$ in q parts $\mathscr{P}_1^{(rs)}, \mathscr{P}_2^{(rs)}, \ldots, \mathscr{P}_q^{(rs)}$ so that, in every one, given any positive integer i, there are infinitely many elements whose first component is i. We have that

$$\sum_{h=1}^{q} \|u_{rr_s}(\cup \{\mathrm{A}_{ij} : (i, j) \in \mathscr{P}_h^{(rs)}\})\| \leqslant \|u_{rr_s}(\mathrm{G})\|$$

and therefore there is a positive integer ℓ, $1 \leqslant \ell \leqslant q$, such that

$$\|u_{rr_s}(\cup \{\mathrm{A}_{ij} : (i, j) \in \mathscr{P}_\ell^{(rs)}\})\| < 1 . \tag{5}$$

We set $\mathscr{P}_\ell^{(rs)} = \mathscr{P}^{(rs)}$ and we continue the construction in the same way. We set $\mathrm{A}_{rr_s} = \mathrm{A}_{11}$ for $r = s = 1$ and H for

$$\cup \{\mathrm{A}_{nn_m} : n, m = 1, 2, \ldots\}.$$

Since (E_n) is an increasing sequence and covers $\ell_0^\infty(\mathrm{X}, \mathscr{A})$ there is a positive integer r such that U_{n_r} absorbs $e(\mathrm{H})$ and therefore there is a positive number λ such that $\lambda e(\mathrm{H}) \subset \mathrm{U}_{n_r}$.

On the other hand,

$$\langle u_{rr_s}, e(\mathrm{H})\rangle = \langle u_{rr_s}, e(\mathrm{A}_{rr_s})\rangle + \sum_{n+n_m<r+r_s} \langle u_{rr_s}, e(\mathrm{A}_{nn_m})\rangle + \langle u_{rr_s}, e(\cup \{\mathrm{A}_{nn_m} : n + n_m > r + r_s\})\rangle$$

and therefore, according to (4) and (5),

$$\begin{aligned}
|\langle u_{rrs}, e(\mathrm{H})\rangle| &\geqslant |\langle u_{rr_s}, e(\mathrm{A}_{rr_s})\rangle| - \sum_{n+n_m<r+r_s} |\langle u_{rr_s}, e(\mathrm{A}_{nn_m})\rangle| \\
&\qquad - |\langle u_{rr_s}, e(\cup \{\mathrm{A}_{nn_m} : n + n_m > r + r_s\})\rangle| \\
&\geqslant r + r_s - \sum_{i+j<r+r_s} |\langle u_{rr_s}, e(\mathrm{A}_{ij})\rangle| \\
&\qquad - |\langle u_{rr_s}(\cup \{\mathrm{A}_{nn_m} : n + n_m > r + r_s\})\rangle| \\
&\geqslant r + r_s - 1 - \|u_{rr_s}(\cup \{\mathrm{A}_{ij} : (i, j) \in \mathscr{P}^{(rs)}\})\| \\
&\geqslant r + r_s - 1 - 1 \geqslant r + s
\end{aligned}$$

and thus

$$\lim_{s \to \infty} |\langle u_{rr_s}, e(\mathrm{H})\rangle| = \infty. \tag{6}$$

On the other hand, since $\lambda e(\mathrm{H}) \in \mathrm{U}_{n_r}$, we apply (4) to obtain

$$|\langle u_{rr_s}, \lambda e(\mathrm{H})\rangle| \leqslant 1$$

which contradicts (6) and therefore there is a positive integer m_0 such that E_{m_0} is a barrelled space.

Next we suppose that E_n is not dense in $\ell_0^\infty(\mathrm{X}, \mathscr{A})$ for $n = 1, 2, \ldots$ Let $\overline{\mathrm{E}}_n$ be the closure of E_n in $\ell_0^\infty(\mathrm{X}, \mathscr{A})$. Let V_n be a closed absolutely convex neighbourhood of the origin in $\overline{\mathrm{E}}_n$. Obviously, $\overline{\mathrm{E}}_n$ is of infinite codimension in $\ell_0^\infty(\mathrm{X}, \mathscr{A})$, hence X does not have property V_n, $n = 1, 2, \ldots$ Following the preceding argument we arrive at contradiction and therefore there is a positive integer n_0 so that E_{n_0} is dense in $\ell_0^\infty(\mathrm{X}, \mathscr{A})$.

The sequence (E_{n_0+r}) is increasing and $\bigcup_{r=1}^{\infty} \mathrm{E}_{n_0+r} = \ell_0^\infty(\mathrm{X}, \mathscr{A})$ and therefore there is a positive integer r_0 so that $\mathrm{E}_{n_0+r_0}$ is barrelled. If $p = n_0 + r_0$, E_p is barrelled and dense in $\ell_0^\infty(\mathrm{X}, \mathscr{A})$.

q.e.d.

Note 2. – If we take natural number N for X in Theorem 1, the set of the parts $\mathscr{P}(\mathrm{N})$ of N for $\mathscr{A}$ and $\mathrm{E}_n = \ell_0^\infty(\mathrm{X}, \mathscr{A})$ we obtain the well known result which asserts the barrelledness of $\ell_0^\infty(\mathrm{N}, \mathscr{P}(\mathrm{N}))$ [3, p. 145].

2. Applications to the space of the bounded finite additive measures on a σ-algebra.

We denote by $\mathrm{H}(\mathscr{A})$ the linear space over K of the K-valued finitely additive bounded measures on $\mathscr{A}$ such that if $\mu \in \mathrm{H}(\mathscr{A})$ its norm is the variation $|\mu|$ of μ. A set M of $\mathrm{H}(\mathscr{A})$ is simply bounded in a subset $\mathscr{B}$ of $\mathscr{A}$ if, for every $\mathrm{A} \in \mathscr{B}$,

$$\sup \{|\mu(\mathrm{A})| : \mu \in \mathrm{M}\} < \infty.$$

Let T be the linear mapping of $\mathrm{H}(\mathscr{A})$ into $(\ell_0^\infty(\mathrm{X}, \mathscr{A}))'$ such that, if $\mu \in \mathrm{H}(\mathscr{A})$, then

$$\langle \mathrm{T}(\mu), e(\mathrm{A})\rangle = \mu(\mathrm{A}), \quad \forall \mathrm{A} \in \mathscr{A}.$$

It is obvious that T is a topological isomorphism between the Banach spaces $H(\mathscr{A})$ and $(\ell_0^\infty(X, \mathscr{A}))'$.

THEOREM 2. – *Let* $(\mathscr{A}_n)$ *be an increasing sequence of subsets of* $\mathscr{A}$ *covering* $\mathscr{A}$. *Then, there is a positive integer* p *such that, if* M *is a subset of* $H(\mathscr{A})$ *simply bounded in* $\mathscr{A}_p$ *then* M *is bounded in* $H(\mathscr{A})$.

Proof. – Let E_n be the subspace of $\ell_0^\infty(X, \mathscr{A})$ generated by $\{e(A) : A \in \mathscr{A}_n\}$. The sequence (E_n) is increasing and covers $\ell_0^\infty(X, \mathscr{A})$. According to Theorem 1 there is a positive integer p such that E_p is a dense barrelled subspace of $\ell_0^\infty(X, \mathscr{A})$. If M is simply bounded in $\mathscr{A}_p$ then its image by T, T(M) is a bounded subset of

$$(\ell_0^\infty(X, \mathscr{A}))' [\sigma((\ell_0^\infty(X, \mathscr{A}))', E_p)]$$

and, since E_p is barrelled, T(M) is bounded in $(\ell_0^\infty(X, \mathscr{A}))'$ and therefore M is a bounded subset of $H(\mathscr{A})$.

q.e.d.

THEOREM 3. – *If* $(\mathscr{A}_n)$ *is an increasing sequence of subsets of* $\mathscr{A}$ *covering* $\mathscr{A}$ *there is a positive integer* p *such that, if* (μ_n) *is a sequence in* $H(\mathscr{A})$ *so that* $(\mu_n(A))$ *is a Cauchy sequence for every* $A \in \mathscr{A}_p$, *then* (μ_n) *is weakly convergent in* $H(\mathscr{A})$.

Proof. – Let p be the positive integer determined by the preceding theorem. Then $(T(\mu_n))$ is a Cauchy sequence in

$$(\ell_0^\infty(X, \mathscr{A}))' [\sigma((\ell_0^\infty(X, \mathscr{A}))', E_p)].$$

Since E_p is barrelled, then $(T(\mu_n))$ converges to an element v in

$$(\ell_0^\infty(X, \mathscr{A}))' [\sigma((\ell_0^\infty(X, \mathscr{A}))', \ell_0^\infty(X, \mathscr{A}))]$$

and thus $(\mu_n(A))$ converges to $T^{-1}(v)(A)$, for every $A \in \mathscr{A}$, and therefore (μ_n) converges weakly in $H(\mathscr{A})$ to $T^{-1}(v)$, [2].

q.e.d.

3. Applications to certain locally convex spaces.

The linear spaces we shall use are defined over the field K of the real or complex numbers. Given the dual pair $\langle E, F \rangle$, $\sigma(E, F)$

denotes the topology on E of the uniform convergence on every finite subset of F. The word "space" will mean "separated locally convex topological linear spaces". Given a space E, its topological dual is E′ and its algebraic dual is E*. A finite additive measure μ with values in E on a σ-algebra $\mathscr{A}$ is bounded if the set $\{\mu(A) : A \in \mathscr{A}\}$ is bounded in E. The finite additive measure μ is exhaustive if given any sequence (A_n) of pairwise disjoints elements of $\mathscr{A}$ the sequence $(\mu(A_n))$ converges to the origin in E. If μ is a countably additive measure then μ is bounded.

A sequence (x_n) in a space E is subseries convergent if for every subset J of the natural numbers N the series $\sum_{n \in J}^{\infty} x_n$ converges. A sequence is bounded multiplier convergent if for every bounded sequence (a_n) in K the series $\sum_{n=1} a_n x_n$ converges. Given a subseries convergent sequence it is possible to associate with it an E-valued measure μ on the σ-algebra $\mathscr{P}(N)$ so that

$$\mu(J) = \sum_{n \in J} x_n, \quad \text{for every} \quad J \in \mathscr{P}(N).$$

In [5] we gave the following definition : a) E *is a* Γ_r*-space if every quasicomplete subspace of* $E^*[\sigma(E^*, E)]$ *intersecting* $E'[\sigma(E', E)]$ *in a dense subspace contains* E′. *The following results are true* [5] b) *If* $f : E \longrightarrow F$ *is a linear mapping with closed graph, f is continuous if* E *is a barrelled space and* F *is a* Γ_r*-space.* c) *If* F *is not a* Γ_r*-space there is a barrelled space* E *and a non-continuous linear mapping* $f : E \longrightarrow F$ *with closed graph.* d) *If* $f : E \longrightarrow F$ *is a continuous linear mapping, being* E *barrelled and* F Γ_r*-space then* f *can be extend in a continuous linear mapping of the completion* $\hat{E}$ *of* E *into* F.

THEOREM 4. – *Let* μ *be a bounded additive measure from a* σ*-algebra* $\mathscr{A}$ *on* X *in a space* E. *Let* (F_n) *be an increasing sequence of* Γ_r*-spaces covering a space* F. *If* $f : E \longrightarrow F$ *is a linear mapping with closed graph there is a positive integer* q *such that* $f \circ \mu$ *is a* F_q*-valued bounded finite additive measure on* $\mathscr{A}$.

Proof. – Let $S : \ell_0^\infty(X, \mathscr{A}) \longrightarrow E$ be the linear mapping defined by $S(e(A)) = \mu(A)$ for every $A \in \mathscr{A}$. Since μ is bounded S is continuous and therefore $T = f \circ S$ is a linear mapping with closed graph. The increasing sequence $(T^{-1}(F_n))$ covers $\ell_0^\infty(X, \mathscr{A})$ and according to Theorem 1 there is a positive integer q such that $T^{-1}(F_q)$ is barrelled and dense in $\ell_0^\infty(X, \mathscr{A})$. Let $T_q = T \mid T^{-1}(F_q)$ and according to d) T_q can be extended continuously $\overline{T}_q : \ell_0^\infty(X, \mathscr{A}) \longrightarrow F_q$. Since T has closed graph there is on F a separated locally convex topology $\mathscr{T}$ (see 4) coarser than the original topology such that $T : \ell_0^\infty(X, \mathscr{A}) \longrightarrow F[\mathscr{T}]$ is continuous. Then T and $\overline{T}_q$ are continuous from $\ell_0^\infty(X, \mathscr{A})$ in $F[\mathscr{T}]$ and coincide on a dense subspace and thus are coincident on $\ell_0^\infty(X, \mathscr{A})$ from which the conclusion follows.

q.e.d.

COROLLARY 1.4. – *Let* (F_n) *be an increasing sequence of* Γ_r-*spaces covering a space* F *and let* $f : E \longrightarrow F$ *be a linear mapping with closed graph, being* E *a space. If* (x_n) *is a subseries convergent sequence in* E *there is a positive integer* q *such that* $(f(x_n))$ *is a bounded sequence of* F_q.

Proof. – It is enough to consider the measure associated with (x_n) and to apply the preceding theorem.

q.e.d.

THEOREM 5. – *Let* (F_n) *be any increasing sequence of* Γ_r-*spaces covering a space* F. *If* (x_n) *is a subseries convergent sequence in* F *there is a positive integer* q *such that* (x_n) *is a sequence of* F_q *which is bounded multiplier convergent.*

Proof. – We set ℓ_0^∞ to denote $\ell_0^\infty(N, \mathscr{P}(N))$. Its completion is ℓ^∞. Let $f : \ell_0^\infty \longrightarrow F$ be the linear mapping defined by $f(e(A) = \sum_{n \in A} x_n$ for every $A \subset N$. It is obvious that $f : \ell_0^\infty[\sigma(\ell_0^\infty, \ell^1)] \longrightarrow F$ is continuous. Proceeding as we did in Theorem 4 there is a positive integer q such that $f^{-1}(F_q)$ is a barrelled dense subspace of ℓ_0^∞. Let g be the restriction of f to $f^{-1}(F_q)$. According to result d) we extend g to a linear continuous mapping $\hat{g} : \ell^\infty \longrightarrow F_q$. Let $\hat{f} : \ell^\infty[\sigma(\ell^\infty, \ell^1)] \longrightarrow \hat{F}$ be the linear extension of f, being $\hat{F}$ the completion of F. The functions $\hat{f}$ and g coincide in $f^{-1}(F_q)$ and therefore are equal.

Given the bounded sequence (a_n) in K we set $v = (a_n)$, $v_p = (b_i)$, $b_i = a_i$, $i = 1, 2, \ldots, p$ and $b_i = 0$, $i = p + 1$, $p + 2, \ldots$ The sequence (v_p) converges to v in $\ell^\infty[\sigma(\ell^\infty, \ell^1)]$ and therefore the sequence $(\hat{f}(v_p)) = \left(\sum_{n=1}^{p} a_n x_n\right)$ converges to $\hat{f}(v) = \sum_{n=1}^{\infty} a_n x_n$ in F_q.

q.e.d.

COROLLARY 1.5. – *Let* (F_n) *be an increasing sequence of spaces covering a space* F. *If for every positive integer* n *there is a topology* $\mathcal{T}_n$ *on* F_n *finer than the original topology such that* $F_n[\mathcal{T}_n]$ *is a* B_r*-complete space, then given a subseries convergent sequence* (x_n) *in* F *there is a positive integer* q *such that* (x_n) *is a bounded multiplier convergent series in* F_q.

Proof. – Since every B_r-complete space is a Γ_r-space [5] it results that $F_n[\mathcal{T}_n]$ is a Γ_r-space and applying c) it is easy to obtain that F_n is a Γ_r-space. We apply now Theorem 5.

q.e.d.

THEOREM 6. – *Let* (F_n) *be an increasing sequence of spaces covering a space* F. *If for every positive integer* n *there is a topology* $\mathcal{T}_n$ *on* F_n *finer than the original topology of* F_n, *such that* $F_n[\mathcal{T}_n]$ *is a* B_r*-complete space not containing* ℓ^∞, *then given a bounded additive measure* μ *on a* σ*-algebra* $\mathcal{A}$ *into* F *there is a positive integer* q *so that* μ *is an additive exhaustive measure on* $\mathcal{A}$ *into* $F_q[\mathcal{T}_q]$.

Proof. – Let $T : \ell_0^\infty(X, \mathcal{A}) \longrightarrow F$ be the linear mapping defined by $T(e(A)) = \mu(A)$ for every $A \in \mathcal{A}$. Since μ is bounded, T is continuous and following the argument of the proof of Theorem 4 there is a positive integer q such that the image of T is contained in F_q. Then T has closed graph in $\ell_0^\infty(X, \mathcal{A}) \times F_q[\mathcal{T}_q]$ and therefore $T : \ell_0^\infty(X, \mathcal{A}) \longrightarrow F_q[\mathcal{T}_q]$ is continuous and thus the set $\{T(e(A)) : A \in \mathcal{A}\} = \{\mu(A) : A \in \mathcal{A}\}$ is bounded in $F_q[\mathcal{T}_q]$. Since μ is bounded in $F_q[\mathcal{T}_q]$ and this space does not contain ℓ^∞ we obtain that μ is exhaustive in $F_q[\mathcal{T}_q]$ [4].

q.e.d.

In [1] the following result is proven and we shall need it later : e) *Let* $f : E \longrightarrow F$ *be a linear mapping with closed graph being* E *a space and* F *a* B_r*-complete space. If* F *does not contain* ℓ^∞, f *maps every subseries convergent sequence of* E *in a subseries convergent sequence of* F.

THEOREM 7. – *Let* (F_n) *be an increasing sequence of spaces covering a space* F. *If for every positive integer* n *there is a topology* $\mathcal{T}_n$ *on* F_n *finer than the original topology such that* $F_n[\mathcal{T}_n]$ *is a* B_r*-complete space not containing* ℓ^∞, *then given a countably additive measure* μ *on a* σ*-algebra* $\mathcal{A}$ *into* F *there is a positive integer* q *so that* μ *is countably additive measure on* $\mathcal{A}$ *into* $F_q[\mathcal{T}_q]$.

Proof. – As we showed in Theorem 4, it is possible to find a positive integer q such that $\mu : \mathcal{A} \longrightarrow F_q$ is a countably additive measure. Let (A_n) be a sequence of pairwise disjoint elements of $\mathcal{A}$. Then $\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$ in F_q. Obviously the sequence $(\mu(A_n))$ is subseries convergent in F_q. If J is the canonical mapping of F_q onto $F_q[\mathcal{T}_q]$, J has closed graph in $F_q \times F_q[\mathcal{T}_q]$ and therefore, according to result e), the sequence $(J(\mu(A_n))) = (\mu(A_n))$ is subseries convergent in $F_q(\mathcal{T}_q)$ and thus $\sum_{n=1}^{\infty} \mu(A_n) = \mu\left(\bigcup_{n=1}^{\infty} A_n\right)$ in $F_q[\mathcal{T}_q]$.

q.e.d.

COROLLARY 1.7. – *Let* (F_n) *be an increasing sequence of spaces covering a space* F. *If for every positive integer* n *there is a topology* $\mathcal{T}_n$ *on* F_n *finer than the original topology such that* $F_n[\mathcal{T}_n]$ *is a* B_r*-complete space not containing* ℓ^∞, *then given a subseries convergent sequence* (x_n) *in* F *there is a positive integer* q *such that* (x_n) *is a subseries convergent sequence in* $F_q[\mathcal{T}_q]$.

Proof. – It suffices to take in Theorem 7 $\mathcal{A} = \mathcal{P}(N)$ and $\mu(A) = \sum_{n \in A} x_n$ for every $A \in \mathcal{A}$.

q.e.d.

Note 4. – Let E be a space containing a subspace F topologically isomorphic to ℓ^∞. Let u be an injective mapping of ℓ^∞ into E such that u is a topological isomorphism of ℓ^∞ onto F. Let

$v : E' \longrightarrow (\ell^\infty)'$ be its transposed mapping. We can find a closed absolutely convex neighbourhood of the origin U in E such that $u^{-1}(U)$ is contained in the closed unit ball of ℓ^∞. We consider ℓ^1 as subspace of $(\ell^\infty)'$ in the natural way. We represent bt (e_n) the element of ℓ^1 having zero components but the n-th which is 1. If $(u^{-1}(U))^0$ is the polar of $u^{-1}(U)$ in $(\ell^\infty)'$ then $e_n \in (u^{-1}(U))^0$, $n = 1, 2, \ldots$. If U^0 is the polar set of U in E' then $v(U^0) = (u^{-1}(U))^0$. Taking $z_n \in U^0$ such that $v(z_n) = e_n$, $n = 1, 2, \ldots$, we define $P : E \longrightarrow F$ in the following way: given $x \in E$ the sequence $(\langle z_n, x\rangle)$ is in ℓ^∞ and we write $P(x) = u((\langle z_n, x\rangle))$. Since U^0 is an equicontinuous set in E' the mapping P is continuous. On the other hand, if $x \in F$ there is a sequence $(t_n) = t$ in K such that $t \in \ell^\infty$ and $u(t) = x$. Then

$$\langle z_n, x\rangle = \langle z_n, u(t)\rangle = \langle v(z_n), t\rangle = \langle e_n, t\rangle = t_n$$

and thus $P(x) = x$. Thus P is a continuous projection of E onto F and thus F has a topological complement in E. As a consequence ℓ^∞ can not be contained in any separable space G. The former property is going to be used to show that "B_r-complete space" can not be substitued by "Γ_r-space" in Corollary 1.7. Indeed, if Z is the subspace of $(\ell^\infty)'$ orthogonal to c_0 we take an element w in Z, $w \neq 0$. Then $\langle w, e(\{n\})\rangle = 0$, $n = 1, 2, \ldots$ Let H be the linear hull of $\ell^1 \cup \{w\}$. Since $L = \ell^\infty[\sigma(\ell^\infty, \ell^1)]$ is separable, $Q = \ell^\infty[\sigma(\ell^\infty, H)]$ is also separable [6]. Since Q has a topology coarser than the topology of ℓ^∞, Q is a Γ_r-space not containing ℓ^∞. Since ℓ_0^∞ is dense in ℓ^∞ there is a subset A in N such that $\langle w, e(A)\rangle \neq 0$ which means that $(e(\{n\}))$ is a subseries convergent sequence in L which is not subseries convergent in Q. If we substitued in Theorem 7 "B_r-complete space" by "sequentially complete Γ_r-space" it can be shown to be valid.

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