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Control systems on semi-simple Lie groups and their homogeneous spaces

CONTROL SYSTEMS ON SEMI-SIMPLE LIE GROUPS AND THEIR HOMOGENEOUS SPACES (*)

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INTRODUCTION

In essence, this paper deals with the accessibility problem for control systems described by ordinary differential equations. The main contribution of the paper is to give conditions for a class of such systems which ensure that the system can be steered from any initial state to any final state by an admissible control.

From the point of view of this paper, a control system, or a polysystem, is a family $\mathscr{F}$ of vector fields on an n-dimensional manifold M. A trajectory of $\mathscr{F}$ is a continuous curve x from an interval $[0, T]$, $T \geqslant 0$ of the real line into M such that for some partition $0 < t_1 < t_2 < \ldots < t_n = T$ there exist vector fields $X_1, \ldots, X_n$ in $\mathscr{F}$ such that on each interval $[t_{i-1}, t_i)$, x is an integral curve of X_i. The accessibility set of $\mathscr{F}$ through a point q in M consists of all points $w \in M$ for which there exists a trajectory x of $\mathscr{F}$ such that $x(0) = q$ and $x(T) = w$. $\mathscr{F}$ is termed transitive if the accessibility set of $\mathscr{F}$ through each point $q \in M$ is equal to M.

Our main objective is to give conditions on $\mathscr{F}$ which ensure that it is transitive. In this paper we concentrate our attention to systems $\mathscr{F}$ which consist of right (resp. left) invariant vector fields on a Lie group G. The transitivity results which we obtain for such systems are directly applicable to the corresponding systems on homogeneous spaces of G.

The prototypes of such systems are the so-called bilinear systems in the control theory literature; these are systems of the form

$$(1) \qquad \frac{dx}{dt} = Ax + \sum_{i=1}^{m} u_i(t)B_i x$$

where A and $B_1, \ldots, B_m$ are $n \times n$ matrices with real entries, $x \in \mathbf{R}^n$ and the controls $u_1, \ldots, u_m$ are unbounded. A very natural way to the accessibility properties of (1) is to consider the matrix differential equation :

$$(2) \qquad \frac{dX}{dt} = \left(A + \sum_{i=1}^{m} u_i(t) \right) X(t)$$

where $X \in GL_n(\mathbf{R})$.

The vector fields $X \to AX$ and $X \to B_i X$ $i = 1, 2, \ldots, m$ are right invariant on $GL_n(\mathbf{R})$ or on any closed subgroup G of $GL_n(\mathbf{R})$ provided that A and $B_1, \ldots, B_m$ lie in the Lie algebra $L(G)$ of G. The accessibility set of (1) through any point $q \in \mathbf{R}^n$ is simply the action of the accessibility set through the identity of (2) applied to q.

When the controls $u = (u_1, \ldots, u_m)$ consist of all piece-wise constant functions on $[0, \infty)$ taking values in some set $\Omega \subset \mathbf{R}^m$, then the set of accessibility of (2) through the identity is a semi-group in G generated by $\bigcup_{M \in \Gamma} \{e^{Mt} : t \geqslant 0\}$ where

$$(3) \qquad \Gamma = \left\{ A + \sum_{i=1}^{m} u_i B_i : u \in \Omega \right\}.$$

In the more general situation when $\mathscr{F}$ is any family of right (resp. left) invariant vector fields on a group G, the situation is quite similar in that the accessibility set of $\mathscr{F}$ through the identity is a semi-group generated by $\bigcup_{M \in \Gamma} \{e^{Mt} : t \geqslant 0\}$ where Γ is the subset of the Lie algebra of G constituted by the values of the elements of $\mathscr{F}$ at the identity. We call such semi-group $S(\Gamma)$, and our main problem in this paper is to find conditions on Γ such that $S(\Gamma)$ is equal to G, for then $\mathscr{F}$ is transitive on G.

The requirement that Lie (Γ), the Lie algebra generated by the set Γ, be equal to $L(G)$ means that the group generated by $S(\Gamma)$ is equal to G.

However, there are many distinct semi-groups of G with this property and hence this condition is not a sufficient condition for transitivity.

Our results are based on the following technique. Rather than working with the family $\mathscr{F}$ we work with $\mathscr{L}S(\mathscr{F})$, which in this paper we call the Lie saturate of $\mathscr{F}$. $\mathscr{L}S(\mathscr{F})$ is the largest (in the sense of set inclusion) family of vector fields such that :

i) The closures of the sets of accessibility of $\mathscr{F}$ and $\mathscr{L}S(\mathscr{F})$ through each point $q \in M$ are the same.

ii) $\mathscr{L}S(\mathscr{F})$ is contained in the Lie algebra generated by $\mathscr{F}$.

The most useful property of $\mathscr{L}S(\mathscr{F})$ is that when it spans the tangent space of M through each point in M then $\mathscr{F}$ is transitive.

In this paper we construct a sequence of operations on a given system $\mathscr{F}$ which permits us to conclude that $\mathscr{L}S_q(\mathscr{F}) = T_qM \;\; \forall q \in M$.

In particular we use the following facts :

i) If $V \subset \mathscr{L}S(\mathscr{F})$, then cov (V), the closed (in the C^∞ topology of vector fields on M) convex positive cone spanned by V is contained in $\mathscr{L}S(\mathscr{F})$, and

ii) If V and $-V$ are contained in $\mathscr{L}S(\mathscr{F})$, then the Lie algebra generated by V is contained in $\mathscr{L}S(\mathscr{F})$.

In the context of invariant vector fields the above ideas admit the following description : If $\Gamma(\mathscr{F})$ is the subset of L(G) defined by the values of the elements of $\mathscr{F}$ at the identity, then $\Gamma(\mathscr{L}S(\mathscr{F}))$, which we in this paper simply call LS(Γ), is the largest subset of L(G) such that

a) The closure of S(Γ) = the closure of S(LS(Γ)), and

b) LS(Γ) is contained in the Lie algebra generated by Γ.

In this setting, $\mathscr{L}S(\mathscr{F})$ spans the tangent space of G at each point if and only if LS(Γ) = L(G).

In order to introduce our main results and to elucidate the techniques used, we next mention certain well known results of transitivity in the context of the preceding formalism.

a) G compact, $\mathscr{F}$ either right or left invariant family. Then, if $M \in L(G)$,

$$\{e^{Mt} : t \leqslant 0\} \subset \text{Cl}\{e^{Mt} : t \geqslant 0\}.$$

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Hence, $-\Gamma \subset LS(\Gamma)$, and thus by ii) $LS(\Gamma) = \text{Lie}(\Gamma)$, the Lie algebra generated by Γ. Thus, Γ is transitive if and only if $\text{Lie}(\Gamma) = L(G)$ (V. J. Sussmann [6], Lobry [9]).

  b) For systems of the form

$$\mathscr{F} = \left\{ X + \sum_{i=1}^{m} u_i Y_i : u \in \Omega \right\}$$

we have the following transitivity results. We distinguish between the bounded control case and the unbounded one.

  b1) The bang-bang case $\Omega = \{ u : |u_i| = 1 \quad i = 1, 2, \ldots, m \}$. Then, $\mathscr{F}_0 = \{ X, X \pm Y_i : i = 1, 2, \ldots, m \} \subseteq \mathscr{L}S(\mathscr{F})$ since $\mathscr{F}_0$ is contained in $\text{cov}(\mathscr{F})$. If in addition X happens to be Poisson stable, then the negative trajectories of X lie in the closure of the positive trajectories of X and hence $-X \in \mathscr{L}S(\mathscr{F})$. Thus, $\text{cov}(\{-X\} \cup \mathscr{F}_0)$ is equal to the vector space generated by $\{ \pm X, \pm Y_1, \ldots, \pm Y_m \}$. Hence, in this case $\mathscr{L}S(\mathscr{F}) = \text{Lie}(\mathscr{F})$, and thus $\mathscr{F}$ is transitive whenever the Lie algebra spans the tangent space of M at each point [9].

  b2) The unbounded controls case, $\Omega = \mathbf{R}^m$. In this case, $\{ X, \pm Y_1, \ldots, \pm Y_m \}$ belongs to $\text{cov}\,\mathscr{F}$, and hence to $\mathscr{L}S(\mathscr{F})$. If $\mathscr{F}_0 = \{ \pm Y_1, \ldots, \pm Y_m \}$ then $-\mathscr{F}_0 = \mathscr{F}$; hence, by ii) above $\text{Lie}(\mathscr{F}_0)$, the Lie algebra generated by $\mathscr{F}_0$, is contained in $\mathscr{L}S(\mathscr{F})$. Thus, $\mathscr{F}$ is transitive whenever $\text{Lie}(\mathscr{F}_0)$ spans the tangent space of M at each point of M. In this connection, there are several papers aimed at proving that the set of all pairs (Y_1, Y_2) of vector fields on a manifold M whose Lie algebra generated by $\{ Y_1, Y_2 \}$ spans the tangent space of M at each of its points is open and dense (for instance, [10]). Thus when $m \geqslant 2$ transitivity of systems of the form $\{ X, \pm Y_1, \ldots, \pm Y_m \}$ is generic. Certainly, this is the case where all the vector fields which define the system are invariant on a semi-simple Lie group G. In this sense, the most interesting case is the scalar control case, i.e., the case when $m = 1$.

  This paper is essentially devoted to such a case, and it is a generalization of our previous paper [5] dealing with matrix systems in $SL_n(\mathbf{R})$ to systems on a general semi-simple Lie group with a finite center.

  In order to state our main results we will need several concepts related to semi-simple Lie algebras. Let L be a real semi-simple Lie algebra, $L_c = L \otimes_{\mathbf{R}} \mathbf{C}$ its complexification, $\text{ad} : L \to \text{End}(L)$, $\text{ad}_c : L_c \to \text{End}(L_c)$ the corresponding adjoint representations.

  We shall say that an element B in L is *strongly-regular* if : 1) all

nonzero eigenvalues of $ad_c B$ are simple. 2) The generalized zero-eigensplace $\bigcup\limits_{n=1}^{\infty} \mathrm{Ker}\, ad_c^n B$, $(ad_c^n B = ad_c B \circ ad_c^{n-1} B)$, does not contain any nontrivial ideal of L_c. The set of all these elements is an open dense semi-algebraic subset of L.

We will use the following notations throughout the paper : $\mathrm{Sp}(B)$ will be the set of all nonzero eigenvalues of $ad\, B$, $\mathrm{Sp}_+(B)$ [resp. $\mathrm{Sp}_-(B)$, $\mathrm{Sp}_0(B)$] the set of all those $a \in \mathrm{Sp}(B)$ such that $\mathrm{Re}\, a > 0$ (resp. $\mathrm{Re}\, a < 0$, $\mathrm{Re}\, a = 0$).

$L_c(a)$ will denote the a-eigenspace $\mathrm{Ker}\,(ad_c B - aI)$ of $ad_c B$ and $L(a) = (L_c(a) + L_c(\bar a)) \cap L$ the corresponding « real eigenspace » ($\bar a$ complex conjugate of a). Then $L(\bar a) = L(a)$. L is the direct sum $L(0) \oplus \{L(a)|a \in \mathrm{Sp}(B), \mathrm{Im}\, a \geqslant 0\}$. Any $A \in L$ has a unique decomposition $A(0) + \Sigma A(a)$, $A(0) \in L(0)$, $A(a) \in L(a)$. $A(a)$ will be called the real a-component of A.

We also introduce the following spaces :

$$N_+ = \Sigma\{L(a)|a \in \mathrm{Sp}_+(B)\} \qquad N_- = \Sigma\{L(a)|a \in \mathrm{Sp}_-(B)\}$$
$$K = \Sigma\{L(a)|a \in \mathrm{Sp}_0(B)\} + \Sigma\{[L(a), L(-a)]|a \in \mathrm{Sp}_0(B)\}$$
$$L_+ = N_+ \oplus K \qquad L_- = N_- \oplus K.$$

All these spaces are subalgebras of L. K is reductive in L and semi-simple. It normalizes N_+ and N_-. We also have the direct sum representations :

$$L = L_+ \oplus \mathcal{O} \oplus N_- \qquad L = L_- \oplus \mathcal{O} \oplus N_+$$

where $\mathcal{O} \subset L(0)$ is the centralizer of $K + \mathbf{R}B$. Call π_+ (resp. π_-) the projection $L \to L$ with kernel $\mathcal{O} \oplus N_-$ (resp. $\mathcal{O} + N_+$) and image L_+ (resp. L_-). It commutes with $ad_c B$ and is in fact the unique projection with image L_+ (resp. L_-) commuting with $ad_c B$.

The next concept we need is :

DEFINITION 0. — *A subset* Γ *of* L *containing the space* $\mathbf{R}B$, B *strongly regular, is said to satisfy the strong rank condition with respect to* B *if the Lie subalgebras of* L *generated respectively by* $\pi_+(\Gamma) + \mathbf{R}B$ *and* $\pi_-(\Gamma) + \mathbf{R}B$ *contain* L_+ *and* L_- *respectively.*

This is also equivalent to the condition : let $R_+(\Gamma)$ (resp. $R_-(\Gamma)$) be the

set of all $a \in Sp_+(B) \cup Sp_0(B)$ (resp. $Sp_-(B) \cup Sp_0(B)$) such that there is an $A \in \Gamma$ with $A(a) \neq 0$. Then the space $\Sigma\{L(a)|a \in R_+(\Gamma)\}$ (resp. $\Sigma\{L(a)|a \in R_-(\Gamma)\}$) should generate L_+ (resp. L_-).

Now we can state our theorem in the simple case where the statement is simpler.

THEOREM 0. — *Let* G *be a real connected Lie group with a* finite center *whose Lie algebra* L *is a real form of a simple Lie algebra* L_c.

A subset $\Gamma \subset L$ *is transitive if* :

1) Γ *contains a one dimensional space* **RB** *where* B *is strongly regular*.

2) Γ *satisfies the strong rank condition with respect to* B. (*)
Let $s = \sup\{Re\ a|a \in Sp(B)\}$.

3) *In the case where* $s \notin Sp(B)$ *there are* $a_1, a_2 \in Sp(B)$ *and* $A_1, A_2 \in \Gamma$ *such that* $Re\ a_1 = s = -Re\ a_2$ *and* $A_1(a_1) \neq 0$, $A_2(a_2) \neq 0$.

4) *In the case where* $s \in Sp(B)$, *there are* $A_1, A_2 \in \Gamma$ *such that* Trace $(ad\ A_1(s) \circ ad\ A_2(-s)) < 0$.

This theorem is a particular case of the next one, dealing with the semi-simple case. We could state it by decomposing L into its simple ideals and asking that the conditions of Theorem 0 be satisfied in each ideal. But this would mask the important role played by the restriction of the adjoint repr. of L to K.

Since B normalizes K, K + **RB** is a subalgebra of L. In proposition 9-(0) we show that K + **RB** is reductive in L and that L is the direct sum of the centralizer of K + **RB** in L and the non trivial, ad(K + **RB**)-irreducible, submodules of L. A representation module is called trivial if the representation on the module is zero.

$$L = \mathcal{C} \oplus \bigoplus \{T|T \subset L,\ T \text{ non trivial, ad}(K + \mathbf{RB})\text{-irreducible}\}.$$

Also either $T \subset N_+$ or $T \subset K$ or $T \subset N_-$.

Notation. — If $A \in L$, we denote by $A(T)$ the component of A in the irreducible module T with respect to the above direct sum decomposition.

(*) In an interesting paper to be published in SIAM Journal of Control, Bornard and Gauthier have succeded in replacing condition (2) by the usual rank condition in case $G = SL(n; \mathbf{R})$ and $\Gamma = \mathbf{R}_+A + \mathbf{RB}$.

DEFINITION 1. — *An* ad $(K+RB)$*-irreducible submodule* T *of* L *is called maximal (resp. minimal) if it is contained in the center of* N_+ *(resp.* N_-*). This is equivalent to each of the following properties*

(1) T *is* ad N_+ *(resp.* ad N_-*) invariant*

(2) T *is* ad L_+ *(resp.* ad L_-*) invariant.*

THEOREM 1. — *Let* G *be a real connected semi-simple Lie group with a* finite center *whose Lie algebra is called* L.

A subset $\Gamma \subset L$ *is transitive if :*

1) Γ *contains a space* **RB**, B *strongly regular.*

2) Γ *satisfies the strong rank condition with respect to* **B**.

3) *If* T *is a maximal or minimal submodule of* L, *which does not contain any* $L(x)$, $x \in Sp\,(B) \cap \mathbf{R}$, *then some* $A \in \Gamma$ *should have a nonzero component* $A(T)$ *in* T.

4) *If* T *is as in 3) but contains a space* $L(r)$, $r \in Sp(B) \cap \mathbf{R}$, *then there should exist* A_1, $A_2 \in \Gamma$ *such that* Trace $(ad\,A_1(r) \circ ad\,A_2(-r)) < 0$.

Remark. — Our methods prove easily the extension of theorem 1 to the case where : 1) G is a reductive connected real Lie group such that the integral subgroup generated by the semi-simple factor of Lie (G) has finite center and 2) Γ satisfies the conditions 1-4 and moreover : 5) : the projection along the semi-simple factor of the positive cone generated by Γ onto the center of Lie (G) is equal to this center.

CHAPTER I

GENERALITIES

0. Notations and Definitions.

Throughout this paper we will use the following notations :

M : m-dimensional real C^∞ or C^ω (analytic) connected manifold.
TM : tangent bundle of M.
$T_q M$: the tangent space of M at $q \in$ M.
F(M) : the vector space of all C^∞ or C^ω, vector fields on M.
$\mathscr{D}$(M) : the pseudo group of all local C^∞ or C^ω diffeomorphisms of M. An element of $\mathscr{D}$(M) is a couple (U,φ) where U is an open subset of M and φ : U → M is a C^∞ or C^ω diffeomorphism.
Diff (M) : the group of all C^∞ or C^ω diffeomorphisms of M.
Cl(M) : the class of all closed subsets of M.

If X ∈ F(M), we recall the following elementary fact from the theory of differential equations : there exist an open subset Δ_X of M × **R** which is a neighborhood of M × {0}, and a function $\Phi_X : \Delta_X \to$ M which satisfy :

a) if $(q,s) \in \Delta_X$, and if $(\Phi_X(q,s),t) \in \Delta_X$, then $(q, s+t) \in \Delta_X$ and

$$\Phi_X(q, s+t) = \Phi_X(\Phi_X(q,s), t).$$

b) $\dfrac{\partial}{\partial t} \Phi_X = X \circ \Phi_X$ on Δ_X.

c) For each $(q,t) \in \Delta_X$ there exists a neighborhood U of q in M such that $(U, \Phi_X|_{U \times \{t\}}) \in \mathscr{D}$(M).

To any X ∈ F(M) we attach the set $\mathscr{P}$(X) of all local diffeomorphisms (U,φ) where there exists $t \in$ **R** such that $U \times \{t\} \in \Delta_X$ and $\varphi = \Phi_X\big|_{U \times \{t\}}$.

$\mathscr{P}$(X) is a sub-pseudo group of $\mathscr{D}$(M). Of particular interest in this exposition will be the subset $\mathscr{P}_+$(X) defined by all local diffeomorphisms (U,φ) where $U \times \{t\} \in \Delta_X$ for some non-negative $t \in$ **R** and where $\varphi = \Phi_X|U \times \{t\}$. $\mathscr{P}_+$(X) is a sub-pseudo semi-group of $\mathscr{D}$(M).

DEFINITION 0. — *A subset of* F(M) *will be called a polysystem* ([8]).

DEFINITION 1. — *If $\mathscr{F}$ is a polysystem, $\mathscr{P}_+(\mathscr{F})$ will denote the pseudo-sub-semi-group of $\mathscr{D}(M)$ generated by $\{\mathscr{P}_+(X), X \in \mathscr{F}\}$.*

DEFINITION 2. — *a) The accessibility mapping of a polysystem $\mathscr{F}$ is the mapping $A_{\mathscr{F}} : M \to 2^M$ defined by : if $q \in M$, $A_{\mathscr{F}}(q)$ is the orbit $\mathscr{P}_+(\mathscr{F})(q)$ of $\mathscr{P}_+(\mathscr{F})$ at q.*

b) The closed accessibility mapping of $\mathscr{F}$ denoted by $\bar{A}_{\mathscr{F}}$ is the mapping $M \to Cl(M)$ defined by $\bar{A}_{\mathscr{F}}(q) = Cl\, A_{\mathscr{F}}(q)$.

DEFINITION 3. — *a) Two polysystems $\mathscr{F}$ and $\mathscr{C}$ are said to be weakly equivalent if $\bar{A}_{\mathscr{F}} = \bar{A}_{\mathscr{C}}$.*

b) A polysystem $\mathscr{F}$ is said to be saturated if for any polysystem $\mathscr{C}$ which is weakly equivalent to $\mathscr{F}$, $\mathscr{C} \subset \mathscr{F}$.

c) If $\mathscr{F}$ is a given polysystem we shall denote by Sat $(\mathscr{F})$ the set $\bigcup \{\mathscr{C} : \mathscr{C}$ weakly equivalent to $\mathscr{F}\}$, and we shall refer to it as the saturate of $\mathscr{F}$.

DEFINITION 4. — *The normalizer of a polysystem $\mathscr{F}$ is the set of all diffeomorphisms $\varphi \in Diff(M)$ such that : $\varphi[\bar{A}_F(\varphi^{-1}(q))] \subset \bar{A}_F(q)$ for all $q \in M$. This set will be denoted by Norm(F).*

In particular, if $\varphi \in Diff(M)$ is such that $\varphi(q) \in \bar{A}_F(q)$ and $\varphi^{-1}(q) \in \bar{A}_F(q)$ for all $q \in M$, then φ and φ^{-1} belong to Norm (F).

LEMMA 0. — 1) *The normalizer is a subsemigroup of Diff(M).*
2) *It does not depend on $\mathscr{F}$ but on Sat $(\mathscr{F})$:*

$$\text{Norm}(\mathscr{F}) = \text{Norm}(\text{Sat}(\mathscr{F})).$$

The proof of this lemma is easy and left to the reader. The next lemma is trivial but useful.

LEMMA 1. — 1) *The weak accessibility correspondance $A : 2^{F(M)} \to Map(M, Cl(M))$ is order preserving for the inclusion order on $2^{F(M)}$ and the order induced by the inclusion order on $Cl(M)$, in $Map(M, Cl(M))$.*

2) *If $p, q \in M$ and $p \in A_{\mathscr{F}}(q)$ then $A_{\mathscr{F}}(p) \subset A_{\mathscr{F}}(q)$.*

1. Invariance of closed accessibility mapping under certain changes of the polysystem.

In the next proposition we collect facts, some of which are more or less known. The vector space F(M), being a subset of the space $C^\infty(M, TM)$

of all C^∞ mappings $M \to TM$, inherits the classical C^∞-topology of that space. It is a locally-convex topology on $F(M)$ and it is complete if $F(M)$ is the set of all C^∞ vector-fields.

PROPOSITION 0. — 1) *For any polysystem* $\mathscr{F}$, *the polysystem* $\mathscr{F}' = \{\varphi_*(X) | \varphi \in \text{Norm}(\mathscr{F}), X \in \mathscr{F}\}$ *is weakly equivalent to* $\mathscr{F}$. *In particular* $\varphi_*(\text{Sat}(\mathscr{F})) \subset \text{Sat}(\mathscr{F})$ *for all* $\varphi \in \text{Norm}(\mathscr{F})$.

2) *Given a polysystem* $\mathscr{F}$, *the closed convex cone in* $F(M)$, *generated by* $\mathscr{F}$, *is weakly equivalent to* $\mathscr{F}$.

The proof of this proposition is long but not difficult. Hence we omit it. Let us remark only that the proof of assertion 2 is based on the well known fact that weak convergence of controls implies the uniform convergence of trajectories.

2. A general procedure for checking whether a polysystem is transitive.

Notation. — Given a polysystem $\mathscr{F}$, let us denote by $\text{Lie}(\mathscr{F})$ the Lie subalgebra of $\mathscr{F}(M)$ generated by $\mathscr{F}$.

The procedure is based on the following quite easy proposition.

PROPOSITION 1. — *Let* $\mathscr{F}$ *be a polysystem such that* : 1) *for all* $q \in M$, $\text{Lie}(\mathscr{F})(q) = T_q M$. 2) *for all* $q \in M$, $\bar{A}_{\mathscr{F}}(q) = M$. *Then* $\mathscr{F}$ *is transitive on* M.

Proof. — It is well known ([12], [7]) that condition 1 implies that the closure of the interior of $P_+(\mathscr{F})q$ is $\bar{A}_{\mathscr{F}}(q)$ for any $q \in M$. Since $\text{Lie}(-\mathscr{F}) = \text{Lie}(\mathscr{F})$, $\text{int}(P_+(-\mathscr{F})(q)) \neq \varnothing$ for all $q \in M$.

1) and 2) imply that for every $q \in M$ $\text{int}(P_+(\mathscr{F})(q))$ is open dense in M. If p and q belong to M, $\text{int}(P_+(\mathscr{F})(p)) \cap \text{int}(P_+(-\mathscr{F})(q)) \neq \varnothing$. Let r be a point of this intersection. There is a $\varphi \in P_+(\mathscr{F})$ and a $\psi \in P_+(-\mathscr{F})$ such that $\varphi(p) = r$ and $\psi(q) = r$. Hence $\psi^{-1} \circ \varphi(p) = q$ but $\psi^{-1} \in P_+(\mathscr{F})$.

DEFINITION 5. — *Given a polysystem* $\mathscr{F}$ *we denote by* $\mathscr{L}S(\mathscr{F})$ *the polysystem* $\text{Lie}(\mathscr{F}) \cap \text{Sat}(\mathscr{F})$.

COROLLARY TO PROPOSITION 1. — *A polysystem* $\mathscr{F}$ *is transitive if* $\mathscr{L}S(\mathscr{F})(q) = T_q M$ *for all* $q \in M$.

PROPOSITION 2. $-$ 1) $\mathscr{L}S \circ \mathscr{L}S = \mathscr{L}S$.

2) $\mathscr{L}S(\mathscr{F})$ is a convex cone in $F(M)$. It is closed if and only if $Lie(\mathscr{F})$ is closed in $F(M)$.

3) The subset of all $X \in \mathscr{L}S(\mathscr{F})$ such that $-X \in \mathscr{L}S(\mathscr{F})$ is a Lie subalgebra of $F(M)$ and it is the largest subalgebra of $F(M)$ contained in $\mathscr{L}S(\mathscr{F})$.

4) If X and $-X$ belong to $\mathscr{L}S(\mathscr{F})$ and if X is complete then $\exp(X)_*(\mathscr{L}S(\mathscr{F})) \subset \mathscr{L}S(\mathscr{F})$.

This proposition follows almost immediately from the definition and Proposition 0.

Remark. $-$ In general if $\varphi \in Norm(\mathscr{F})$, $\varphi_*(\mathscr{L}S(\mathscr{F}))$ is not contained in $\mathscr{L}S(\mathscr{F})$.

The procedure to check if a given polysystem $\mathscr{F}$ is transitive consists of the transfinite repetition of the following two steps.

Step 1. $-$ Given $\mathscr{F}$, one constructs the closed convex cone $cov(\mathscr{F})$ generated by $\mathscr{F}$ in $F(M)$.

Step 2. $-$ Given $\mathscr{F}$, one constructs the polysystem

$$Lie(\mathscr{F}) \cap \{\varphi_*(X) | \varphi \in Norm(\mathscr{F}), X \in \mathscr{F}\}.$$

Starting from a polysystem $\mathscr{F}$, one constructs a transfinite sequence $\{\mathscr{F}_\alpha | \alpha \text{ ordinal}\}$ of polysystems in $F(M)$ as follows : if α is an even or a limit ordinal, $\mathscr{F}_\alpha$ is step 1 applied to $\bigcup_{\beta < \alpha} \mathscr{F}_\beta$; if α is an odd ordinal, $\mathscr{F}_\alpha$ is step 2 applied to $\mathscr{F}_{\alpha-1}$.

All these $\mathscr{F}_\alpha$ are contained in $\mathscr{L}S(\mathscr{F})$. If this transfinite sequence becomes stationary at some ordinal α and if for all $q \in M$, $\mathscr{F}_\alpha(q) = T_q M$ then the procedure is successful and $\mathscr{F}$ will be transitive.

In practice usually, $Norm(\mathscr{F}_\alpha)$ is not known, just a subset of it. So when the procedure is applied in practice, step 2 is replaced by step 2'.

Step 2'. $-$ Given $\mathscr{F}$ and given a subset N of $Norm(F)$ containing the identity, one constructs the polysystem

$$Lie(\mathscr{F}) \cap \{\varphi_*(X) | X \in \mathscr{F}, \varphi \in N\}.$$

This procedure seems very effective in many classical situations, for

example those mentioned in the introduction. It can also be used to prove
the well known controllability results for the linear systems. As a way of
illustration of the technique we outline this case in the following :

Example 1 (The linear systems). — Let $\mathscr{F} = \left\{ X + \sum\limits_{i=1}^{m} u_i Y_i : u_i \in \mathbf{R} \right\}$
be a polysystem on $M = \mathbf{R}^n$ where X is a linear field and where
$Y_1, \ldots, Y_m$ are constant vector fields. Let $X(q) = Aq$, and let
$Y_i(q) = b_i$, $i = 1, \ldots, m$ where A is $n \times n$ matrix and where
$b_1, \ldots, b_m$ are vectors in $\mathbf{R}^n$. A basis for $\mathrm{Lie}(\mathscr{F})$ is given by
$\{ X \} \cup \{ \mathrm{ad}^k X(Y_i), k = 0, \ldots, n-1, i = 1, 2, \ldots, m \}$. We shall now show
that $\mathrm{LS}(\mathscr{F}) = \mathrm{Lie}(\mathscr{F})$.

Step 1. — $\mathscr{F}_1 = \{ X, \pm Y_1, \pm Y_2, \ldots, \pm Y_m \}$ is weakly equi-
valent to $\mathscr{F}$ because $\mathscr{F}_1 \subset \mathrm{cov}(\mathscr{F})$. For each integer $i = 1, 2, \ldots, m$
and for each real $te^{Y_i t} \in \mathrm{Norm}(\mathscr{F})$.

Thus, $e_*^{Y_i t}(\mathscr{F}_1) = \{ e_*^{Y_i t}(X), X, \pm Y_1, \ldots, \pm Y_m \}$ is weakly equivalent
to $\mathscr{F}_1$. Also, $e_*^{Y_i t}(X) = \mathrm{Id} \pm tAb_i = \mathrm{Id} \pm t \, \mathrm{ad} \, X(Y_i)$ where Id is the
identity mapping on $\mathbf{R}^n$.

Step 2. — By passing to the closed convex cone generated by the fields
$e_*^{Y_i t}(\mathscr{F}_1)$ we get that

$$\mathscr{F}_2 = \{ X, \pm Y_1, \ldots, \pm Y_m, \pm \mathrm{ad} X(Y_1), \ldots, \pm \mathrm{ad} X(Y_m) \}$$

is weakly equivalent to $\mathscr{F}$.

Now, $e_*^{t \, \mathrm{ad} X(Y_i)}(X) \in \mathrm{LS}(\mathscr{F})$ for each $t \in \mathbf{R}$ and each $i = 1, 2, \ldots, m$.
This shows that $\mathrm{ad}^2 X(Y_i) \in \mathrm{LS}(\mathscr{F})$, and hence it follows by an easy
induction that $\mathrm{ad}^k X(Y_i) \in \mathrm{LS}(\mathscr{F})$ for each integer k and for all
$i = 1, 2, \ldots, m$. Thus, $\mathrm{LS}(\mathscr{F}) = \mathrm{Lie}(\mathscr{F})$.

Hence, the linear systems are transitive if and only if $\mathrm{Lie}(\mathscr{F})$ spans $\mathbf{R}^n$
at each point $q \in \mathbf{R}^n$. And this happens exactly when

$$\{ b_1, \ldots, b_m, Ab_1, \ldots, Ab_m, \ldots, A^{n-1} b_1, \ldots, A^{n-1} b_m \}$$

contains n linearly independent vectors. If B is the matrix whose columns
are $b_1, \ldots, b_m$, then the preceding rank condition is equivalent to saying
that the rank of $n \times (n \times m)$ matrix $(B, AB \ldots A^{n-1} B)$ is equal to n.
Thus our methods give yet another proof of the celebrated rank condition
for the linear systems.

3. Polysystems associated to group action.

A. Let G be a connected Lie group whose Lie algebra will be denoted by L. Let $\theta : G \times M \to M$ be a C^∞ or C^ω action of G on the manifold M.

θ induces a linear mapping $\theta_* : L \to F(M)$.

DEFINITION 4. — *A polysystem $\mathscr{F}$ on M will be called subordinated to the action θ if $\mathscr{F}$ is contained in the image of θ_*. If $\mathscr{F} = \theta_*(\Gamma)$, $\mathscr{F}$ will be said induced by Γ.*

DEFINITION 5. — *In particular if $M = G$ and θ is either the left or the right translation action of G on itself, given a subset Γ of L, the polysystem induced by Γ under the right (resp. left) translation action will be called the left (resp. right) invariant polysystems induced by Γ and will be denoted by Γ_ℓ (resp. Γ_r).*

The mapping $G \to G, x \to x^{-1}$ transforms the left action into the right action. It transforms a left invariant polysystem Γ_ℓ induced by Γ into the right invariant polysystem $(-\Gamma)_r$ induced by $-\Gamma$. Hence one needs only to consider one type of action.

PROPOSITION 3. — *Let Γ be a subset of L and let $\bar{S}(\Gamma)$ denote the closed semi-group of G generated by the one-parameter semi-groups $\{e^{tX} | t \in \mathbf{R}_+\}$ for all $X \in \Gamma$. Then :*

1) *For any $g \in G$, $A_{\Gamma_\ell}(g) = g S(\Gamma)$, $A_{\Gamma_r}(g) = S(\Gamma)g$. In particular $A_{\Gamma_\ell}(e) = A_{\Gamma_r}(e) = S(\Gamma)$.*

2) *If $\Gamma, \Delta \subset L, \Gamma_\ell$ and Δ_ℓ (resp. Γ_r and Δ_r) are weakly equivalent if and only if $\bar{S}(\Gamma) = \bar{S}(\Delta)$. We will say then that Γ and Δ are weakly equivalent.*

3) *The union, $\mathrm{Sat}(\Gamma)$, of all subsets of L weakly equivalent to Γ induces both*

$\mathrm{Sat}(\Gamma_\ell) \cap \{Left\ invariant\ vector\ fields\ on\ G\}$ *and*

$\mathrm{Sat}(\Gamma_r) \cap \{Right\ invariant\ vector\ fields\ on\ G\}$.

4) $\mathrm{Sat}(\Gamma) = \{X | X \in L, e^{tX} \in \bar{S}(\Gamma)\ for\ all\ t \geqslant 0\}$. *It is a closed convex cone.*

5) *If ρ_ℓ (resp. ρ_r) : $G \to \mathrm{Diff}(G)$ denotes the homomorphism associated*

to the left (resp. right) translation action, then

$$\rho_r^{-1}(\mathrm{Norm}(\Gamma_\ell)) = \{g \,|\, g \in G, g\bar{S}(\Gamma)g^{-1} \supset \bar{S}(\Gamma)\}$$
$$\rho_\ell^{-1}(\mathrm{Norm}(\Gamma_r)) = \{g \,|\, g \in G, g\bar{S}(\Gamma)g^{-1} \subset \bar{S}(\Gamma)\}.$$

PROPOSITION 4. — *Let* $\theta : G \times M \to M$ *be a* C^∞ *or* C^ω *action of* G *on* M, Γ *a subset of* L *and* $\mathscr{F} = \theta_*(\Gamma)$ *the induced polysystem. Then*

1) $P_+(\mathscr{F})$ *is the pseudo-semi-group generated by* $\rho_\theta(S(\Gamma))$ *where* $\rho_\theta : G \to \mathrm{Diff}(M)$ *is the homomorphism associated to the action* θ.

2) *For any* $q \in M$, $A_{\mathscr{F}}(q) \supset \theta(S(\Gamma) \times \{q\})$.

3) $\theta_*(\mathrm{Sat}(\Gamma)) \subset \mathrm{Sat}(\mathscr{F})$.

COROLLARY. — *If* G *is transitive on* M, *and* Γ *is transitive on* G, *then* $\mathscr{F} = \theta_*(\Gamma)$ *is transitive on* M.

Proposition 4 is straightforward. Proposition 3 is a direct consequence of proposition 0.

B. *A criterion for the transitivity of right and left invariant polysystems.*

DEFINITION 6. — *Given a subset* Γ *of* L, *we call* LS *saturation of* Γ *and we denote by* $\mathrm{LS}(\Gamma)$, *the intersection* $\mathrm{Sat}(\Gamma) \cap \mathrm{Lie}(\Gamma)$ *where* $\mathrm{Lie}(\Gamma)$ *is the Lie subalgebra* L *generated by* Γ.

PROPOSITION 5. — (0) *Given* $\Gamma \subset L$, $\mathrm{LS}(\Gamma)$ *induces both* $\mathscr{L}S(\Gamma_\ell)$ *and* $\mathscr{L}S(\Gamma_r)$.

1) $\mathrm{LS} \circ \mathrm{LS} = \mathrm{LS}$.

2) $\mathrm{LS}(\Gamma)$ *is a closed convex cone.*

3) *The subset of all* $X \in \mathrm{LS}(\Gamma)$ *such that* $-X \in \mathrm{LS}(\Gamma)$ *is a Lie subalgebra of* L *and is the largest subalgebra of* L *contained in* $\mathrm{LS}(\Gamma)$. *It is also the largest vector subspace of* L *contained in* $\mathrm{LS}(\Gamma)$.

4) *If* X *and* $-X$ *belong to* $\mathrm{LS}(\Gamma)$ *then* $e^{\mathrm{ad}X}(\mathrm{LS}(\Gamma)) \subset \mathrm{LS}(\Gamma)$ *where* $\mathrm{ad} : L \to \mathrm{End}(L)$ *is the adjoint representation of* L.

5) *If* $X \in \mathrm{LS}(\Gamma)$ *and* X *is compact, then* $RX \subset \mathrm{LS}(\Gamma)$.

Proof. — (0) is obvious. 1-2-3-4 follow immediately from Proposition 2. As for 5, X compact means that the group $\{e^{tX} | t \in \mathbf{R}\}$ is relatively compact in G. Then the semi-group $\{e^{tX} | t \in \mathbf{R}_+\}$ is relatively compact in G. Its closure 'T is then a compact semi-group in G. Hence it is a group

and $T \supset \{e^{tX} | t \in \mathbf{R}\}$. Since $\bar{S}(\Gamma)$ is closed, $T \subset \bar{S}(\Gamma)$. $\bar{S}(\Gamma) \supset \{e^{tX} | t \in \mathbf{R}\}$. By Proposition 3-4) this implies that $-X \in LS(\Gamma)$.

PROPOSITION 6. — *Given a subset* Γ *of* L, *the right (resp. left) invariant polysystem* Γ_r *(resp.* Γ_ℓ*) induced by* Γ *on* G *is transitive if* $LS(\Gamma) = L$.

This proposition is a consequence of the corollary to Proposition 1 and of the assertion (0) of Proposition 5.

DEFINITION 7. — *Given a subset* Γ *of* L *we call* End (Γ) *the subset of* End (L) *of all linear endomorphisms* $A : L \to L$ *such that* $A(\Gamma) \subset \Gamma$. End $(\Gamma) \neq \varnothing$ *for* $Id_L \in$ End (Γ).

LEMMA 2. — 1) End (Γ) *is a sub-semi-group of* End (L).

2) *If* Γ *is a closed convex cone*, End (Γ) *is a closed convex cone.*

3) *If* Γ *is a closed convex cone, any projection operator* $P : L \to L$ *such that* Ker $P \subset \Gamma$ *belongs to* End (Γ).

4) *If* Γ *is* LS-*saturated (i.e.* $\Gamma = LS(\Gamma)$*) and if* H *is the integral subgroup generated by the largest Lie subalgebra contained in* Γ, *then* adj H $\subset$ End (Γ) : *that is if* $h \in H$, $h \Gamma h^{-1} \subset \Gamma$.

Remark. — Since $h^{-1} \in H$, $h^{-1} \Gamma h \subset \Gamma$ and hence $h \Gamma h^{-1} = \Gamma$.

The proof of this lemma is easy and left to the reader.

CHAPTER II

0. Some preliminary results and definitions.

Let L denote a real finite dimensional semi-simple Lie algebra, $L_c = L \otimes_R C$ its complexification, $\sigma : L_c \to L_c$ the anti-involution of L_c associated to L ($\sigma = \text{Id} \otimes$ conjugate). Let ad : $L \to \text{End}(L)$. $\text{ad}_c :$ $L_c \to \text{End}(L_c)$, denote the adjoint representations and Kil : $L \times L \to R$, $\text{Kil}_c : L_c \times L_c \to C$ the respective Killing forms $\text{Kil}_c(X,Y) = \text{Trace}$ $(\text{ad}_c X \circ \text{ad}_c Y)$.

The next proposition collects some known facts about semi-simple Lie algebra.

PROPOSITION 7. — *Let* $B \in L$ *be such that :* 1) *all nonzero eigenvalues of* $\text{ad}_c B$ *are simple.*

2) $\bigcup_{n \geqslant 1} \text{Ker } \text{ad}_c^n B$ *does not contain any ideal of* L *other than* 0.

$$(\text{ad}_c^n B = \underbrace{\text{ad}_c B \circ \text{ad}_c B \circ \ldots \circ \text{ad}_c B}_{n \ times}).$$

Let $\text{Sp}(B) = \{a | a \in C, a \neq 0, \text{Ker}(\text{ad}_c B - a\,\text{Id}) \neq \{0\}\}$,

$$L_c(a) = \text{Ker}(\text{ad}_c B - a\,\text{Id})$$

the eigenspace of a *and* $L_c(0) = \bigcup_{n \geqslant 1} \text{Ker } \text{ad}_c^n B$ *the generalized eigenspace of* 0.
Then :

1) $L_c = L_c(0) \oplus \bigoplus_{a \in \text{Sp}(B)} L_c(a)$ *(direct sum !).*

2) $L_c(0)$ *is stable under* σ *and* $\sigma\{L_c(a)) = L_c(\bar{a})$ *($\bar{a}$ complex conjugate of* a*).*

3) $L = L(0) \oplus \bigoplus \{L(a) | a \in \text{Sp}(B), \text{Im } a \geqslant 0\}$ *where*

$$L(0) = L_c(0) \cap L, \qquad L(a) = (L_c(a) \oplus L_c(\bar{a})) \cap L.$$

4) $L_c(0) = \dot{\text{Ker }} \text{ad}_c B$ *and it is a Cartan algebra of* L_c.

5) *For any a , $b \in \mathrm{Sp}(B)$ such that $a + b \to 0$ ·*

$$[L_c(a), L_c(b)] = \begin{cases} L_c(a+b) \\ \text{or} \\ \{0\} \end{cases} ,$$

$$[L(a), L(b)] = \begin{cases} L(a+b) + L(a+\bar{b}) \\ \text{or} \\ \{0\} . \end{cases}$$

6) $\mathrm{Sp}(B)$ *is invariant under conjugation and under the symmetry* $a \to - a$.

7) $[L_c(a), L_c(-a)]$ *is a one dimensional vector space contained in* $L_c(0)$ *for every* $a \in \mathrm{Sp}(B)$ *. The union*

$$\cup \{[L_c(a), L_c(-a)] \mid a \in \mathrm{Sp}(B)\}$$

generates $L_c(0)$ *as a vector space.*

Proof. — 1), 2), 3) are trivial. By Theorem 1.9.3 of [3] or Prop. 10, Chap. 7 of [1], applied to the commutative algebra **CB**, we see that $L_c(0)$ is a subalgebra, $[L_c(0), L_c(a)] \subset L_c(a)$ and $[L_c(a), L_c(b)] \subset L_c(a+b)$, where by definition $L_c(d)$ is taken to be $\{0\}$ if $d \notin \{0\} \cup \mathrm{Sp}(B)$. It follows immediately that $L_c(0)$ is orthogonal to $\bigoplus \{L_c(a) \mid a \in \mathrm{Sp}(B)\}$ with respect to Kil_c (see Prop. 1.9.5 of [3]) : Kil_c being non-degenerate, its restriction to $L_c(0) \times L_c(0)$ is non-degenerate. This implies that $L_c(0)$ is reductive (see Prop. 1.7.3 of [3] or Prop. 3.2 of [2]). Hence $L_c(0) = Z \oplus [L_c(0), L_c(0)]$ where Z is the center of $L_c(0)$.

If $\rho : L_c(0) \to \mathrm{End}(L_c)$ is the restriction to $L_c(0)$ of ad_c, the spaces $L_c(a)$, $a \in \mathrm{Sp}(B)$ are ρ-invariant. Since they are one dimensional $\rho([L_c(0), L_c(0)])L_c(a) = \{0\}$ for every $a \in \mathrm{Sp}(B)$. This shows that $[L_c(0), L_c(0)]$ is an ideal in L_c. By condition 2 on B, it is $\{0\}$. $L_c(0) = Z$. Since $B \in L_c(0)$, $Z = \mathrm{Ker} \, \mathrm{ad}_c B$ and Z is its own normalizer. Hence $L_c(0)$ is a Cartan algebra (see 1.9.1 of [3]).

Let $L_c(0)^*$ be the dual space of $L_c(0)$ and $\Delta \subset L_c(0)^*$ be the set of roots of $L_c(0)$ (see 1.9.10 of [3]). Condition 1 on B implies that the mapping $\Delta \to C$, $\alpha \to \alpha(B)$ is a bijection of Δ onto $\mathrm{Sp}(B)$ (see Thm. 1.10.2 of [3]). Thus, Theorem 1.10.2 and Proposition 1.10.7 of [3] imply 5, 6, 7.

Notation. — Because of the bijection $\Delta \to \mathrm{Sp}(B)$ $(\alpha \to \alpha(B))$ we will indifferently use the notations $L_c(\alpha)$, $L(\alpha)$ instead of $L_c(\alpha(B))$, $L(\alpha(B))$.

Remark. — If B satisfies the further condition : 3) For any α, β, $\gamma \in \Delta$ such that $\alpha + \beta + \gamma \neq 0$, then $\alpha(B) + \beta(B) + \gamma(B) \neq 0$, property 5) above can be replaced by the stronger :

8) *For any a, $b \in \mathrm{Sp}(B)$ such that $a + b \in \mathrm{Sp}(B)$,*

$$[L_c(a), L_c(b)] = L_c(a+b).$$

This is a consequence of Prop. 1.10.7-(V) of [3].

PROPOSITION 8. — *The assumptions and notations are the same as in Proposition 7. Moreover let G be a real connected semi-simple Lie group with* finite center *having L as Lie algebra.*

1) If $a \in \mathrm{Sp}(B) \cap \mathbf{R}\sqrt{-1}$ *and* $X \in L_c(a)$, $X \neq 0$, *then the element* $\sqrt{-1}[X, \sigma X]$ *belongs to L(0) and is compact.*

In other words every element in $[L(a), L(-a)]$ is compact.

2) *If* $a \in \mathrm{Sp}(B) \cap \mathbf{R}$, $X \in L(a)$, $Y \in L(-a)$ *and* $\mathrm{Kil}(X,Y) < 0$, *then* $X + Y$ *is a compact element in G.*

Proof. — Let Int(L) be the adjoint group of L. It is sufficient to consider the case $G = \mathrm{Int}(L)$. In fact if G is any connected group with Lie algebra L, let $\mathrm{adj} : G \to \mathrm{Int}(L)$ be the adjoint representation of G. adj is surjective and its kernel is the center of G. If this center is finite, adj is a finite covering map. Hence a subset $K \subset G$ is compact if and only if $\mathrm{adj}(K)$ is compact. This shows that an $X \in L$ is compact in Int(L).

An element $X \in L$ will be compact in Int(L) if and only if all eigenvalues of $\mathrm{ad}X$ are purely imaginary.

Now (see [11]) there is a unique anti-involution $\tau : L_c \to L_c$ called the Weyl anti-involution such that σ and τ commute and $\tau(L_c(a)) = L_c(-a)$ for all $a \in \mathrm{Sp}(B)$ and the form $H : L_c \times L_c \to \mathbf{C}$, $H(X,Y) = \mathrm{Kil}_c(X, \tau(Y))$ is hermitian negative-definite. Then :

$$H(\sigma X, \sigma Y) = \mathrm{Kil}_c(\sigma X, \tau\sigma(Y)) = \mathrm{Kil}_c(\sigma X, \sigma\tau(Y)) = \overline{\mathrm{Kil}_c(X, \tau(Y))},$$

$$H(\sigma X, \sigma Y) = \overline{H(X,Y)}.$$

Hence H is real negative-definite on $L \times L$.

If $Z \in L_c$ the adjoint of $\mathrm{ad}_c Z$ with respect to H is $-\mathrm{ad}_c\tau(Z)$. In case 1), let $Z = \sqrt{-1}[X, \sigma X]$. It is easy to see that $\sigma(Z) = Z$. Hence

$$Z \in L(0), \qquad \tau(X) \in L_c(-a) = L_c(\bar{a}) = \sigma L_c(a)$$

since $a \in \mathbf{R}\sqrt{-1}$. So $\tau(X) = u\sigma X$ where $u \in \mathbf{C}$.

$$\tau(Z) = -\sqrt{-1}\,[\tau(X),\tau\sigma(X)] = -\sqrt{-1}\,[\tau(X),\sigma\tau(X)] = |u|^2 Z.$$

Since $\tau^2(Z) = Z$, $\tau(Z) = Z$. Then $\mathrm{ad}\,Z$ is antisymmetric with respect to H. Its spectrum lies on $\mathbf{R}\sqrt{-1}$.

In case 2) let $Z = X + Y$. $\tau(X) \in L_c(-a)$. Since

$$\sigma\tau(X) = \tau\sigma(X) = \tau(X), \qquad \tau(X) \in L(-a).$$

Hence $\tau(X) = uY$ where $u \in \mathbf{R}$.

$$H(X,X) = \mathrm{Kil}\,(X,\tau X) = u\,\mathrm{Kil}\,(X,Y).$$

Since $H(X,X) < 0$, $u > 0$. If $u = 1$, $\tau(Z) = Z$ and as above Z is compact. If $u \neq 1$, let $T = [X,Y]$. Then (see 1.10.2 in [3]),

$$[T,X] = tX, \quad [T,Y] = -tY \quad \text{where} \quad t \in \mathbf{R}. \quad \text{Let} \quad v = \frac{1}{t}\log u \quad \text{and}$$

$Z_1 = e^{v\,\mathrm{ad}\,T}(Z)$. It is easy to see that $\tau(Z_1) = Z_1$. Hence the spectrum of $\mathrm{ad}\,Z_1$ lies in $\mathbf{R}\sqrt{-1}$. But $\mathrm{ad}\,Z_1$ is conjugate to $\mathrm{ad}\,Z$: $\mathrm{ad}\,Z_1 = e^{v\,\mathrm{ad}\,T} \circ \mathrm{ad}\,Z \circ e^{-v\,\mathrm{ad}\,T}$. This proves 2).

In the proposition and its proof $\mathrm{ad}_c : L_c \to \mathrm{End}\,(L_c)$ (resp. $\mathrm{ad} : L \to \mathrm{End}\,(L)$) denotes the adjoint representation of L_c (resp. L).

Proposition 9. — (0) $K_c + CB$ *(resp. $K + RB$) is a subalgebra, reductive in L_c (resp. L). Hence an $\mathrm{ad}_c(K_c + CB)$ (resp. $\mathrm{ad}\,(K + RB)$)-submodule of L_c (resp. L) is irreducible if and only if it is $\mathrm{ad}_c K_c$ (resp. $\mathrm{ad}\,K$)-irreducible and $\mathrm{ad}_c B$ (resp. $\mathrm{ad}\,B$)-stable. Moreover :*

(i) *Any $\mathrm{ad}_c(K_c + CB)$ (resp. $\mathrm{ad}\,(K + RB)$)-submodule M of L is the direct sum of $M \cap \mathcal{A}$ and its non trivial, $\mathrm{ad}_c(K_c + CB)$ (resp. $\mathrm{ad}\,(K + RB)$)-irreducible submodules. ($\mathcal{A}$ is the centralizer of $K_c + CB$ (resp. $K + RB$) in L_c (resp. L)).*

(ii) *For any $a \in \mathrm{Sp}\,(B)$, the $\mathrm{ad}_c K_c$ (resp. $\mathrm{ad}\,K$)-module generated by $L_c(a)$ (resp. $L(a)$) is irreducible.*

(1) *Let R_0 be a conjugation invariant subset of $\mathrm{Sp}_0(B)$ such that $\Sigma\{L(a)\,|\,a \in R_0\}$ generates K. If $a, b \in \mathrm{Sp}(B)$ and the $\mathrm{ad}\,K$-module generated by $L(a)$ contains $L(b)$, then there exist $a_1, a_2, \ldots, a_n \in R_0$ such that*

(i) *$a = b_0 + a_1 + a_2 + \cdots + a_n$ where $b_0 = b$ or $\bar{b}$,*

(ii) *if* $b_j = b_0 + a_1 + a_2 + \cdots + a_j,$ $b_j \in \mathrm{Sp}\,(B)$ *and*
$[L_c(b_{j-1}),L_c(a_j)] = L_c(b_j)$ *for* *all* $j,$ $1 \leqslant j \leqslant n.$

(2) *Let* R *be a conjugation invariant subset of* $\mathrm{Sp}_-(B) \cup \mathrm{Sp}_0(B)$ *(resp.*
$\mathrm{Sp}_+(B) \cup \mathrm{Sp}_0(B))$ *such that* $\Sigma\,\{L(a)\,|\,a \in R\}$ *generates* L_- *(resp.* L_+*). If*
$a \in \mathrm{Sp}_+(B)$ *(resp.* $\mathrm{Sp}_-(B))$ *and* a *is not maximal (resp. minimal), then there*
are $b_0,$ $b \in \mathrm{Sp}_+(B)$ *(resp.* $\mathrm{Sp}_-(B))$ *such that :*

(i) $\mathrm{Re}\,b_0 > \mathrm{Re}\,a,$ *(resp.* $\mathrm{Re}\,b_0 < \mathrm{Re}\,a).$
(ii) $L_c(b)$ *belongs to the* $\mathrm{ad}\,K_c$-*module generated by* $L_c(a).$
(iii) $b - b_0 \in R$ *and* $[L_c(b-b_0),L_c(b_0)] = L_c(b).$

Proof. $-$ (0) It is sufficient to give the proof in the complex case since, in
the real case, it is identical up to the notations. If $K_c = 0$, the statement
(0) is trivial. We shall assume that $K_c \neq 0$.

It is clear that B normalizes K_c. Since K_c is reductive in L_c,

$$K_c + CB = K_c \oplus D$$

where D is ad K_c-stable and $\dim_C D = 1$. As $[K_c, K_c + CB] \subset K_c$, D
commutes with K_c. $B = B_2 + B_1$, where B_1 is a basis vector of D and
$B_2 \in K_c$. Since $\mathrm{ad}_c B_2\,|\,K_c = \mathrm{ad}_c B\,|\,K_c$, B_2 is semi-simple. Since B_2 and
B commute, B_1 is semi-simple. This shows that $K_c + CB$ is reductive in
L_c.

Let M be any $\mathrm{ad}_c(K + CB)$ submodule of L_c. If $M \cap \mathcal{A} \neq 0$,
$M \cap \mathcal{A}$ has a unique $\mathrm{ad}_c(K_c + CB)$-stable complement $\bar{M}$:
$M = M \cap \mathcal{A} \oplus \bar{M}$. If T is a non-trivial $\mathrm{Ad}_c(K_c + CB)$-irréductible
submodule of M, then $T \subset \bar{M}$. Hence we can assume that
$M \cap \mathcal{A} = 0.$

$$M = \Sigma\,\{T\,|\,T \subset M,\, T\ \mathrm{ad}_c(K_c + CB)\text{-irreducible}\}.$$

Each T is a direct sum of eigenspaces of $\mathrm{ad}_c B$. So either $T \subset K_c$ or
$T \subset N_{+c} \oplus N_{-c}$. Hence

$$M = \Sigma\,\{T\,|\,T \subset M \cap K_c\} \oplus \Sigma\,\{T\,|\,T \subset M \cap (N_{+c} + N_{-c})\}.$$

If $T \subset K_c \cap M$, T is one of the simple summands of K_c. So the first sum
is direct. In $(N_{+c} + N_{-c})$, $\mathrm{ad}_c B$ has only simple eigenvalues. Hence the
second sum is also direct.

In particular, let M be the ad K_c-module generated by $L_c(a),$

$a \in \mathrm{Sp}(B)$. M is also an $\mathrm{ad}_c(K_c + CB)$-module and $M \cap \mathcal{A} = 0$. (ii) is an immediate consequence of (i).

(1) Let T_c be the ad K_c-module generated by $L_c(a)$. The ad K-module generated by $L(a)$ is $(T_c + \sigma(T_c)) \cap L$. Since the complex space generated by $L(b)$ in L_c is $L_c(b) \oplus L_c(\bar{b})$, at least one of $L_c(b)$, $L_c(\bar{b})$ is in T_c. (1) follows from (0), (ii).

(2) The Weyl involution commuting with σ, shows that $\Sigma\{L_c(a) \mid -a \in R\}$ generates N_{+c} (resp. N_{-c}). Since a is not maximal (resp. minimal), the ad L_+ (resp. ad L_-)-module M generated by $L(a)$ contains at least one $L(x)$ with $\mathrm{Re}\, x > \mathrm{Re}\, a$ (resp. $\mathrm{Re}\, x < \mathrm{Re}\, a$). Then the $\mathrm{ad}_c L_{+c}$ (resp. $\mathrm{ad}_c L_{-c}$)-module M_c generated by $L_c(a)$ contains an $L_c(u)$ with $\mathrm{Re}\, u > \mathrm{Re}\, a$ (resp. $\mathrm{Re}\, u < \mathrm{Re}\, a$). If T_c is the $\mathrm{ad}_c K_c$-module generated by $L_c(a)$, M_c is the ad N_{+c} (resp. ad N_{-c})-module generated by T_c. Hence there is a $b \in \mathrm{Sp}(B)$ and an $a_0 \in R$ such that $L_c(b) \subset T_c$ and

$$[L_c(b), L_c(-a_0)] \neq 0.$$

If $b_0 = b - a_0$, $b_0 \in \mathrm{Sp}(B)$, $\mathrm{Re}\, b_0 > \mathrm{Re}\, a$ and

$$[L_c(b-b_0), L_c(b_0)] = [L_c(a_0), [L_c(-a_0), L_c(b)]] = L_c(b).$$

PROPOSITION 10. — *If L_c is simple the set of all maximal (resp. minimal) $a \in \mathrm{Sp}_+(B)$ (resp. $\mathrm{Sp}_-(B)$) is the set*

$$\{a \mid a \in \mathrm{Sp}(B), \mathrm{Re}\, a = m\} \quad \{resp.\ \{a \mid a \in \mathrm{Sp}(B), \mathrm{Re}\, a = -m\}$$

where $m = \sup\{\mathrm{Re}\, a \mid a \in \mathrm{Sp}(B)\}$. The space $\Sigma\{L_c(a) \mid \mathrm{Re}\, a = m\}$ (resp. $\Sigma\{L_c(a) \mid \mathrm{Re}\, a = -m\}$) is a irreducible ad K_c-module ($K_c = K \otimes_R C$ is the complexification of K).

Proof. — Let, as before, $\Delta \subset L_c(0)^*$ be the set of roots of $L_c(0)$. On Δ we put a total order relation as follows : $\alpha \gg \beta$ if either $\mathrm{Re}\, \alpha(B) > \mathrm{Re}\, \beta(B)$ or $\mathrm{Re}\, \alpha(B) = \mathrm{Re}\, \beta(B)$ and $\mathrm{Im}\, \alpha(B) > \mathrm{Im}\, \beta(B)$. This order is compatible with the additive structure of Δ.

L_c being simple the ad_c representation is simple. By 7.1.6 and 7.2.2 of [3] there is a unique root λ which is maximal for the preceding order. For any $\alpha \in \Delta_+$, the set of positive roots, there is a sequence $\alpha_1, \ldots, \alpha_m \in \Delta_+$ (possibly with repetition) such that

$$\alpha = \lambda - \alpha_1 - \cdots - \alpha_m \quad \text{and} \quad \lambda - \alpha_1 - \cdots - \alpha_j \in \Delta_+$$

for all j, $1 \leqslant j \leqslant m$. For all $\alpha \in \Delta$ Re $\alpha(B) \leqslant$ Re $\lambda(B)$. If
Re $\alpha(B) =$ Re $\lambda(B)$ then $\sum_{j=1}^{m}$ Re $\alpha_j(B) = 0$. Since $\alpha_j \in \Delta_+$ Re $\alpha_j(B) \geqslant 0$.
Hence, Re $\alpha_j(B) = 0$ for all $1 \leqslant j \leqslant m$. Then

$$L_c(\alpha_j(B)) \subset K_c \quad \text{and} \quad L_c(\alpha_j(B)) = \prod_{j=1}^{m} \text{ad } X_j(L_c(\lambda(B)))$$

where $X_j \in L_c(-\alpha_j(B))$. As before in the proof of 1) of Prop. 9 if we choose

$$Y_j \in L_c(\alpha_j(B)) - \{0\}, \quad L_c(\lambda(B)) = \prod_{j=m}^{1} \text{ad } Y_j(L_c(\alpha_j(B))).$$

This proves the case of maximal. The case of minimal is similar.

LEMMA 3. — *If* $r \in \text{Sp}(B) \cap \mathbf{R}$ *and* r *is not maximal (resp. minimal) there is an* $a \in \text{Sp}(B)$ *such that* Re $a > r$ *(resp.* Re $a < r$*) and*

$$[L_c(-a), L_c(r)] = L_c(r-a) \neq \{0\}.$$

Proof: Let $r = \rho(B)$, $\rho \in \Delta$, set of the roots of $L_c(0)$. Then by Proposition 9, there exist $\alpha_1, \ldots, \alpha_n \in \Delta$ and $\beta \in \Delta$ such that:
1) Re $\alpha_j(B) = 0$, $2 \leqslant j \leqslant n$, Re $\alpha_1(B) < 0$. 2) $\beta + \alpha_1 + \cdots + \alpha_n = \rho$.
3) For every j, $1 \leqslant j \leqslant n$, $\beta + \alpha_1 + \cdots + \alpha_j \in \Delta$.

Let σ^*: $L_c(0)^* \to L_c(0)^*$ be the mapping $\varphi \to \overline{\varphi \circ \sigma}$. Then $\sigma^*(\Delta) = \Delta$ and « Re $\alpha(B) = 0$ » is equivalent to $\sigma^*\alpha + \alpha = 0$. Let $\langle \ \rangle$ the scalar product on $L_c(0)^*$ induced by the restriction of $\underline{\text{Kil}_c}$ to $L_c(0) \times L_c(0)$. Then for any φ, $\psi \in L_c(0)^*$, $\langle \sigma^*\varphi, \sigma^*\psi \rangle = \overline{\langle \varphi, \psi \rangle}$.

Since Re $\alpha(B) < 0$, if $\rho - \alpha_1 \in \Delta$ then we can take $a = \rho(B) - \alpha_1(B)$. If $\rho - \alpha_1 \notin \Delta$ then by 1.10.7-(ii)-(iv) of [3], $\langle \rho, \alpha_1 \rangle < 0$. Since $\rho(B)$ is real $\sigma^*\rho = \rho$ and hence $\langle \rho, \sigma^*\alpha_1 \rangle < 0$.

Now $2\langle \beta, \rho \rangle = \langle \beta + \sigma^*\beta, \rho \rangle$ and since

$$\beta = \rho - \alpha_1 - \cdots - \alpha_n, \quad \sigma^*\beta = \rho - \sigma^*\alpha_1 + \alpha_2 + \cdots + \alpha_n$$

so $2\langle \beta, \rho \rangle = \langle \rho - \alpha_1 - \sigma^*\alpha, \rho \rangle = \langle \rho, \rho \rangle - \langle \rho, \alpha_1 \rangle - \langle \rho, \sigma^*\alpha_1 \rangle$. This shows that $\dfrac{2\langle \beta, \rho \rangle}{\langle \rho, \rho \rangle} > 0$. By 1.10.7 (ii) of [3], $\beta - \rho \in \Delta$. Hence we can choose $a = \beta(B)$.

1. Some basic propositions.

We will keep the notations of part (0). In this section let Γ be a subset of L containing a one dimensional space $\mathbf{R}B$ with B strongly regular.

Due to the decomposition $L_c = L_c(0) \oplus \bigoplus\limits_{a \in \mathrm{Sp}(B)} L_c(a)$, every $A \in L_c$ can be written in a unique way $A = A_0 + \Sigma A_a$, $A_0 \in L_c(0)$, $A_a \in L_c(a)$. A_a will be called the complex a-component of A. If $A \in L$, the relation between the real a-component $A(a) \in L(a)$ and A_a is : $A(a) = A_a + \sigma A_a$ and $\sigma A_a = A_{\bar{a}}$. Hence $A_a \neq 0 \Leftrightarrow A(a) \neq 0 \Leftrightarrow A_{\bar{a}} \neq 0$.

PROPOSITION 11. — Let $A \in \Gamma$ and $A = A_0 + \Sigma A_a$ be the decomposition of A. Let $r = \sup \{\mathrm{Re}\ a \,|\, a \in \mathrm{Sp}(B), A_a \neq 0\}$ (resp. $r = \inf \{\mathrm{Re}\ a \,|\, a \in \mathrm{Sp}(B), A_a \neq 0\}$). Then :

a) if $r \notin \mathrm{Sp}(B)$, $\mathrm{LS}(\Gamma) \supset \Sigma\{L(a)\,|\,\mathrm{Re}\ a = r, A_a \neq 0\}$;

b) if $r \in \mathrm{Sp}(B)$, $\mathrm{LS}(\Gamma) \supset \mathbf{R}_+ A_r = \mathbf{R}_+ A(r)$. Also for any $a \in \mathrm{Sp}(B)$ such that $A_a \neq 0$ and $\mathrm{Re}\ a = r$,

$$\mathrm{LS}(\Gamma) \ni A_r \pm (A_a + \sigma A_a) \quad and \quad \mathrm{LS}(\Gamma) \supset A_r \pm \sqrt{-1}\,(A_a - \sigma A_a) ;$$

c) if $r \in \mathrm{Sp}(B)$ and $L(r) \subset \Gamma$ then

$$\mathrm{LS}(\Gamma) \supset \Sigma\{L(a)\,|\,A_a \neq 0, \mathrm{Re}\ a = r\} ;$$

d) if $A \in \Gamma$ and $-A \in \Gamma$, then $\mathrm{LS}(\Gamma) \supset \Sigma\{L(a)\,|\,A_a \neq 0\}$.

Proof. — It is sufficient to consider the case $r = \sup \{\mathrm{Re}\ a \,|\, A_a \neq 0\}$. The case of inf is similar or it can be deduced from the sup case using Weyl's involution.

By Lemma 2-4) of Chapter I, $e^{v\,\mathrm{ad}\,B}(A) \in \mathrm{LS}(\Gamma)$ for all $v \in \mathbf{R}$. Since $\mathrm{LS}(\Gamma)$ is a convex closed cone, for any $T > 0$ and any non-negative continuous function g on the real line :

$$\frac{1}{T} \int_T^{2T} e^{-rv}\, g(v) e^{v\,\mathrm{ad}B}(A)\, dv \in \mathrm{LS}(\Gamma).$$

If $\lim\limits_{T \to +\infty} \dfrac{1}{T} \displaystyle\int_T^{2T} e^{-rv}\, g(v) e^{v\,\mathrm{ad}B}(A)\, dv$ exists, it belongs to $\mathrm{LS}(\Gamma)$.

Calling $e^{v\,\mathrm{adB}}(A)$, $A(v)$, for simplicity

$$A(v) = A_1(v) + A_2(v) \quad \text{where} \quad A_1(v) = \sum_{\mathrm{Re}\,a = r} e^{av} A_a,$$

$A_2(v) = \sum_{\mathrm{Re}\,a < r} e^{av} A_a$. Let

$$\varepsilon = \inf\{r - \mathrm{Re}\,a \mid a \in \mathrm{Sp}(B), \mathrm{Re}\,a < r\}.$$

Then $\varepsilon > 0$ and $e^{-rv} A_2(v) = O(e^{-\varepsilon v})$ as $v \to +\infty$.

If we take g a quasi-periodic positive function, then :

$$\lim_{T \to \infty} \frac{1}{T} \int_T^{2T} e^{-rv} g(v) A_1(v)\, dv = \sum_{\mathrm{Re}\,a = r} c(\mathrm{Im}\,a, g) A_a$$

$$\frac{1}{T} \int_T^{2T} e^{-rv} g(v) A_2(v)\, dv = O(e^{-\varepsilon T})$$

where $c(\lambda, g) = \lim_{T \to +\infty} \int_T^{2T} e^{i\lambda v} g(v)\, dv$ is a Fourier coefficient of g.

This shows that $\sum_{\mathrm{Re}\,a = r} c(\mathrm{Im}\,a, g) A_a \in \mathrm{LS}(\Gamma)$, for any quasi-periodic positive g.

If we choose $g = 1 + \eta \cos(\mu v)$ (resp. $g = 1 + \eta \sin(\mu v)$) with $|\eta| \leqslant 1$ and $\mu \in \mathbf{R}_+$ we get :

$$c(g, \lambda) = \begin{cases} 1 & \lambda = 0 \\ \dfrac{1}{2}\eta & |\lambda| = \mu \\ 0 & \text{otherwise} \end{cases} \qquad \text{resp. } c(g, \lambda) = \begin{cases} 1 & \lambda = 0 \\ \dfrac{1}{2}\eta\sqrt{-1} & \lambda = \mu \\ -\dfrac{1}{2}\eta\sqrt{-1} & \lambda = -\mu \\ 0 & \text{otherwise.} \end{cases}$$

Hence if $r \notin \mathrm{Sp}(B)$, taking $\mu = \mathrm{Im}\,a$ for $a \in \mathrm{Sp}(B)$ with $\mathrm{Re}\,a = r$ we get that $\eta(A_a + \sigma A_a)$ and $\sqrt{-1}\,\eta(A_a - \sigma A_a)$ belong to $\mathrm{LS}(\Gamma)$ for all η $|\eta| \leqslant 1$. Since $\{A_a + \sigma A_a, \sqrt{-1}(A_a - \sigma A_a)\}$ is a basis of $L(a)$ if $A_a \neq 0$, we get that $L(a) \subset \mathrm{LS}(\Gamma)$. This is a).

If $r \in \mathrm{Sp}(B)$, taking $\eta = 0$ we get $A_r \in \mathrm{LS}(\Gamma)$. Then taking $\mu = \mathrm{Im}\, a$, $a \in \mathrm{Sp}(B)$, $\mathrm{Re}\, a = r$, as above, we get the rest of $b)$. $c)$ and $d)$ are consequences of $a)$ and $b)$ and of the fact that $\mathrm{LS}(\Gamma)$ is a closed convex cone.

LEMMA 5. — *Let r be a positive real number set*

$$E = \Sigma\{L(a)\,|\,|\mathrm{Re}\, a| > r, a \in \mathrm{Sp}(B)\}$$

and

$$F = \Sigma\{L(a)\,|\,|\mathrm{Re}\, a| \leqslant r, a \in \mathrm{Sp}(B)\} + L(0).$$

Assume that $\mathrm{LS}(\Gamma) \supset E$. *Let* $P : L \to L$ *be the projection with kernel* E *and image* F. *Then for any* $Z \in E$, $P \circ \mathrm{ad}\, Z \circ P \in \mathrm{End}\,(\mathrm{LS}(\Gamma))$.

Proof. — By Lemma 2-1), 3) and Prop. 5-4), for any $Z \in E$, $P \circ e^{\mathrm{ad}\, Z} \circ P \in \mathrm{End}\,(\Gamma)$.

Choose $Z \in L(a)$, $a \in \mathrm{Sp}(B)$ $|\mathrm{Re}\, a| > r$. Then by Prop. 7-5), for any integer $n \geqslant 1$, and any $x \in \mathrm{Sp}(B)$:

$$\mathrm{ad}^n Z(L(x)) \subset \sum_{p+q=n} L(x + pa + q\bar{a}).$$

If $|\mathrm{Re}\, x| < r$, $\mathrm{Re}\,(x + pa + q\bar{a}) > (n-1)r$ if $\mathrm{Re}\, a > r$ and $\mathrm{Re}\,(x + pa + q\bar{a}) < -(n-1)r$ if $\mathrm{Re}\, a < -r$, for all $n \geqslant 2$. As

$$e^{\mathrm{ad}\, Z} = \sum_{n=0}^{\infty} \frac{1}{n!}\, \mathrm{ad}^n Z, \quad P \circ e^{\mathrm{ad}\, Z} \circ P = P + P \circ \mathrm{ad}\, Z \circ P.$$

Now for any $Z \in L(a)$ and any $v \in \mathbf{R}$,

$$P + vP \circ \mathrm{ad}\, Z \circ P = P \circ e^{\mathrm{ad}\, vZ} \circ P \in \mathrm{LS}(\Gamma).$$

Then $\quad P \circ \mathrm{ad}\, Z \circ P = \lim\limits_{v \to +\infty} \dfrac{1}{v}(P + vP \circ \mathrm{ad}\, Z \circ P) \in \mathrm{LS}(\Gamma).$ Since $\mathrm{End}\,(\mathrm{LS}(\Gamma))$ is a convex cone and any $Z \in E$ is a sum of $Z(a)$, $Z(a) \in L(a)$, with $\mathrm{Re}\, a > r$, we get the lemma.

PROPOSITION 12. — *We use the same notations and assumptions as in Lemma 5.*

1) *Given any* $a \in \mathrm{Sp}(B)$ *with* $|\mathrm{Re}\, a| \leqslant r$, *assume there is a* $b \in \mathrm{Sp}(B)$ *and* *an* $A \in \mathrm{LS}(\Gamma)$ *such that* $|\mathrm{Re}\, b| > r$, $a - b \in \mathrm{Sp}(B)$, $[L_c(a-b), L_c(b)] \neq 0$ *and* $A_{a-b} \neq 0$. *Then* $L(a) \subset \mathrm{LS}(\Gamma)$.

2) *Given any* $e \in \mathrm{Sp}(B)$ *with* $|\mathrm{Re}\, e| \leqslant r$, *assume there is an* $f \in \mathrm{Sp}(B)$ *and an* $A \in LS(\Gamma)$ *such that* $A_e \neq 0$, $|\mathrm{Re}\, f| > r$ *and* $[L_c(e), L_c(-f)] \neq 0$. *Then* $L(e) \subset LS(\Gamma)$.

Proof of 1). — If $|\mathrm{Re}(a-b)| > r$, $L(a-b) \subset LS(\Gamma)$ and by prop. 7-5 $L(a) \subset [L(a-b), L(b)] \subset LS(\Gamma)$. If $|\mathrm{Re}(a-b)| \leqslant r$, apply lemma 5. For any $Z \in L(b)$, $P[Z, P(A)] \in LS(\Gamma)$. From Proposition 11-d) it follows that $LS(\Gamma) \supset \Sigma \{L(x) \,|\, (P[Z,P(A)])_x \neq 0\}$. Let X be a generator of $L_c(b)$. Then $\quad L(b) = \{uX + \bar{u}\sigma X \,|\, u \in \mathbf{C}\}$. Taking $\quad Z = uX + \bar{u}\sigma X$, $(P[Z,P(A)])_a = u[X, A_{a-b}] + \bar{u}[\sigma X, A_c]$ where $c = a - \bar{b}$ and $A_c = 0$ if $c \notin \mathrm{Sp}(B)$.

From the Assumption and Prop. 7-5) $[X, A_{a-b}] \neq 0$. As $\dim L_c(a) = 1$, $[\sigma X, A_c] = z[X, A_{a-b}]$ where $z \in \mathbf{C}$. Hence

$$(P[Z,P(A)])_a = (u + z\bar{u})[X, A_{a-b}].$$

For a suitable choice of u, $(P[Z,P(A)])_a \neq 0$. This proves 1).

Proof of 2). — Since $[L_c(e), L_c(-f)] \neq 0$, $L_c(e-f) \neq 0$. If $|\mathrm{Re}(e-f)| \leqslant r$, part 1 applied to $a = e - f$ and $b = -f$ shows that $L(e-f) \subset LS(\Gamma)$. If $|\mathrm{Re}(e-f)| > r$, $L(e-f) \subset LS(\Gamma)$ already.

By prop. 5-3), $|L(e-f), L(f)| \subset LS(\Gamma)$. By [4]-20.5.12,

$$[L_c(e-f), L_c(f)] = [[L_c(e), L_c(-f)], L_c(f)] \neq 0.$$

Hence $L(e) \subset [L(e-f), L(f)] \subset LS(\Gamma)$.

LEMMA 6. — *Assume that* a, $b \in \mathrm{Sp}(B)$, $L(a) \subset LS(\Gamma)$ *and* $[L_c(b), L_c(-a)] \neq 0$. *If there is an* $U \in LS(\Gamma)$ *such that* $U_{b-a} \neq 0$, *then there is an* $A \in LS(\Gamma)$ *such that* $A_b \neq 0$.

Proof. — Let a be complex first. Choose a basis vector X of $L_c(a)$. Call $Z : \mathbf{C} \to L(a)$ the function $Z(v) = vX + \bar{v}\sigma X$. By Proposition 5-4) and Lemma 2-4), $A(v) = e^{\mathrm{ad}\, Z(v)}(A)$ belongs to $LS(\Gamma)$ for all $v \in \mathbf{C}$.

Since $Z(v)$ is nilpotent, the function $A(v) : \mathbf{C} \to L$ is polynomial in $\mathrm{Re}\, v$ and $\mathrm{Im}\, v$. The homogeneous component of degree 1 of $A(v)_a$ is $v[X, A_{b-a}] + \bar{v}[\sigma X, A_{b-\bar{a}}]$. By the assumption this component is not zero. Hence the polynomial $A(v)_a$ is not identically 0. So for an open dense set of v's in $\mathbf{C}$, $A(v)_a \neq 0$.

2. Proof of the theorems.

Theorem 0 follows from Theorem 1 and Proposition 10.

Proof of Theorem 1. — We can assume that $\Gamma = LS(\Gamma)$. Denote by $R_+(\Gamma)$ (resp. $R_-(\Gamma),R_0(\Gamma)$) the set of all

$$a \in Sp_+(B) \cup Sp_0(B) \quad (\text{resp. } Sp_-(B) \cup Sp_0(B),Sp_0(B))$$

such that there exists an $A \in \Gamma$ with $A(a) \neq 0$.

The proof will be an induction. We shall assume that there is a positive r such that $\Gamma \supset \Sigma\{L(a) \,|\, |\text{Re } a| > r\}$ and show that $\Gamma \supset \Sigma\{L(a) \,|\, |\text{Re } a| = r\}$. As before let $P : L \to L$ be the projection with kernel $\Sigma\{L(a) \,|\, |\text{Re } a| > r\}$ and image $\Sigma\{L(a) \,|\, |\text{Re } a| \leqslant r\} + L(0)$. For the proof we need two lemmas :

LEMMA 7. — *Assume that either* $r \notin Sp(B)$ *or* $r \in Sp(B)$ *and* $L(r) \subset \Gamma$ *or there is no* $A \in \Gamma$ *with* $A(r) \neq 0$. *Let* $a \in Sp_+(B)$ *and* $\text{Re } a = r$. *If the* ad K*-module generated by* $L(a)$ *contains an* $L(b)$ *such that* $L(b) \subset \Gamma$, *then* $L(a) \subset \Gamma$.

LEMMA 8. — *Assume that* $r \in Sp_+(B)$ *and that either* r *is maximal or it is not but there is an* $A \in \Gamma$ *such that* $A(r) \neq 0$. *Then* $L(r) \subset \Gamma$.

Proof of the induction step. — It is sufficient to show that $\Gamma \supset \Sigma\{L(a) \,|\, \text{Re } a = r\}$. The case of $-r$ is symmetric.

Lemma 8 shows that either $r \notin Sp(B)$ or $L(r) \subset \Gamma$ or there is no $A \in \Gamma$ such that $A(r) \neq 0$. Take an $a \in Sp_+(B)$ with $\text{Re } a = r$.

Case 1. — a is maximal. By Proposition 11-a), b), c),

$$\Gamma \supset \Sigma\{L(x) \,|\, \text{Re } x = r \text{ and } \exists A \in \Gamma \text{ with } A_x \neq 0\}.$$

By condition 3 of Thm. 1, the ad K-module generated by $L(a)$ contains an $L(b)$ such that $L(b) \subset \Gamma$. By Lemma 7, $L(a) \subset \Gamma$.

Case 2. — a is not maximal. By prop. 9-2 there are b_0, $b \in Sp_+(B)$ such that : 1) $\text{Re } b_0 > \text{Re } b = \text{Re } a = r$, 2) $L(b)$ is contained in the ad K-module generated by $L(a)$. 3) $b - b_0 \in R_-(\Gamma)$.

This last condition shows that there is an $A \in \Gamma$ such that $A_{b-b_0} \neq 0$. Then prop. 12-1 implies that $L(b) \subset \Gamma$. By lemma 7, $L(a) \subset \Gamma$.

The induction shows that $\Sigma\{L(a)\,|\,\mathrm{Re}\,a \neq 0\} \subset \Gamma$. Hence

$$\Sigma\{[L(a),L(-a)]\,|\,\mathrm{Re}\,a \neq 0\} \subset \Gamma.$$

Now $L = K + \Sigma\{L(a)+[L(a),L(-a)]\,|\,|\mathrm{Re}\,a| \neq 0\}$. Let $Q : L \to L$ be a projection with kernel in Γ and image in K, $K = H_K \oplus \sum\limits_{\mathrm{Re}\,a=0} L(a)$ where

$$H_K = L(0) \cap K = \sum_{\mathrm{Re}\,a=0} [L(a),L(-a)].$$

By Proposition 8-1), $[L(-a),L(-a)]$ is made up of compact elements for every $a \in \mathrm{Sp}_0(B)$. Since H_K is a commutative algebra, all elements in H_K are compact. Let $A \in \Gamma$. Then $Q(A) \in \Gamma$. $Q(A) = A_0 + \sum\limits_{\mathrm{Re}\,a=0} A(a)$, $A_0 \in H_K$, $A(a) \in L(a)$. Proposition 11-b) shows that $A_0 \in \Gamma$. Since A_0 is compact, $RA_0 \subset \Gamma$. By proposition 11-b), c), $RA(a) \subset \Gamma$. In particular for every $a \in R_0(\Gamma)$, $L(a) \subset \Gamma$. Since $\Sigma\{L(a)\,|\,a \in R_0(\Gamma)\}$ generates K, $K \subset \Gamma$. This shows that $\Gamma = L$ and Γ is transitive.

Proof of Lemma 7. — By Proposition 9-2) there is a sequence $b_0, b_1, b_2, \ldots, b_n$ such that : 1) $b_n = a$, 2) $b_{j+1} - b_j = a_j \in R_0(\Gamma)$ for $0 \leqslant j \leqslant n - 1$, 3) $L_c(b_{j+1}) = [L_c(b_j),L_c(a_j)]$, 4) $b_0 = b$ or $b_0 = \bar{b}$. We show by induction on j that $L(b_j) \subset \Gamma$.

For $j = 0$, $L(b_0) = L(b) \subset \Gamma$. Since $a_j \in R_0(\Gamma)$, there is a $U \in \Gamma$ such that $U_{a_j} \neq 0$. If $L(b_j) \subset \Gamma$, by Lemma 6 there is an $A \in \Gamma$ with $A_{b_{j+1}} \neq 0$. By Proposition 11-a), b), c), since $\mathrm{Re}\,b_{j+1} = r$, $L(b_{j+1}) \subset \Gamma$.

Proof of Lemma 8. — If r is maximal, by condition 4 of Theorem 1 there are A_1, $A_2 \in \Gamma$ such that Trace $(\mathrm{ad}\,A_1(r) \circ \mathrm{ad}\,A_2(-r)) < 0$. Now $P(A_1)$, $P(A_2) \in \Gamma$. Hence by Proposition 11-b) $A_1(r)$, $A_2(-r) \in \Gamma$. For any real positive u, v, $uA_1(r) + vA_2(-r) \in \Gamma$ and it is a compact element by Proposition 8-2). Hence $\Gamma \supset \{uA_1(r)+vA_2(-r)\,|\,u,v \in \mathbf{R},uv > 0\}$. Since Γ is convex, $\Gamma \supset L(r) + L(-r)$.

If r is not maximal, by Lemma 3 there is an $a \in \mathrm{Sp}(B)$, $\mathrm{Re}\,a > r$ such that $[L_c(r),L_c(-a)] \neq 0$. By Proposition 12-2 applied to $e = r$, $f = a$, $L(r) \subset \Gamma$.

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