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CLOSED CONVEX HULL OF SET OF MEASURABLE FUNCTIONS, RIEMANN-MEASURABLE FUNCTIONS AND MEASURABILITY OF TRANSLATIONS

by Michel TALAGRAND (*)

1. Introduction.

Let G be a compact group. For $f \in L^\infty(G)$, and $t \in G$, let $L_t f \in L^\infty(G)$ be given by $L_t f(x) = f(tx)$. It has been conjectured by G.A. Edgar and J. Rosenblatt that f is equivalent to a Riemann-measurable (i.e., is continuous at each point of a set of full Haar measure) if and only if the map $t \longrightarrow L_t f$ from G into $L^\infty(G)$ is scalarly measurable, i.e., for $\varphi \in L^\infty(G)^*$ the map $t \longrightarrow \varphi(L_t f)$ is measurable for the Haar measure.

We prove here that under Martin's axiom this conjecture holds. Generalizing this result we characterize those $f \in L^\infty(G)$ for which $t \longrightarrow L_t f$ is scalarly measurable when G is assumed to be only locally compact. We also show in both cases the surprising fact that the map $t \longrightarrow L_t f$ is scalarly measurable as soon as $t \longrightarrow \theta(L_t f)$ is measurable for each character (i.e., multiplicative linear functional) on $L^\infty(G)$. Using a beautiful result of D.H. Fremlin we also show that the map $t \longrightarrow L_t f$ is Borel measurable if and only if f is uniformly continuous.

These results are based on powerful ideas of measure theory. Among the new measure-theoretic results proven here, the following seems of independent interest: Under Martin's axiom, if a pointwise

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bounded set of measurable functions on a compact metric space with a Radon measure is pointwise separable and pointwise relatively compact in the set of measurable functions, the same is true of its convex hull.

The author thanks G.A. Edgar and J. Rosenblatt for bringing their conjecture to his attention, and for communication of partial results. The author is also indebted to J. Bourgain for bringing (independently) the problem of the convex hull of compact sets of measurable functions to his attention.

2. Preliminaries.

Let (Ω, Σ, μ) be a complete probability space. We denote by μ^* the outer measure, by $M(\Sigma)$ the set of all measurable real-valued functions on Ω.

The pointwise convergence topology on the set of all real-valued functions on a set X is denoted by τ_p. The τ_p-closed convex hull of a set T of functions is denoted by $\overline{c}(T)$.

We need to consider two set-theoretical assumptions. The strongest one is CH, the continuum hypothesis, i.e., the assumption that the first uncountable cardinal has the power of continuum. The second one is MA, Martin's axiom, whose nature is more complicated, and which is explained in [9]. This axiom will be used (through Lemma 1, but not directly in this work) through the following consequence, more familiar to the measure theorist: If Σ is the completion of a countably generated algebra a union of less than the continuum of negligible sets is negligible.

The following lemma is the cornerstone of this work. It is used in our two main results, theorems 5 and 17. It is due to D.H. Fremlin, and independently, to J. Bourgain and F. Delbaen.

LEMMA 1. – *Let* $(\Omega_i, \Sigma_i, \mu_i)_{i \leq q}$ *be complete probability spaces. Suppose that for each* $s = (s_1, \ldots, s_q)$ *is given a measurable set* A_s *of* $\Omega_1^{s_1} \times \ldots \times \Omega_q^{s_q}$, *of measure* 1 *for the product measure. Suppose either*

a) *Each* Σ_i *is the completion of a countably generated algebra, and* MA *holds.*

b) *Each* Σ_i *is the completion of an algebra of cardinality* $\leqslant$ card $\mathbf{R}$, *and* CH *holds.*

Then there exist sets $U_i \subset \Omega_i$, $\mu_i^*(U_i) = 1$, *such that for all* $s = (s_1, \ldots, s_q)$, *and all families* $(u_{i,j})_{i \leqslant q, j \leqslant s_q}$ *of distinct elements of* U_i, *we have* $(u_{1,1}, \ldots, u_{1,s_1}, u_{2,1}, \ldots, u_{2,s_2}, \ldots, u_{q,n_q}) \in A_s$.

Proof. – In either case, Σ_i is the completion for μ_i of a σ-algebra Σ_i' which has the power of continuum. Let γ be the first ordinal of cardinality of the continuum, and let $(V_\alpha^i)_{\alpha<\gamma}$ be an enumeration of the sets of positive measure of Σ_i'.

It is clear that one can assume each A_s invariant for each $i \leqslant q$ by permutation of the componants which belong to Ω_i. For finite sets $F_i = (u_{i,j})_{j \leqslant r_i}$ and s with $r_i \leqslant s_i$ for $i \leqslant q$, let

$$A_s(F_1, \ldots, F_q) = \{(u_{1,r_i+1}, \ldots, u_{1,s_1}, u_{2,r_2+1}, \ldots, u_{q,r_q+1}, \ldots, u_{q,s_q}) \\ \in \Omega_1^{s_1-r_1} \times \ldots \times \Omega_q^{s_q-r_q} \, ; \, (u_{1,1}, \ldots, u_{1,s_1}, u_{2,1}, \ldots, u_{q,1}, \ldots, u_{q,s_q}) \\ \in A_s\}$$

with the convention that if $r_i = s_i$, we omit the factor Ω_i. By induction over $\beta < \gamma$ we construct $(u_\beta^i)_{\beta<\gamma}$ such that

i) For $i \leqslant q$, the $(u_\beta^i)_\beta$ are distinct

ii) $u_\beta^i \in V_\beta^i$

iii) For each finite sets $F_i \dot{\subset} \{u_\alpha^i\}_{\alpha \leqslant \beta}$ and for all s with $s_i \geqslant$ card F_i for all i, we have

$$\bigotimes_{i \leqslant q} \mu_i^{\otimes s_i - \text{card } F_i)} (A_s(F_1, \ldots, F_q)) = 1$$

with the convention that if $s_i =$ card F_i for all i, this means that, if $F_i = (u_{i,j})_{j \leqslant s_i}$ we have

$$(u_{1,1}, \ldots, u_{1,s_1}, u_{2,1}, \ldots, u_{q,1}, \ldots, u_{q,s_q}) \in A_s .$$

This construction is straightforward since condition a) or b) and Fubini's theorem ensures that if $(u_\alpha^i)_{\alpha<\beta}$ are constructed, $(u_\beta^i)_{i \leqslant q}$ can be picked anywhere in a subset of $\prod_{i \leqslant q} V_\beta^i$ of full measure (hence non-empty).

It is then clear that $U_i = \{u_\beta^i\}_{\beta<\gamma}$ satisfy the requirements.

Q.E.D.

For two subsets A, B of Ω, not necessarily measurable, we write $A \overset{\text{ess}}{\subset} B$ if $B \backslash A$ is negligible.

We now turn to the definition and elementary properties of Riemann measurability. Let X a compact space and μ a Radon measure on it. A function $f: X \longrightarrow \mathbf{R}$ will be said Riemann-measurable if it is bounded and if its set of points of discontinuity is negligible. The following is elementary.

LEMMA 2. – *Let* $f: X \longrightarrow \mathbf{R}$ *be a bounded function. The following are equivalent.*

a) *f is Riemann-measurable.*

b) *For all* $\epsilon > 0$, *there exist* $f_1, f_2, X \longrightarrow \mathbf{R}$ *continuous such that* $f_1 \leqslant f \leqslant f_2$ *and* $\int (f_2 - f_1)\, d\mu \leqslant \epsilon$.

Proof. – $a \Longrightarrow b$. For $x \in X$, let

$$\mathrm{Os}_x(f) = \mathrm{Inf}\,\{\mathrm{Sup}\,\{f(x_2) - f(x_1)\,;\, x_1, x_2 \in V\}\,;\; V \text{ neighborhood of } x\}.$$

Let $\eta > 0$. The set $F = \{x \in X\,;\, \mathrm{Os}_x(f) \geqslant \eta\}$ is negligible and closed. Let U and V be open, with $F \subset U \subset \overline{U} \subset V$ and $\mu(V) \leqslant \eta$. On $Y = X \backslash U$ define

$$g_1(x) = \liminf_{y \to x} f(y), \quad g_2(x) = \limsup_{y \to x} f(y).$$

Then $g_2 - g_1 \leqslant \eta$, so $g_2 - \eta < g_1 + \eta$. Since $g_2 - \eta$ is upper semi-continuous, and $g_1 + \eta$ lower semi-continuous, there exists $h: Y \longrightarrow \mathbf{R}$, continuous, with

$$g_2 - \eta \leqslant h \leqslant g_1 + \eta, \text{ i.e., } h - \eta \leqslant g_1 \leqslant f \leqslant g_2 \leqslant h + \eta$$

on Y. Hence there exist continuous functions $f_1, f_2: X \longrightarrow \mathbf{R}$, with $f_1 = \inf\{f(x)\,;\, x \in X\}$ on $\overline{U}$, $f_2 = \sup\{f(x)\,;\, x \in X\}$ on $\overline{U}$, $f_1 \leqslant h - \eta$, $h + \eta \leqslant f_2$ on Y, $f_1 = h - \eta$, $f_2 = h + \eta$ on $Y \backslash V$. So $f_1 \leqslant f \leqslant f_2$, and $\int (f_2 - f_1)\, d\mu \leqslant 2\eta\,(\|f\|_\infty + |\mu|(X))$, which is arbitrarily small.

$b \Longrightarrow a$ since $\{x\,;\, \mathrm{Os}_x(f) \geqslant \eta\} \subset \{x\,;\, f_2(x) - f_1(x) \geqslant \eta\}$ has a measure at most ϵ/η for all ϵ, the former set is negligible.

For a measurable set A, let ess cl A be the smallest closed set such that $A \overset{\text{ess}}{\subset} \text{ess cl } A$, i.e., the set of points x such that each neighborhood of x meets A in a set of positive measure. (This set is also the support of the restriction of μ to A.)

LEMMA 3. – *Let* $f \in L^\infty(\mu)$. *The following are equivalent.*

a) *There exists a Riemann-measurable function* h *in the class of* f.

b) *For all real numbers* $a < b$, *if* $M_a = \text{ess cl}\,\{f \leqslant a\}$, $N_b = \text{ess cl}\,\{f \geqslant b\}$, *then* $\mu(M_a \cap N_b) = 0$.

Proof. – $a \Longrightarrow b$ is clear since no point of continuity of h belongs to $M_a \cap N_b$.

$b \Longrightarrow a$ We can suppose that X is the support of μ. Define $h(x) = \text{Inf}\,\{t \in \mathbf{R}\,;\, f \leqslant t \text{ a.e. on } V,\ V \text{ neighborhood of } x\}$.

For two rationals $r < s$, let $F_{r,s} = \overline{\{h \leqslant r\}} \cap \overline{\{h \geqslant s\}}$. Let $r < a < b < s$. If $x \in \overline{\{h \leqslant r\}}$, each neighborhood V of x contains a point $y \in \{h \leqslant r\}$, and hence an open W on which $h \leqslant a$ a.e. so $W \overset{\text{ess}}{\subset} \{f \leqslant a\}$, so $x \in M_a$. If $x \in \overline{\{h \geqslant s\}}$, each neighborhood V of x contains a point $y \in \{h \geqslant s\}$, and hence a set of positive measure on which $f \geqslant b$, so $x \in N_b$. So $F_{r,s} \subset M_a \cap N_b$, and hence $\mu(F_{r,s}) = 0$ by hypothesis.

Since h is continuous in the complement of $\cup_{r,s} F_{r,s}$ h is Riemann-measurable. Moreover, since $\overline{\{h \leqslant r\}} \subset M_a$, we have $\mu(\{h \leqslant r\} \cap N_b) = 0$ for $b > r$. It follows easily that h is in the class of f.

3. Convex hull of sets of measurable functions.

Before we start explaining our result, we give a brief survey of known results connected to it. A general problem is the following. Let X a set, and T be a uniformly bounded (to simplify) set of functions on T, which is compact for τ_p. Suppose T consists of functions with some regularity. Let $\overline{c}(T)$ be the τ_p-closure of the convex hull of T. Does it follow that $\overline{c}(T)$ consists of regular functions ? The following results are known :

a) If X is compact, and T consists of continuous functions, then $\overline{c}(T)$ also consists of continuous functions. (This is essentially the classical Krein's theorem).

b) If X is Polish, and T consists of functions of first Baire class, so does $\overline{c}(T)$. [1].

c) If X is Polish, and T is *separable,* and consists of Baire fonction of class $\leqslant \alpha$, then $\overline{c}(T)$ consists of functions of Baire class $\leqslant \alpha + 1$ [1]. (It is not known if $\overline{c}(T)$ always consists of functions of class $\leqslant \alpha$.)

d) If CH holds, even if $X = [0,1]$, there exists a T which consists of functions of Baire class two, but such that $\overline{c}(T)$ contains a non-Lebesgue measurable function.

e) If MA holds, there exists a probability space (Ω, Σ, μ) (large in some sense) and a separable compact $T \subset M(\Sigma)$ such that $\overline{c}(T) \not\subset M(\Sigma)$. [4]. (D.H. Fremlin told us he can modify this example to remove MA.)

It is interesting to see how the situation deteriorates when one increases the generality of the class of functions considered. The examples d) and e) does not tell us what happens when both $T \subset M(\Sigma)$ is separable and Ω is not pathological. For further applications, we need an assumption on T which is less restrictive than separability.

DEFINITION 4. – a) *A family* U *of sets of* Σ *will be called* nice *if for all* n, *and for all measurable set* $H \subset \Omega^n$, *there exists* $H' \subset \Omega^n$, $H' \overset{ess}{\subset} H$ *such that for all* $A_1, \ldots, A_n \in U$, $\prod_{i=1}^{n} A_i \overset{ess}{\subset} H$ *implies* $\prod_{i=1}^{n} A_i \subset H'$.

b) *A family of functions* $T \subset M(\Sigma)$ *will be called* nice *if the family of sets* $\{f > a\}$, $\{f < a\}$ *for* $f \in T$, $a \in \mathbf{R}$ *is* nice (T *is not assumed to be* τ_p*-compact.*)

Examples. –

1) Countable families are nice.

2) If U_1 is nice, and each member of U is a union of members of U_1, U is nice.

3) It is easy to see that if μ is a Radon measure on a compact space S, the family of open sets is nice, and the family of continuous functions is nice.

4) An important example of nice family is given by proposition 10 in the next section.

Let $T \subset M(\Sigma)$ a set which is not necessarily τ_p-compact, and $\overline{T}$ its τ_p-closure. We want an estimate of roughly speaking "how far $\overline{T}$ is to be contained in $M(\Sigma)$". For $f : \Omega \longrightarrow \mathbf{R}$, let $f^* = \operatorname{ess\,inf}\{h : \Omega \longrightarrow \mathbf{R} ; h \geqslant f,\ h \text{ measurable}\}$, $f_* = -(-f)^*$. Since f^* is valued in $\mathbf{R} \cup \{\infty\}$, $f^* - f_*$ is well defined, and if $\int^*$ denotes the upper integral of positive measurable functions, $\int^* (f^* - f_*)\, d\mu$ gives a measure of how far f is to be measurable. In fact, if $\int^* (f^* - f_*)\, d\mu = 0$, then $f^* = f_* = f$ a.e. and hence f is measurable, the converse being obvious.

Define $O(T) = \sup_{f \in \overline{T}} \left(\int^* (f^* - f_*)\, d\mu\right)$. Then $\overline{T} \subset M(\Sigma)$ if and only if $O(T) = 0$.

THEOREM 5. – *Let* (Ω, Σ, μ) *be a complete probability space. Suppose either*

a) MA *holds and* Σ *is the completion of a countably generated algebra*

b) CH *holds and* Σ *is the completion of a* σ*-algebra* Σ_0 *which has the power of continuum.*

Let $T \subset M(\Sigma)$ *be a nice family of functions such that for all* $\omega \in \Omega$, $g(\omega) = \sup\{|f(\omega)| ; f \in T\} < +\infty$. *Then*

α) *We have* $O(c(T)) \leqslant O(T)$, *where* $c(T)$ *is the convex hull of* T.

β) *If* $\overline{T} \subset M(\Sigma)$ *then* $\overline{c}(T) \subset M(\Sigma)$.

Remark that T is not assumed to be τ_p-compact.

We need a simple lemma before starting the proof. A sequence (Y_r) of measurable sets will be said *irreducible* if for all subsequence (r_ℓ) we have

$$\mu(\limsup_r Y_r) = \mu(\limsup_\ell Y_{r_\ell})$$

where $\limsup_r Y_r = \bigcap_p \bigcup_{r \geqslant p} Y_r$.

LEMMA 6. – a) *Each sequence of measurable sets contains a subsequence which is irreducible*

b) *Let* $(\Omega_i, \Sigma_i^*, \mu_i)_{i\leqslant q}$ *be probability spaces. For* $i \leqslant q$, *let* $(Y_{i,r})_r$ *be an irreductible sequence of* Ω_i. *Then*

$$\limsup_r \prod_{i\leqslant q} Y_{i,r} \overset{\text{ess}}{=} \prod_{i\leqslant q} \limsup_r Y_{i,r}.$$

Proof. – a) It is standard. We define by induction sequences $(t_\ell^p)_\ell$ of integers, such that t_ℓ^{p+1} is a subsequence of t_ℓ^p, and $\mu(\limsup_\ell Y_{t_\ell^{p+1}}) \leqslant 2^{-p} + \operatorname{Inf}\{\mu(\limsup_s Y_{q_s}) ; (q_s)$ is a subsequence of $t_\ell^p\}$.

Let $q_\ell = t_\ell^\ell$. For each subsequence q'_ℓ of q_ℓ, since for each $p, (q'_\ell)_{\ell\geqslant p}$ is a subsequence of $(t_\ell^p)_\ell$, we have

$$\mu(\limsup_\ell Y_{q_\ell}) \leqslant \mu(\limsup_\ell Y_{t_\ell^{p+1}}) \leqslant 2^{-p} + \mu(\limsup_\ell Y_{q'_\ell})$$

whence $\mu(\limsup_\ell Y_{q_\ell}) = \mu(\limsup_\ell Y_{q'_\ell})$, i.e. (Y_{q_ℓ}) is irreducible.

b) Suppose first that $n = 2$. Let $x_1 \in \limsup Y_{1,r}$. There exists a sequence of integers (r_ℓ) such that $x_1 \in Y_{1,r_\ell}$. Then:

$$\limsup_\ell Y_{2,r_\ell} \subset \{x_2 \subset \Omega_2 ; (x_1, x_2) \in \limsup Y_{1,r} \times Y_{2,r}\} \subset \limsup Y_{2,r}$$

hence this set has the same measure as $\limsup Y_{2,r}$. Then Fubini's theorem shows that

$$\mu_1 \otimes \mu_2 (\limsup (Y_{1,r} \times Y_{2,r})) \geqslant \mu_1 \otimes \mu_2 (\limsup Y_{1,r} \times \limsup Y_{2,r}).$$

The result follows since of course

$$\limsup Y_{1,r} \times Y_{2,r} \subset \limsup Y_{1,r} \times \limsup Y_{2,r}.$$

Notice that this result implies that $(Y_{1,r} \times Y_{2,r})_r$ is irreducible.

The general case now follows by an obvious induction on n.

Proof of Theorem 5. – If $\overline{T}$ is τ_p-compact, T is pointwise bounded, $O(T) = 0$, i.e. the hypothesis of α) are satisfied. It is then enough to prove α). One can obviously suppose that the measure space is diffuse (i.e. atomless).

1st Step. – We reduce to the case where T is uniformly bounded. For all n, let $A_n = \operatorname{ess\,sup}_{f\in T} \{|f| > n\}$. Since T is nice, there exist $A'_n \in \Sigma, A'_n \overset{\text{ess}}{\subset} A_n$ such that for all $f \in T, \{|f| > n\} \subset A'_n$. Since for all n, and all $f \in T, \{|f| > n\} \subset \{g > n\}$ the set $A_n \backslash \{g > n\}$

is negligible, and hence it is the case for $\overline{A'_n}\backslash\{g > n\}$ and so $\bigcap_n A'_n$ is negligible.

Let $T^n = \{f\chi_{\Omega\backslash A'_n} ; f \in T\}$. It is clear that this is a nice family. Also $\overline{c}(T^n) = \{f\chi_{\Omega\backslash A'_n} ; f \in \overline{c}(T)\}$, so, since $\lim_n \mu(A'_n) = 0$, we have $O(c(T)) = \lim O(c(T^n))$ for if $f \in \overline{c}(T)$ we have

$$\int^* (f^* - f_*)\, d\mu = \lim_n \int^* (f^* - f_*)\, \chi_{\Omega\backslash A'_n}\, d\mu = \lim_n \int^* ((f\chi_{\Omega\backslash A'_n})^* - (f\chi_{\Omega\backslash A'_n})_*)\, d\mu$$

and of course $c(T^n) \leqslant c(T)$.

Since $O(T^n) \leqslant O(T)$ and T^n is uniformly bounded, this concludes this step. We suppose now on that $f \in T \Longrightarrow 0 \leqslant f \leqslant 1$.

Let $a = O(c(T))$ and $q \in \mathbf{N}$. We want to show that $O(T) \geqslant a - 10q^{-1}$. Let $h \in \overline{c}(T)$, with $\int^* (h^* - h_*)\, d\mu \geqslant a - q^{-1}$.

2nd Step. – For each sequence

$$s = (s_1, \ldots, s_q, s_{q+1}, \ldots, s_{2q}) \in \mathbf{N}^{2q},$$

let $|s| = \sum_{i \leqslant 2q} s_i$. For $f \in T$, let

$$B_s(f) = \prod_{1 \leqslant i \leqslant q} \{f > iq^{-1}\}^{s_i} \times \prod_{q < i \leqslant 2q} \{f < (i-1)q^{-1}\}^{s_i} \subset X^{|s|}.$$

Let $A_s = \operatorname{ess\,sup}_{f \in T} B_s(f)$. Since T is nice, by hypothesis there exist $A'_s \subset X^{|s|}$ with $A'_s \overset{\text{ess}}{\subset} A_s$ and $B_s(f) \subset A'_s$ for all $f \in T$.

Let p be an integer, and $\eta = (8\, qp \text{ card } \{s \in \mathbf{N}^{2q} ; |s| \leqslant p\})^{-1}$. Then there exist $f_1, \ldots, f_n \in T$ such that if $C_s = \bigcup_{i \leqslant n} B_s(f_i)$, we have $\mu^{\otimes |s|}(A'_s \backslash C_s) \leqslant \eta$ for all s with $|s| \leqslant p$ (where $\mu^{\otimes |s|}$ is the product measure on $\Omega^{|s|}$).

For each integer $\ell \leqslant p$, the set E_ℓ of points of Ω^ℓ which have any two coordinates equal is negligible, since μ is diffuse. Hence there exists a finite subalgebra $\mathscr{A}$ of Σ, which contains the sets $\{f_r > iq^{-1}\}$, $\{f_r < (i-1)q^{-1}\}$ for $1 \leqslant i \leqslant q$ and $r \leqslant n$ and such that each E_ℓ is contained in a set of the product algebra $\mathscr{A}^p$ of measure $\leqslant \eta$. We can suppose that the atoms $Z_1, \ldots, Z_k$ of $\mathscr{A}$ have positive measure. For $i \leqslant k$, let μ_i be the normalization of the restriction of μ to Z_i. Let $Z = (\prod_{i \leqslant k} Z_i)^2$, provided with the measure $\lambda = (\bigotimes_{i \leqslant k} \mu_i)^{\otimes 2}$. A point $z \in Z$ will be denoted by $(z_i)_{i \leqslant 2k}$ where $z_i \in Z_i$ if $i \leqslant k$, $z_i \in Z_{i-k}$ if $i > k$.

For $z \in Z$, let

$$g(z) = \sum_{1 \leqslant i \leqslant k} h^*(z_i)\, \mu(Z_i) - \sum_{k < i \leqslant 2k} h_*(z_i)\, \mu(Z_i).$$

A straightforward calculation shows that

$$\int g d\lambda = \int (h^* - h_*)\, d\mu \geqslant a - q^{-1}.$$

Since $g \leqslant 1$, (for $h^* \leqslant 1$) we have

$$\lambda \{z \,;\, g(z) \geqslant a - 2q^{-1}\} \geqslant \frac{1}{3}\, q^{-1}. \tag{3.1}$$

For $z \in Z$, let $P_z = \{z_i \,;\, i \leqslant k\}$, provided with the measure ν_z^1 which gives weight $\mu(Z_i)$ to z_i, and let $Q_z = \{z_i \,;\, k < i \leqslant 2k\}$, provided with the measure ν_z^2 which gives weight $\mu(Z_{i-k})$ to z_i. Let $c = \sum_{1 \leqslant i \leqslant q} s_i$, $d = \sum_{q < i \leqslant 2q} s_i$, and $R_{u,s} = P_u^c \times Q_u^d$, provided with the measure $\nu_{z,s} = (\nu_z^1)^{\otimes c} \otimes (\nu_z^2)^{\otimes d}$. This is a measure on $\Omega^{|s|}$.

3rd Step. – We show that for each measurable $G \subset \Omega^{|s|}$, where $|s| \leqslant p$, we have

$$\int_{z \in Z} \nu_{z,s}(G)\, d\lambda(z) \leqslant \eta + \mu^{\otimes |s|}(G). \tag{3.2}$$

For a sequence $\sigma = (\sigma_i)_{i \leqslant |s|}$ of integers $\leqslant k$, and $z \in Z$, let

$$z_\sigma = ((z_{\sigma_i})_{i \leqslant c}\, , \, (z_{k+\sigma_i})_{c < i \leqslant |s|}) \in \Omega^{|s|}. \tag{3.3}$$

We have

$$\nu_{z,s} = \sum_\sigma \prod_{i \leqslant |s|} \mu(Z_{\sigma_i})\, \delta_{z_\sigma} \tag{3.4}$$

where the summation is taken over all possible choices of σ and where δ_u is the Dirac measure at u. If all the integers $(\sigma_i)_{i \leqslant |s|}$ are distinct, then

$$\int_{z \in Z} \prod_{i \leqslant |s|} \mu(Z_{\sigma_i})\, \delta_{z_\sigma}(G)\, d\lambda(z) = \mu^{\otimes |s|}(G \cap \prod_{i \leqslant |s|} Z_{\sigma_i})$$

as easily seen, and otherwise

$$\int_{z \in Z} \prod_{i \leqslant |s|} \mu(Z_{\sigma_i})\, \delta_{z_\sigma}(G)\, d\lambda(z) \leqslant \mu^{\otimes |s|}(\prod_{i \leqslant |s|} Z_{\sigma_i}).$$

The union of the sets $\prod_{i \leqslant |s|} Z_{\sigma_i}$ where two of the σ_i are equal is the smallest set of $\mathscr{A}^{|s|}$ which contains $E_{|s|}$, so by the choice of $\mathscr{A}$, it is of measure $\leqslant \eta$, which proves the claim.

4th Step. – Hence, if $|s| \leqslant p$, we have

$$\lambda\{z \in Z\,;\, \nu_{z,s}(A'_s \backslash C_s) \geqslant p^{-1}\} \leqslant p \int_{z \in Z} \nu_{z,s}(A'_s \backslash C_s)\, d\lambda(z) \leqslant 2\eta p\,. \tag{3.5}$$

Let Y be the set of $z \in Z$ such that $\nu_{z,s}(A'_s \backslash C_s) \leqslant p^{-1}$ for $|s| \leqslant p$. It follows from (3.1), (3.5) and the choice of η that the set of $z \in Y$ with $g(z) \geqslant a - 2q^{-1}$ has positive measure.

It is clear that $g = \theta^*$, where

$$\theta(z) = \sum_{1 \leqslant i \leqslant q} h(z_i)\, \mu(Z_i) - \sum_{q < i \leqslant 2q} h(z_i)\, \mu(Z_{i-q})\,.$$

If we had $\theta(z) \leqslant a - 3q^{-1}$ on Y, we would have $g(z) = \theta^*(z) \leqslant a - 3q^{-1}$ a.e. on Y. Since it is not true, there exists $u \in Z$ with $\theta(u) \geqslant a - 3q^{-1}$ and

$$\nu_{u,s}(A'_s \backslash C_s) \leqslant p^{-1} \quad \text{for } |s| \leqslant p\,. \tag{3.6}$$

Since $h \in \overline{c}(T)$, there exists $f \in c(T)$ with

$$\sum_{1 \leqslant i \leqslant q} f(u_i)\, \mu(Z_i) - \sum_{q < i \leqslant 2q} f(u_i)\, \mu(Z_{i-q}) \geqslant a - 4q^{-1}\,. \tag{3.7}$$

For each integer $\ell \leqslant q$, let

$$\begin{aligned} H_\ell &= \cup\,\{Z_i\,;\, i \leqslant q\,,\, f(u_i) > \ell q^{-1}\} \\ K_\ell &= \cup\,\{Z_{i-q}\,;\, i > q\,,\, f(u_i) < (\ell - 1)\, q^{-1}\}\,. \end{aligned} \tag{3.8}$$

An easy calculation gives

$$\sum_{1 \leqslant i \leqslant q} f(u_i)\, \mu(Z_i) - q^{-1} \sum_{\ell \leqslant q} \mu(H_\ell) \leqslant q^{-1}$$

$$\sum_{q < i \leqslant 2q} f(u_i)\, \mu(Z_{i-q}) - q^{-1} \sum_{\ell \leqslant q} (1 - \mu(K_\ell)) \geqslant -q^{-1}\,.$$

From (3.7) we get

$$q^{-1}\Big(\sum_{\ell \leqslant q} \mu(H_\ell) + \sum_{\ell \leqslant q} \mu(K_\ell)\Big) \geqslant 1 + a - 6q^{-1}\,.$$

We now show the fundamental fact: for each $s \in \mathbf{N}^{2q}$ with $|s| \leqslant p$, we have

$$\mu^{\otimes |s|}\Big(\Big(\prod_{1\leqslant \ell \leqslant q} H_\ell^{s_\ell} \times \prod_{1\leqslant \ell \leqslant q} K_\ell^{s_{\ell+q}}\Big) \setminus A_s\Big) \leqslant p^{-1} .$$

In fact, it is enough to show the corresponding assertion where A_s is replaced by C_s. For a sequence $(\sigma_i)_{i\leqslant |s|}$ of integers $\leqslant k$, let $Z_\sigma = \prod_{i\leqslant |s|} Z_{\sigma_i}$. Each atom of $\mathcal{A}^{|s|}$ is of this form. If

$$Z_\sigma \subset \prod_{i\leqslant q} H_i^{s_i} \times \prod_{i\leqslant q} K_i^{s_{i+q}} ,$$

then the point u_σ (given by (3.3)) belongs to $B_s(f) \subset A'_s$ by (3.8). Since C_s is a union of atoms of $\mathcal{A}^{|s|}$, if moreover $Z_\sigma \not\subset C_s$, we have $u_\sigma \in A'_s \setminus C_s$. It follows from (3.4) that

$$\mu^{\otimes |s|}\Big(\Big(\prod_{i\leqslant q} H_i^{s_i} \times \prod_{i\leqslant q} K_i^{s_{i+q}}\Big) \setminus C_s\Big) \leqslant \nu_{u,s}(A'_s \setminus C_s) \leqslant p^{-1} .$$

5^th^ Step. – With an obvious notation, we have shown that for each integer p, there exist measurable sets $(H_{i,p})_{i\leqslant q}(K_{i,p})_{i\leqslant q}$ such that

for all $s \in \mathbf{N}^{2q}$ with $|s| \leqslant 2^p$

$$\mu^{\otimes |s|}\Big(\Big(\prod_{i\leqslant q} H_{i,p}^{s_i} \times \prod_{i\leqslant p} K_{i,p}^{s_{i+q}}\Big) \setminus A_s\Big) \leqslant 2^p \tag{3.9}$$

$$q^{-1}\Big(\sum_{i\leqslant q} \mu(H_{i,p}) + \sum_{i\leqslant q} \mu(K_{i,p})\Big) \geqslant 1 + a - 6q^{-1} . \tag{3.10}$$

By lemma 6a), there is a sequence p_ℓ such that the sequences $(H_{i,p_\ell})_\ell$, $(K_{i,p_\ell})_\ell$ are irreducibles for all $i \leqslant q$. Let

$$L_i = \limsup_\ell H_{i,p_\ell} , \quad M_i = \limsup_\ell K_{i,p_\ell} .$$

Of course we have

$$q^{-1}\Big(\sum_{i\leqslant q} \mu(L_i) + \sum_{i\leqslant q} \mu(M_i)\Big) \geqslant 1 + a - 6q^{-1}. \tag{3.11}$$

From lemma 2b), and (3.9), one deduces that

$$\forall s \in \mathbf{N}^{2q} , \prod_{i\leqslant q} L_i^{s_i} \times \prod_{i\leqslant q} M_i^{s_{i+q}} \overset{\text{ess}}{\subset} A_s . \tag{3.12}$$

Let (f_n) be a sequence of T such that if $D_s = \bigcup_n B_s(f_n)$, $A_s \overset{\text{ess}}{\subset} D_s$ for each s.

It follows from lemma 1 that there exist sets $(U_i)_{i \leq q}$, $(V_i)_{i \leq q}$ with $U_i \subset L_i$, $V_i \subset M_i$, $\mu^*(U_i) = \mu(L_i)$, $\mu^*(V_i) = \mu(M_i)$, such that for all sequences $s \in \mathbf{N}^{2q}$, and all families $(u_{i,j})_{j \leq s_i}$ of distinct elements of U_i, all families $(v_{i,u})_{j \leq s_{i+q}}$ of distinct elements of V_i, we have

$$(u_{1,1}, \ldots, u_{1,s_1}, u_{2,1}, \ldots, u_{2,s_2}, \ldots, u_{q,s_q}, v_{1,1}, \ldots, v_{1,s_{q+1}}, \ldots, v_{q,s_{2q}}) \in D_s .$$

This means that there exists $n \in \mathbf{N}$ with $f_n(u_{i,j}) > iq^{-1}$ for $i \leq q$, $j \leq s_i$, $f_n(v_{i,j}) < (i-1)q^{-1}$ for $i \leq q$, $j \leq s_{i+q}$. Hence there exists a τ_p-cluster point f of the f_n such that $f \geq iq^{-1}$ on U_i, $f \leq (i-1)q^{-1}$ on V_i. Hence $f^* \geq iq^{-1}$ a.e. on L_i, $f_* \leq (i-1)q^{-1}$ a.e. on M_i. So as is easily seen

$$\int f^* d\mu \geq q^{-1} \sum_{i \leq q} \mu(L_i)$$

$$\int f_* d\mu \leq 1 - q^{-1} \sum_{i \leq q} \mu(M_i).$$

From (3.11) this gives $O(T) \geq \int (f^* - f_*)\, d\mu \geq a - 6q^{-1}$, and concludes the proof.

Remarks. – 1) Let $\Omega = [0,1]$, $T = \{a\chi_I \,;\, a \in \mathbf{R},\ I \text{ interval}\}$. Then $O(T) = 0$ but $\overline{c}(T)$ is the set of all functions $[0,1] \longrightarrow \mathbf{R}$. This shows that a hypothesis of the type of pointwise boundedness is necessary.

2) It is not hard to extract the following result from the above proof: under the hypothesis of the theorem, $O(T) = \sup\{O(D)\,;\, D \subset T,\ D \text{ countable}\}$.

Although this is not the main aim of this paper, we give an application to integration in Banach space. If E is a Banach space, a function $\eta : \Omega \longrightarrow E$ is said to be *scalarly measurable* if for each $x^* \in E^*$, the function $x^* \circ \eta$ is measurable.

PROPOSITION 7. – *Suppose either a) or b) of theorem 5 hold. Let* η *a bounded function* $\Omega \longrightarrow \ell^\infty$. *If for each character (i.e. multiplicative linear functional)* θ *on* ℓ^∞, $\theta \circ \eta$ *is measurable, then* η *is scalarly measurable.*

Proof. – The set $T = \{\theta \circ \eta \, ; \, \theta$ character$\}$ is a separable compact subset of $M(\Sigma)$. For each linear functional $x^* \in \ell^{\infty *}$, positive and of norm 1, $x^* \circ \eta \in \overline{c}(T)$ hence $x^* \circ \eta$ is measurable. So by additivity, $x^* \circ \eta$ is measurable for all $x^* \in \ell^{\infty *}$.

The interested reader will find several connected results in [4].

We now prove a (much easier) analogous of theorem 5 for Riemann-measurability.

THEOREM 8. – *Let* X *be a compact metrizable space, and* μ *a Radon probability on* X. *Let* T *be a uniformly bounded* τ_p-*compact set of Riemann-measurable functions. Then* $\overline{c}(T)$ *consists of Riemann-measurable functions.*

Proof. – For $g \in \overline{c}(T)$ it is easy to see that there exists a Radon measure ν on T such that for all $x \in X$

$$g(x) = \int_{f \in T} f(x) \, d\nu(f). \tag{2.1}$$

Consider the measure $\lambda = \mu \otimes \nu$ on $X \times T$. Let d a distance defining the topology of X. For each n, define the function φ_n on $X \times T$ by

$$\varphi_n(x, f) = \sup\{|f(y) - f(z)|, \; d(y, x) < n^{-1}, \; d(z, x) < n^{-1}\}.$$

It is easy to see that φ_n is lower semi-continuous, hence measurable on $X \times T$. This shows that $\varphi = \operatorname{Inf}_n \varphi_n$ is also measurable. For each f, $\{x \in X \, ; \, \varphi(x, f) > 0\}$ is negligible since f is continuous a.e.. By Fubini's theorem there exists a set of full measure $Y \subset X$ such that if $y \in Y$, $\nu\{f \in T; \varphi(x, f) > 0\} = 0$. It follows then by Lebesgue's theorem that g is continuous at each point of Y.

Remark. – Let $(q_n)_{n \geq 2}$ be an enumeration of the rationals of $[0, 1]$.

The set $\{0\} \cup \{2^n \chi_{\{q_n\}}\}$ is τ_p-compact, pointwise bounded, and consists of Riemann-measurable functions, but its closed convex hull contains $\chi_{\mathbb{Q} \cap [0,1]}$ which is not Riemann-measurable.

4. Consistent liftings.

Let (Ω, Σ, μ) a complete measure space. A linear lifting of $L^\infty(\mu)$ is a positive linear map ρ from $L^\infty(\mu)$ into $\mathcal{L}^\infty(\mu)$, such that $\rho(1) = 1$, and that for $f \in L^\infty(\mu)$, $\rho(f)$ belongs to the class of f. If moreover ρ is multiplicative we say simply that ρ is a lifting. See [5] as a basic reference.

For each n, let Σ^n be the completion of the product σ-algebra for $\mu^{\otimes n}$.

DEFINITION 9. – *A lifting ρ of $L^\infty(\mu)$ is said to be consistent if for each n there exists a lifting ρ^n of $L^\infty(\mu^{\otimes n})$ such that for each $A_1, \ldots, A_n \in \Sigma$, we have*

$$\rho^n(A_1 \times \ldots \times A_n) = \rho(A_1) \times \ldots \times \rho(A_n).$$

The motivation for this definition is the following result.

PROPOSITION 10. – *Let ρ be a consistent lifting of the probability space (Ω, Σ, μ). Then*

a) *The family U of sets $A \subset \Sigma$ such that $A \subset \rho(A)$ is nice (see Definition 5).*

b) *The family T of functions $f \in M(\Sigma)$ such that $\rho(f) = f$ is nice.*

Proof. – a) Let $n \in \mathbf{N}$ and ρ^n the lifting of $L^\infty(\mu^{\otimes n})$ as in Definition 8. Let $H \in \Sigma^n$, and $H' = \rho^n(H)$. We have $H' \overset{\text{ess}}{\subset} H$. For $A_1, \ldots, A_n \in U$, we have, if $A_1 \times \ldots \times A_n \overset{\text{ess}}{\subset} H$

$$A_1 \times \ldots \times A_n \subset \rho(A_1) \times \ldots \times \rho(A_n) = \rho^n(A_1 \times \ldots \times A_n)$$
$$\subset \rho^n(H) = H'$$

b) It is enough to show that if $f \in T$ and $a \in \mathbf{R}$, then $\{f > a\}$ and $\{f < a\}$ belong to U. Now we have, if $b = \|f\|_\infty$

$$f \leqslant a + 2b\chi_{\{f>a\}} \quad \text{so} \quad f = \rho(f) \leqslant a + 2b\chi_{\rho\{f>a\}}$$

hence if $f(t) > a$, $t \in \rho\{f > a\}$, i.e. $\{f > a\} \subset \rho\{f > a\}$. And we have $\{f < a\} = \{-f > -a\} \subset \rho\{-f > -a\} = \rho\{f < a\}$ since $-f \in T$.

The following lemma is a useful tool to construct consistent liftings.

LEMMA 11. – *Let* (Ω, Σ, μ) *be a measure space. Suppose that for each* n *there exist a lifting* ρ_n *of* $L^\infty(\mu)$ *and a lifting* σ_n *of* $L^\infty(\mu^{\otimes n})$ *such that the following conditions are satisfied:*

(4.1) *For all* $k \leqslant n$, *for all* $A \in \Sigma^k$, *there exists* $A' \in \Sigma^k$ *such that* $\sigma_n(A \times \Omega^{n-k}) = A' \times \Omega^{n-k}$.

(4.2) *For* $A_1, \ldots, A_n \in \Sigma$, $\sigma_n(A_1 \times \ldots \times A_n) = \rho_n(A_1) \times \ldots \times \rho_n(A_n)$. *Let* $\mathcal{U}$ *be any non-trivial ultrafilter on* $\mathbf{N}$. *Then the lifting* ρ *of* $L^\infty(\mu)$ *given for* $\omega \in \Omega$ *by*

(4.3) $\rho(f)(\omega) = \lim_{n \to \mathcal{U}} \rho_n(f)(\omega)$

is consistent.

Proof. – Since each $\rho_k(f)$ is equal a.e. to f, it is clear that (4.3) defines a lifting. From condition (4.1) there exists for $k \leqslant n$ a lifting ρ_n^k of Σ^k such that $\rho_n^k(A) \times \Omega^{n-k} = \sigma_n(A \times \Omega^{n-k})$ for $A \in \Sigma^k$. Let ρ^k given by $\rho^k(f)(\omega') = \lim_{n \to \mathcal{U}} \rho_n^k(f)(\omega')$ for $\omega' \in \Omega^k$, $f \in M(\Sigma^k)$.

For $A_1, \ldots, A_k \in \Sigma$, $\omega \in \Omega^k$, we have

$$\rho^k(A_1 \times \ldots \times A_k)(\omega) = \lim_{n \to \mathcal{U}} \rho_n^k(A_1 \times \ldots \times A_k)(\omega).$$

But

$$\begin{aligned}\rho_n^k(A_1 \times \ldots \times A_k) \times \Omega^{n-k} &= \sigma_n(A_1 \times \ldots \times A_k \times \Omega^{n-k}) \\ &= \rho_n(A_1) \times \ldots \times \rho_n(A_k) \times \Omega^{n-k}\end{aligned}$$

so

$$\rho_n^k(A_1 \times \ldots \times A_k) = \rho_n(A_1) \times \ldots \times \rho_n(A_k).$$

It then follows that

$$\rho^k(A_1 \times \ldots \times A_k) = \rho(A_1) \times \ldots \times \rho(A_k),$$

which concludes the proof.

The following result answers a natural question, but it will not be used in the sequel, so we don't give all the details of the proof.

THEOREM 12. – *For any complete probability space* (Ω, Σ, μ), $L^\infty(\mu)$ *has a consistent lifting.*

Proof. – It closely follows the method of [5]. Given n, we are going to construct liftings $\rho = \rho_n$ of $L^\infty(\mu)$ and $\sigma = \sigma_n$ of $L^\infty(\mu^{\otimes n})$ which satisfy (4.1) and (4.2).

Let $\aleph$ be the cardinal of Σ, and $(A_\alpha)_{\alpha<\aleph}$ be an enumeration of Σ. Let Σ_α be the completion of the σ-algebra generated by $(A_\beta)_{\beta<\alpha}$ and for $k \leqslant n$, Σ_α^k the completion of its product on Ω^k. We shall construct by induction on α liftings ρ_α of $L^\infty(\mu_\alpha)$ and σ_α of $L^\infty(\mu_\alpha^{\otimes n})$ where μ_α and $\mu_\alpha^{\otimes n}$ denote the restrictions of μ to Σ_α and $\mu^{\otimes n}$ to Σ_α^n respectively, which satisfy the following condition:

(4.4) For $\beta < \alpha$, $A \in \Sigma_\beta$, $B \in \Sigma_\beta^n$, $\rho_\alpha(A) = \rho_\beta(A)$, $\sigma_\alpha(B) = \sigma_\beta(B)$, and the conditions corresponding to (4.1), (4.2) hold.

Suppose ρ_β and σ_β have been constructed for all $\beta < \alpha$. If α has uncountable cofinality, since $\Sigma_\alpha = \bigcup_{\beta<\alpha} \Sigma_p$, $\Sigma_\alpha^n = \bigcup_{\beta<\alpha} \Sigma_\beta^n$, it is enough to set $\rho_\alpha(A) = \rho_\beta(A)$ if $A \in \Sigma_\beta$, $\sigma_\alpha(B) = \sigma_\beta(B)$ if $B \in \Sigma_\beta^n$, and (4.4) shows that this does not depend on β.

Suppose now $\alpha = \sup_p \alpha_p$, where $\alpha_p < \alpha$. To simplify the notation, let $\Sigma_p = \Sigma_{\alpha_p}$, $\Sigma_p^n = \Sigma_{\alpha_p}^n$, $\rho_p = \rho_{\alpha_p}$, $\sigma_p = \sigma_{\alpha_p}$. Let $\mathfrak{U}$ be an ultrafilter on $\mathbf{N}$, and for $f \in L^\infty(\mu_\alpha)$, $g \in L^\infty(\mu_\alpha^{\otimes n})$, let

$$\rho'_\alpha(f) = \lim_{p\to\mathfrak{U}} \rho_p(E^{\Sigma_p}(f))\,;\quad \sigma'_\alpha(g) = \lim_{p\to\mathfrak{U}} \sigma_p(E^{\Sigma_p^n}(g))$$

where for a σ-algebra Σ', $E^{\Sigma'}$ denotes the conditional expectation with respect to Σ'. We define in this way two linear liftings of $L^\infty(\mu_\alpha)$ and $L^\infty(\mu_\alpha^{\otimes n})$ respectively. Let

$$k \leqslant n,\ A \in \Sigma_\alpha^k,\ A_{k+1}, \dots, A_n \in \Sigma_\alpha .$$

For all p, we have, if $\omega = (\omega_1, \dots, \omega_n) \in \Omega^n$

$$E^{\Sigma_p^n}(\chi_{A\times A_{k+1}\times\dots\times A_n})(\omega) = E^{\Sigma_p^n}(\chi_A)(\omega_1, \dots, \omega_k) \times \prod_{i=k+1}^{n} E^{\Sigma_p}(\chi_{A_i})(\omega_i),$$

so, from (4.1) and (4.2)

$$\sigma'_\alpha(A \times A_{k+1} \times \dots \times A_n) = \sigma'_\alpha(A \times \Omega^{n-k}) \cap (\Omega^k \times \prod_{i=k+1}^{n} \rho'_\alpha(A_i)). \tag{4.5}$$

Moreover $\sigma'_\alpha(A \times \Omega^{n-k})(\omega)$ depends only on the first k coordinates of ω, hence we can write $\sigma'_\alpha(A \times \Omega^{n-k}) = \theta^k(A) \times \Omega^{n-k}$ where $\theta^k(A)$ is a linear lifting of $L^\infty(\mu_\alpha^{\otimes n})$, (of course $\theta^1 = \rho'_\alpha$).

Now we show how to modify ρ'_α and σ'_α to make them multiplicative. For $\omega = (\omega_1, \ldots, \omega_n) \in \Omega^n$, $k \leqslant n$, and $\omega^k = (\omega_1, \ldots, \omega_k)$, let $\mathcal{F}^k_\omega = \{A \in \Sigma^k_\alpha ; \theta^k(A)(\omega^k) = 1\}$. It is easy to see that $\mathcal{F}^k_\omega$ is a filter, which depends only on $\omega_1, \ldots, \omega_k$. For each $\omega_1 \in \Omega$, let $\mathcal{V}_{\omega_1}$ be an ultrafilter containing $\mathcal{F}^1_{\omega_1}$. For each $\omega = (\omega_1, \ldots, \omega_n) \in \Omega^n$, we construct by induction on $k \leqslant n$, ultrafilters $\mathcal{U}^k_\omega$ on Ω^k with the following properties:

(4.6) For $C \in \mathcal{U}^{k-1}_\omega \cap \Sigma^{k-1}_\alpha$, $A \in \mathcal{V}_{\omega_k} \cap \Sigma_\alpha$, we have $C \times A \in \mathcal{U}^k_\omega$

(4.7) $\mathcal{F}^k_\omega \subset \mathcal{U}^k_\omega$

(4.8) $\mathcal{U}^k_\omega$ depends only on $\omega_1, \ldots, \omega_k$.

We take $\mathcal{U}^1_\omega = \mathcal{V}_{\omega_1}$. Then (4.6) to (4.8) are automatically satisfied.

Suppose now $\mathcal{U}^{k-1}_\omega$ has been constructed, satisfying (4.6) to (4.8). The family of sets $C \times A$ on Ω^k, where $C \in \mathcal{U}^{k-1}_\omega \cap \Sigma^{k-1}_\alpha$, $A \in \mathcal{V}_{\omega_k} \cap \Sigma_\alpha$, is a filter. For $B \in \mathcal{F}^k_\omega$, we have

(4.9) $\mu^{\otimes k}_\alpha (C \times A \cap B) > 0$.

Indeed, otherwise

$$0 = 1 - \theta^k(B)(\omega^k) \geqslant \theta^k(C \times A)(\omega^k)$$

so

$$0 = \theta^k(C \times A)(\omega^k) = \sigma'_k(C \times A \times \Omega^{n-k})(\omega)$$
$$= \sigma'_k(C \times \Omega^{n-k+1})(\omega)\, \rho'_k(A)(\omega_k)$$

by (4.5). Since $A \in \mathcal{V}_{\omega_k}$ we cannot have $\rho'_\alpha(A)(\omega_k) = 0$, for otherwise $\rho'_\alpha(\Omega \backslash A)(\omega_k) = 1$, i.e. $\Omega \backslash A \in \mathcal{F}^1_{\omega_k} \subset \mathcal{V}_{\omega_k}$, a contradiction. So we have $\sigma'_\alpha(C \times \Omega^{n-k+1})(\omega) = 0$, hence

$$\sigma'_\alpha((\Omega^{k-1} \backslash C) \times \Omega^{n-k+1})(\omega) = 1$$

so $\Omega^{k-1} \backslash C \in \mathcal{F}^{k-1}_\omega$, which contradicts the fact that $C \in \mathcal{U}^{k-1}_\omega$. It is now clear that there exists an ultrafilter $\mathcal{U}^k_\omega$ on Ω^k which contains $\mathcal{F}^k_\omega$ and all the sets $C \times A$ for $C \in \mathcal{U}^{k-1}_\omega \cap \Sigma^{k-1}_\alpha$, $A \in \mathcal{V}_{\omega_k} \cap \Sigma_\alpha$, and of course $\mathcal{U}^k_\omega$ can by choosen depending on $\omega_1, \ldots, \omega_k$ only. This concludes the construction.

It is clear that $\mathcal{V}_\tau$ and $\mathcal{U}_\omega = \mathcal{U}^n_\omega$ contain no negligible sets. Hence one can define ρ_α by

$$\rho_\alpha(f)(\tau) = \lim_{\tau' \to \mathcal{V}_\tau} f(\tau'), \quad \sigma_\alpha(g)(\omega) = \lim_{\omega' \to \mathcal{U}_\omega} g(\omega')$$

for $f \in L^\infty(\mu_\alpha)$, $g \in L^\infty(\mu_\alpha^{\otimes n})$, $\tau \in \Omega$, $\omega \in \Omega^n$, since these quantities depend only on the class of f and g. These maps are linear and multiplicative. If $A \in \Omega$, let

$$A' = \{\tau \in \Omega \, ; \, \rho'_\alpha(A)(\tau) = 1\}$$
$$A'' = \{\tau \in \Omega \, ; \, \rho'_\alpha(A)(\tau) = 0\}.$$

For $\tau \in A'$, we have $A' \in \mathfrak{F}_\tau$, hence $A' \in \mathcal{V}_\tau$, i.e. $\rho_\alpha(A)(\tau) = 1$. Similarly $\rho_\alpha(A)(\tau) = 0$ for $\tau \in A''$, which shows ρ_α, and similarly σ_α, are liftings. Moreover if $A \in \Sigma_\beta$ for $\beta < \alpha$, $\rho'_\alpha(A) = \rho_\beta(A)$, so $\rho_\alpha(A) = \rho_\beta(A)$, so (4.4) holds. It remains to check (4.1) and (4.2). Let $k \leqslant n$, $\omega \in \Omega^n$, and $A \in \Sigma_\alpha^k$. Then, from condition (4.6) $\sigma_\alpha(A \times \Omega^{n-k})(\omega) = 1 \iff A \times \Omega^{n-k} \in \mathcal{U}_\omega \iff A \in \mathcal{U}_\omega^k$ and the last condition depends on $\omega_1, \ldots, \omega_k$ only, so (4.1) is proved. Similarly condition (4.6) gives, for $A \in \Sigma$

$$\sigma_\alpha(\Omega^{k-1} \times A \times \Omega^{n-k-1})(\omega) = 1 \iff \Omega^{k-1} \times A \times \Omega^{n-k-1} \in \mathcal{U}_\omega$$
$$\iff A \in \mathcal{V}_{\omega_k},$$

so $\sigma_\alpha(\Omega^{k-1} \times A \times \Omega^{n-k-1})(\omega) = \rho_\alpha(A)(\omega_k)$, which implies (4.2). This concludes the construction if α is a limit ordinal.

Suppose finally that $\alpha = \beta + 1$. Then Σ_α is generated by Σ_β and A_β. Let

$$B_1 = \text{ess sup}\,\{C \in \Sigma_\beta \, , \, C \subset A_\beta\}$$

and

$$B_2 = \text{ess sup}\,\{C \in \Sigma_\beta \, , \, C \cap A_\beta = \phi\}.$$

Then set $\rho_\alpha(A_\beta) = (A_\beta \cup \rho_\beta(B_1)) \backslash \rho_\beta(B_2)$. It is easy to see that the only candidate for ρ_α which satisfies this condition and (4.4) is actually a lifting. For $B \in \Sigma_\beta^k$, set

$$\sigma_\alpha(A \cap \prod_{i \leqslant n} \epsilon_i A_\alpha) = \sigma_\beta^k(A) \cap \prod_{i=1}^{n} \epsilon_i \rho_\alpha(A_\alpha)$$

where $\epsilon_i = \pm 1$ and for a set C of Ω and $+C = C$, $-C = \Omega \backslash C$. It is easy to check this is a lifting, and that σ_α, ρ_α satisfy (4.1), (4.2), (4.4). The construction is completed. The theorem follows with $\alpha = \aleph$.

Let G be a locally compact group, and dx a left-invariant Haar measure. Since all these measures are proportional, the object

$L^\infty(G)$ is well defined. (When G is not σ-compact, $L^\infty(G)$ is the set of bounded measurable functions on G modulo the locally negligible functions). There is a natural operation of G on $L^\infty(G)$, given, for $f \in L^\infty$, by $L_t f(x) = f(tx)$ for $x \in G$ (the class of $L_t f$ depends only of the class of f). A lifting ρ of $L^\infty(G)$ is said to be left-invariant if $\rho(L_t f) = L_t \rho(f)$ for all $f \in L^\infty$. In [6], the existence of a left invariant lifting is shown for any locally compact group. We find very likely that there exist *consistent* left invariant liftings on each locally compact group. However, we don't see how this could be proved without revisiting the proof of [6], and, to be frank, this proof is so long that we don't have this courage. So we shall only prove it in a much simpler way for a large class of groups (which contains all the metrizable abelian groups).

THEOREM 13. – *Let* G *be a (metrizable) group which satisfies the following conditions, where* $|\cdot|_k$ *denotes the product on* G^k *of the Haar measure on* G:

(4.10) "*There exists a decreasing sequence* V_n *of neighborhoods of the unit such that for all* k *and* $A \subset G^k$ *measurable* $\lim_{n\to\infty} |V_n^k|_k^{-1}\, |A \cap xV_n^k|_k = \chi_A(x)$ *for almost all* $x \in G^k$".
Then G *has a left invariant consistent lifting.*

Proof. – Let $\mathcal{U}$ be an ultrafilter on $\mathbf{N}$. Consider the class $\mathcal{A}$ of measurable sets of G such that $\lim_{\mathcal{U}} |V_n|_1^{-1}\, |A \cap V_n|_1 = 1$. It is closed under finite intersections. Let θ be a character of L^∞ such that $\theta(A) = 1$ if $A \in \mathcal{A}$. Note that

$$\theta(A) = 1 \Longrightarrow \lim_{\mathcal{U}} |V_n|_1^{-1}\, |A \cap V_n|_1 > 0$$

for otherwise $G\backslash A \in \mathcal{A}$.

For $f \in L^\infty$, let $\rho(f)(t) = \theta(L_t f)$. For A measurable, $t \in G$, we have

$$\lim_{\mathcal{U}} |V_n|_1^{-1}\, |A_n \cap t\, V_n| = 1 \quad (\text{resp } 0) \Longrightarrow \rho(A)(t) = 1 \quad (\text{resp } 0)$$

hence by hypothesis and an obvious density argument $\rho(f)$ is in the class of f, so ρ is a lifing, of course invariant since

$$\rho(L_u f)(t) = \theta(L_t L_u f) = \theta(L_{ut} f) = \rho f(ut) = L_u \rho(f)(t).$$

For $k \geq 1$, the class $\mathcal{A}_k$ of sets of the form $A_1 \times \ldots \times A_k \cap A$ where $A_i \in \mathcal{A}$ for $i \leq n$ and $\lim_{\mathcal{U}} |V_n^k|_k^{-1} \, |A \cap V_n^k|_k = 1$ is closed under finite intersections, and none of its elements is empty since $\lim_{\mathcal{U}} |V_n^k|_k^{-1} \, |A_1 \times \ldots \times A_k \cap V_n^k|_k > 0$. Hence if θ_k is a character on $L^\infty(G^k)$ such that $\theta_k(B) = 1$ if $B \in \mathcal{A}_k$, we can as above define an invariant lifting ρ_k of G^k for which it is easily seen that $\rho_k(A_1 \times \ldots \times A_k) = \prod_{i \leq k} \rho(A_i)$.

5. Measurability of translations.

Let G be a locally compact group. We fix a left invariant Haar measure dx. This paragraph is mostly devoted to a study of the following question: For which functions $f \in L^\infty(G)$ is the map $t \longrightarrow L_t f$ from G to $L^\infty(G)$ scalarly measurable, i.e. such that for each $\varphi \in L^{\infty *}$, $t \longrightarrow \varphi(L_t f)$ is measurable ?

The Haar measure of a measurable set A of G is denoted by $|A|$ and the product measure on G^n is similarly denoted by $|\cdot|_n$. A *character* θ on $L^\infty(G)$ is a linear functional which is also multiplicative. We say that a character θ on $L^\infty(G)$ (resp a $\varphi \in L^{\infty *}$) is *localizable* if there is a compact set $K \subset G$ such that $\theta(\chi_K) = 1$ (resp for all $\epsilon > 0$, there is a compact K of G with $|\varphi|(\chi_{G \setminus K}) \leq \epsilon$).

We denote by $\mathcal{L}^\infty(G)$ the set of bounded measurable functions on G. Each $f \in L^\infty(G)$ is a set in $\mathcal{L}^\infty(G)$. For $h \in \mathcal{L}^\infty(G)$, its class in $L^\infty(G)$ is denoted by $\dot{h}$. We denote by τ_p the topology of pointwise convergence on the set of real-valued functions on G. For $h \in \mathcal{L}^\infty(G)$, and $t \in G$, let $L_t h(x) = h(tx)$, $R_t h(x) = h(xt)$.

LEMMA 14. – *Let* $h \in \mathcal{L}^\infty(G)$,

a) *Let* θ *be a character on* $L^\infty(G)$. *Then the function* $t \longrightarrow \theta(L_t h)$ *belongs to the* τ_p*-closure of* $\{R_t h \, ; \, t \in G\}$. *If* K *is a compact of* G *with* $\theta(\chi_K) = 1$, *it belongs to the* τ_p*-closure of* $\{R_t h \, ; \, t \in K\}$.

b) *Let* $\varphi \in L^{\infty *}(G)$, $\varphi(1) = 1$, $\varphi \geq 0$. *Then the function* $t \longrightarrow \varphi(L_t h)$ *belongs to* $\overline{c}\{R_t h \, ; \, t \in G\}$. *If* K *is a compact subset of* G *with* $\varphi(\chi_K) = 1$, *it belongs to* $\overline{c}\{R_t h \, ; \, t \in K\}$.

Proof. – Since a) is similar to b) but simpler, we prove only b). Let $h_1, \ldots, h_n \in \mathcal{L}^\infty(G)$. Let $\eta : G \longrightarrow \mathbf{R}^n$ given by $u \longrightarrow (h_i(u))_{i \leqslant n}$. It follows from the theorem of Hahn-Banach that $(\varphi(h_i))_{i \leqslant n}$ belongs to $\overline{c}(\eta(G))$. Hence for each ϵ there exists $\alpha_1, \ldots, \alpha_p \geqslant 0$, $\sum_{j \leqslant p} \alpha_j = 1$, and $u_1, \ldots, u_p \in G$ with $\left| \sum_{j \leqslant p} \alpha_j h_i(u_j) - \varphi(h_i) \right| \leqslant \epsilon$ for $i \leqslant n$. Let us apply this with $h_i = L_{t_i} h$ for a family $t_1, \ldots, t_n \in G$. We have for $i \leqslant n$

$$\left| \sum_{j \leqslant p} \alpha_j L_{t_i} h(u_j) - \varphi(L_{t_i} h) \right| = \left| \sum_{j \leqslant p} \alpha_j R_{u_j} h(t_i) - \varphi(L_{t_i} h) \right| \leqslant \epsilon$$

which concludes the proof.

For $A, B \subset G$, we write

$$AB = \{ab \,;\, a \in A,\ b \in B\}, \quad A^{-1} = \{a^{-1} \,;\, a \in A\}.$$

LEMMA 15. – *Suppose* G *is metrizable. Let* $A_1, \ldots, A_n$ *be relatively compact measurable sets. For each* $i \leqslant n$, *let* $F_i = \operatorname{ess\,cl} A_i$. *Then*

$$|\{(t_1, \ldots, t_n) \in G^n \,;\, |\bigcap_{i \leqslant n} t_i F_i| > 0,\ |\bigcap_{i \leqslant n} t_i A_i| = 0\}|_n = 0.$$

Of course the interest of this lemma is that F_i is in general much bigger than A_i.

Proof. – Suppose first $n = 2$, and let ρ be a left-invariant lifting of $L^\infty(G)$ [6].

First for A, B measurable we have

$$t \in \rho(A)\,\rho(B)^{-1} \Longleftrightarrow t\,\rho(B) \cap \rho(A) \neq \emptyset \Longleftrightarrow |tA \cap B| > 0$$

so $\rho(A)\,\rho(B)^{-1}$ is open.

Let $L = \{t \in G \,;\, |A_1 \cap tA_2| = 0\}$. This set is closed.

Since ρ is left translation-invariant

$$t \in L \Longrightarrow \rho(A_1) \cap t\rho(A_2) = \emptyset$$

hence $\rho(A_1) \cap L\rho(A_2) = \emptyset$.

Since ρ is left translation invariant, ρ is strong, i.e. $\rho(A) \subset A$ for A closed. Since L is closed, $\rho(L) \subset L$, hence

$$\rho(A_1) \cap \rho(L)\,\rho(A_2) = \emptyset.$$

Let H_n be a closed set with $H_n \subset \rho(A_2)$ and $|A_2 \backslash H_n| \leqslant n^{-1}$. Since ρ is strong, and H_n^{-1} closed, $\rho(H_n^{-1})^{-1} \subset H_n \subset \rho(A_2)$, so for all n, $\rho(A_1) \cap \rho(L)\, \rho(H_n^{-1})^{-1} = \phi$. Since the second set is open, and meets $\rho(A_1)$ in a negligible set, the definition of F_1 shows that $F_1 \cap \rho(L)\ \rho(H_n^{-1})^{-1} = \phi$. Since $|(\bigcup_n \rho(H_n^{-1})^{-1}) \Delta A_2| = 0$, for each $t \in L$ except in the negligible set $L \backslash \rho(L)$ we have $|F_1 \cap tA_2| = 0$. Changing t in t^{-1} the same proof shows that

$$|\{t \in G\,;\, |F_1 \cap tF_2| > 0\,,\ |A_1 \cap tA_2| = 0\}| = 0$$

which implies the result by Fubini's theorem.

Suppose now it has been proved by induction that the result holds until $n-1$. It is clear that there exists an open subgroup G' of G which is separable, and contains $A_1, \ldots, A_n$. It is easy to see that one can in fact suppose $G = G'$, i.e. that G is separable. Let (V_p) be a basis of open sets for the topology. We are first going to show that for all p and for almost all $(t_1, \ldots, t_{n-1})$ in G^{n-1}, $|V_p \cap \bigcap_{i \leqslant n-1} t_i F_i| = 0$ when $|V_p \cap \bigcap_{i \leqslant n-1} t_i A_i| = 0$. In fact, if

$$V_p \cap \bigcap_{i \leqslant n-1} t_i F_i = \bigcap_{1 \leqslant n-1} t_i (F_i \cap t_i^{-1} V_p)$$

is not negligible, there exist $k_1, \ldots, k_{n-1}, k_1', \ldots, k_{n-1}'$ with $\overline{V}_{k_i} \subset V_{k_i'} \subset t_i^{-1} V_p$ and $|\bigcap_{i \leqslant n-1} t_i (F_i \cap \overline{V}_{k_i})| > 0$. Of course we have $|\bigcap_{i \leqslant n-1} t_i (A_i \cap V_{k_i'})| = 0$. Now each open set meeting $F_i \cap \overline{V}_{k_i}$ meets $A_i \cap V_{k_i'}$ in a set of positive measure, so

$$F_i \cap \overline{V}_{k_i} \subset \text{ess cl}\,(A_i \cap V_{k_i'})\,.$$

Hence for all $k_1, \ldots, k_{n-1}, k_1', \ldots, k_{n-1}'$ the set of $(t_i)_{i \leqslant n-1}$ such that $|\bigcap_{i \leqslant n-1} t_i (F_i \cap \overline{V}_{k_i})| > 0$ and $|\bigcap_{i \leqslant n-1} t_i (A_i \cap V_{k_i'})| = 0$ is negligible by induction hypothesis, which proves the assertion.

It follows that for almost all $s = (t_i)_{i \leqslant n-1} \in G^{n-1}$, each open set U which meets $\bigcap_{i \leqslant n-1} t_i F_i$ in a set of positive measure meets $\bigcap_{i \leqslant n-1} t_i A_i$ in a set of positive measure. It follows that if

$$H_s = \text{ess cl}\,(\bigcap_{i \leqslant n-1} t_i A_i)$$

we have $\bigcap_{i \leqslant n-1} t_i F_i \overset{\text{ess}}{\subset} H_s$, and hence that for all $t_n \in G$,

$$|\bigcap_{i \leqslant n-1} t_i F_i \cap t_n F_n| \leqslant |H_s \cap t_n F_n|\,.$$

The result follows then by the case $n = 2$ and Fubini's theorem.

THEOREM 16. – *Suppose that* MA *holds. Let* G *be a locally compact group, and* $f \in L^\infty(G)$. *The following are equivalent*

a) *There exists a Riemann-measurable* $h \in \mathcal{L}^\infty(G)$, $h \in f$

b) *For all localizable* $\varphi \in L^{\infty *}(G)$, *the map* $t \longrightarrow \varphi(L_t f)$ *is measurable*

c) *For all localizable character* θ *on* $L^\infty(G)$, *the map* $t \longrightarrow \theta(L_t f)$ *is measurable*

d) *There exist* $h \in \mathcal{L}^\infty(G)$, $h \in f$ *such that all compact* K *of* G, $\overline{c}\{R_t h\ ;\ t \in K\}$ *consists of Riemann-measurable functions*

e) *For each compact* K *of* G, *there exist* $h \in \mathcal{L}^\infty(G)$, $h \in f$ *such that* $\{R_t h\ ;\ t \in K\}$ *is* τ_p*-relatively compact in* $\mathcal{L}^\infty(G)$.

Proof. – From lemma 14, $e \Longrightarrow c$. From lemma 14 and obvious approximations, $d \Longrightarrow b$. It is obvious that $b \Longrightarrow c$ and $d \Longrightarrow e$. Let us show that $a \Longrightarrow d$. Let $g \in \overline{c}\{R_t h\ ;\ t \in K\}$ where h is Riemann-measurable. Let Y be a compact of G. Let $X = YK$. Let $\epsilon > 0$. From lemma 3, there exist two continuous functions h_1, h_2 on G with

$$h_1 \leqslant h \leqslant h_2\,,\ \int_X (h_2(x) - h_1(x))\,dx \leqslant \epsilon\,,\ h_1 = h_2 = 0\,,$$

outside a compact neighbourhood of X. We know that g is a pointwise limit along an ultrafilter of functions of the form $\Sigma\,\alpha_i R_{t_i h}$ for $\alpha_i \geqslant 0$ $\Sigma\alpha_i = 1$, $t_i \in K$. Let μ be the Radon measure on K which is the limit of the $\Sigma\,\alpha_i\,\delta_{t_i}$ along the same ultrafilter. Then for $x \in Y$ we have $g_1(x) \leqslant g(x) \leqslant g_2(x)$, where

$$g_1(x) = \int_K h_1(xt)\,d\mu(t)\,,\ g_2(x) = \int_K h_2(xt)\,d\mu(t)\,.$$

Hence g_1 and g_2 are continuous on X, and a straightforward computation shows $\int_Y (g_2(x) - g_1(x))\,dx \leqslant \epsilon\ \sup\{\Delta(t^{-1})\,;\ t \in K\}$ where Δ is the modular function, so by lemma 2, g is Riemann-measurable.

It remains to show that $c \Longrightarrow a$. It is clear that G can be partitioned $G = \bigcup_i H_i$ where each H_i is a relatively compact set, and the union of the frontiers of the H_i is a closed negligible set. It is then enough to show that for each compact set X of G, there exist $h \in \mathcal{L}^\infty$, $h \in f$, whose restriction to X is Riemann-measurable. Indeed, if $h_i \in f$ is such that its restriction to $\overline{\mathrm{H}}_i$ is Riemann-measurable, we can suppose $\|h_i\| \leqslant \|f\|$. Then the function $g \in \mathscr{L}^\infty$ which coincide with h_i on H_i is Riemann-measurable and in the class of f. One can hence assume that G is σ-compact. Since f can be factored through a metrizable quotient of G, it is easy to show that one can suppose G metrizable.

So let X be a compact of G. Let $a < b$, $\mathrm{A} = \{f \leqslant a\} \cap \mathrm{X}$, $\mathrm{B} = \{f \geqslant b\} \cap \mathrm{X}$. From lemma 4, it is enough to show that if $\mathrm{M} = \operatorname{ess\,cl} \mathrm{A}$, $\mathrm{N} = \operatorname{ess\,cl} \mathrm{B}$, $\mathrm{H} = \mathrm{M} \cap \mathrm{N}$ we have $|\mathrm{H}| = 0$. Let us suppose $|\mathrm{H}| > 0$. Let ρ be a left invariant lifting of G. Let λ denote the normalization of the restriction of μ to $\mathrm{L} = \rho(\mathrm{H})^{-1}$, and e the unit of G. For $s_1, \ldots, s_n \in \mathrm{L}$ we have $e \in \bigcap_{i \leqslant n} s_i \rho(\mathrm{H})$ so $|\bigcap_{i \leqslant n} s_i \mathrm{H}| \geqslant 0$. Hence for $s_1, \ldots, s_n, t_1, \ldots, t_m \in \mathrm{L}^{n+m}$ we have $|\bigcap_{i \leqslant n} s_i \mathrm{M} \cap \bigcap_{i \leqslant m} t_i \mathrm{N}| > 0$. It then follows from lemma 15 that

$$\lambda^{\otimes(n+m)}\{s_1, \ldots, s_n, t_1, \ldots, t_m \in \mathrm{L}^{n+m} ;\ |\bigcap_{i \leqslant n} s_i \mathrm{A} \cap \bigcap_{i \leqslant m} t_i \mathrm{B}| > 0\} = 1 .$$

Since L is metrizable and MA holds, lemma 3a shows that there exists $\mathrm{U}_1, \mathrm{U}_2 \subset \mathrm{L}$, $\lambda^*(\mathrm{U}_1) = \lambda^*(\mathrm{U}_2) = 1$, and such that for $s_1, \ldots, s_n \in \mathrm{U}_1$, $t_1, \ldots, t_m \in \mathrm{U}_2$, we have $|\bigcap_{i \leqslant n} s_i \mathrm{A} \cap \bigcap_{i \leqslant m} t_i \mathrm{B}| > 0$. It is then clear that there exists a character θ on $\mathrm{L}^\infty(\mathrm{G})$, such that $\theta(\mathrm{H}^{-1}\mathrm{X}) = 1$ and $\theta(\chi_{s\mathrm{A}}) = \theta(\chi_{t\mathrm{B}}) = 1$ for $s \in \mathrm{U}_1$, $t \in \mathrm{U}_2$. This shows that $\theta(\mathrm{L}_{t^{-1}} \dot{f}) \leqslant a$ for $t \in \mathrm{U}_1$, and $\theta(\mathrm{L}_{t^{-1}} \dot{f}) \geqslant b$ for $t \in \mathrm{U}_2$, hence $t \longrightarrow \theta(\mathrm{L}_t \dot{f})$ is not measurable, which concludes the proof.

The above theorem is very satisfactory when G is compact. However, when G is not compact, to say that $h \in \mathscr{L}^\infty(\mathrm{G})$ is Riemann-measurable gives information only about the local behaviour of h. Hence we should not in general expect any regularity for the functions $t \longrightarrow \varphi(\mathrm{L}_t \dot{h})$ $\varphi \in \mathrm{L}^{\infty *}(\mathrm{G})$. For example, one has the following easy result.

Proposition 17. – *There exists a Riemann-measurable function $h \in \mathscr{L}^\infty(\mathbf{R})$, such that for all subsets $\mathrm{X} \subset \mathbf{R}$, there exists a character θ on $\mathrm{L}^\infty(\mathbf{R})$, with $\theta(\mathrm{L}_t \dot{h}) = 1$ for $t \in \mathrm{X}$ and $\theta(\mathrm{L}_t \dot{h}) = 0$ for $t \notin \mathrm{X}$.*

Proof. – Let $n \longrightarrow (p(n), q(n))$ a bijection of $\mathbf{N}$ and $\mathbf{N} \times \mathbf{N}$. Let $(\mathrm{I}_q^p)_q$ be an enumeration of all the subsets of $[0, p]$

which are finite unions of intervals with rational endpoints. Let $I = \bigcup_n (a_n + I_{q(n)}^{p(n)})$, where $a_n = \sum_{i \leqslant n} p(i)$. Let $h = \chi_I$. It is clear that h is Riemann-measurable. Let us show that for any $s_1, \ldots, s_m$, $t_1, \ldots, t_r$ distinct real numbers $|\bigcap_{i \leqslant m} (s_i + I) \cap \bigcap_{j \leqslant r} (t_j + \mathbf{R} \backslash I)| > 0$. In fact let $a \in \mathbf{R}$ and $p \in \mathbf{N}$ such that all the $a - s_i$, $a - t_j$, belong to $[0, p]$. There exists q such that $a - s_i \in \overset{\circ}{I}{}_q^p$ for $i \leqslant m$, $a - t_j \in \overset{\circ}{\widehat{[0, p] \backslash I_q^p}}$ for $j \leqslant r$. If n is such that $p = p(n)$, $q = q(n)$, it is clear that $a + a_n$ belongs to the interior of

$$\bigcap_{i \leqslant m} (s_i + I) \cap \bigcap_{j \leqslant r} (t_j + \mathbf{R} \backslash I),$$

which proves the claim. The result follows by a standard argument.

We can however generalize theorem 15 in the following direction.

THEOREM 18. – *Suppose* MA *holds, and let* G *be a locally compact group and* $f \in L^\infty(G)$. *Consider the following conditions:*

a) *For all character* θ *on* $L^\infty(G)$, *the map* $t \longrightarrow \theta(L_t f)$ *is measurable*

b) *For all character* θ *on* $L^\infty(G)$, *the map* $t \longrightarrow \theta(L_t f)$ *is equal almost everywhere to a Riemann-measurable function*

c) *There exist* $h \in \mathcal{L}^\infty(G)$, $h \in f$, *such that* $\{R_t h, t \in G\}$ *is* τ_p*-relatively compact in* $\mathcal{L}^\infty(G)$

a') *For all* $\varphi \in L^\infty(G)^*$, *the map* $t \longrightarrow \varphi(L_t f)$ *is measurable*

b') *For all* $\varphi \in L^\infty(G)^*$, *the map* $t \longrightarrow \varphi(L_t f)$ *is equivalent to a Riemann-measurable function*

c') *There exist* $h' \in \mathcal{L}^\infty(G)$, $h' \in f$, *such that*

$$\overline{c}\{R_t h', t \in G\} \subset \mathcal{L}^\infty(G).$$

Then $a \Longleftrightarrow b \Longleftrightarrow c$ *and* $a' \Longleftrightarrow b' \Longleftrightarrow c'$. *Moreover, if* G *has a consistent lifting which is both left and right invariant (e.g.* G *is abelian) all the conditions are equivalent.*

Note: We dont know when there exist bi-invariant liftings.

Proof. – $c \Longrightarrow a$ follows from lemma 14. To prove $a \Longrightarrow c$, we show that for any left invariant lifting ρ, the function $h = \rho(f)$ satisfies c). Notice that for all $t, u \in G$, we have

$$R_t\rho(f)\,(u) = \rho(f)\,(ut) = L_u\rho(f)\,(t) = \rho(L_u f)\,(t).$$

Since the map $g \longrightarrow \rho(g)\,(t)$ is a character on L^∞, this shows that $\{R_t\rho(f) : t \in G\} \subset \{u \longrightarrow \theta(L_u f);\ \theta$ character on $L^\infty(G)\}$ and the right-hand set is τ_p-compact and contained in $\mathcal{R}^\infty(G)$.

It is obvious that $b \Longrightarrow a$. To prove the converse, notice that for each character θ on $L^\infty(G)$ and each character η on the algebra $\mathfrak{F}(G)$ of bounded real-valued functions on G, we can define a character ξ on $L^\infty(G)$ by $\xi(g) = \eta(u \longrightarrow \theta(L_u g))$. Moreover, if $\varphi(u) = \theta(L_u f)$,

$$\xi(L_t f) = \eta(u \longrightarrow \theta(L_u(L_t f))) = \eta(u \longrightarrow \theta(L_{tu} f)) = \eta(L_t\varphi).$$

Since by hypothesis $t \longrightarrow \xi(L_t f)$ is measurable, $t \longrightarrow \eta(L_t\varphi)$ is measurable. But for each character θ on L^∞ there exists a character η on $\mathfrak{F}(G)$ such that $\eta(h_1) = \theta(h_2)$ whenever $h_2 \in L^\infty(G)$, $h_1 \in \mathcal{R}^\infty(G)$, $h_1 \in h_2$. So theorem 16 shows that φ is equivalent to a Riemann-measurable function. We have shown the equivalence of a, b, c. The proof of the equivalence of a', b', c' is similar. Suppose now G has a consistent lifting ρ which is both right and left invariant. We can factor f through a metrizable quotient of G, and it is easily checked that this quotient still have a consistent lifting (call it again ρ) right and left invariant. So we can suppose G metrizable. We have seen that under a) the set

$$\{R_t\rho(f)\,;\ t \in G\} = \{\rho(R_t(f))\,;\ t \in G\}$$

is τ_p-relatively compact in $\mathcal{R}^\infty(G)$. So theorem 5 proposition 10 and lemma 14 show that $\overline{c}\{R_t\rho(f);\ t \in G\} \subset \mathcal{R}^\infty(G)$, i.e. that a' holds. Q.E.D.

A natural question is to ask if, when the conditions of theorem 18 are satisfied, the map $t \longrightarrow \theta(L_t f)$ is Riemann-measurable. The following result shows that even if $G = \mathbf{R}$, it is not the case.

PROPOSITION 19. – *There exists a function* $f : \mathbf{R} \longrightarrow \{0, 1\}$ *such that*

a) *For all* $\varphi \in L^\infty(\mathbf{R})^*$, $t \longrightarrow \varphi(L_t f)$ *is equivalent to a Riemann-measurable function*

b) *There exists a character* θ *on* $L^\infty(\mathbf{R})$ *such that both sets* $\{t \in \mathbf{R}\,;\ \theta(L_t f) = 0\}$ *and* $\{t \in \mathbf{R}\,;\ \theta(L_t f) = 1\}$ *are dense in* $\mathbf{R}$.

Note that when G is compact, condition a) and MA imply that f is equivalent to a Riemann measurable function, and the proof of Theorem 16 shows that $t \longrightarrow \varphi(L_t f)$ is Riemann measurable.

Proof. – Let (P_n) be an enumeration of the finite subsets of $\mathbf{Q}$. Let ϵ_n a sequence of real numbers > 0 such that $\sum_n \epsilon_n (\operatorname{card} P_n)^2 < +\infty$. Let $A = \bigcup_n (a_n + P_n + [0, \epsilon_n])$, where the sequence (a_n) is chosen by induction such that

$$1 + \sup(\bigcup_{k<n} a_k + P_k + [0, \epsilon_k]) \leqslant a_n + \operatorname{Inf} P_n .$$

Let $f = \chi_A$.

It is enough to show a) when moreover $\varphi \geqslant 0$. If φ is localizable it was shown in theorem 16 that $t \longrightarrow \varphi(L_t f)$ is measurable. Now each positive $\varphi \in L^\infty(G)^*$ can be written $\varphi = \varphi_1 + \varphi_2$, where φ_1 is localizable and $\varphi_2(\chi_K) = 0$ for each compact K of $\mathbf{R}$. To prove a) it is hence enough to prove

c) If $\varphi \in L^\infty(G)^*$, $\varphi \geqslant 0$, $\varphi(\chi_K) = 0$ for all compact K of $\mathbf{R}$, then $\varphi(L_t f) = 0$ for almost all real t.

Suppose that there exists $\alpha > 0$ such that $\varphi(L_t f) > \alpha$ in a non-negligible set. Hence there exists an interval L in $\mathbf{R}$, of length one, such that if $B = \{t \in L ; \varphi(L_t f) > \alpha\}$ the outer measure of B is equal to $\beta > 0$. Let n an integer such that $n\alpha \geqslant 2$. For $(t_1, \ldots, t_n) \in B^n$, we have $\varphi\left(\sum_{i \leqslant n} L_{t_i} f\right) \geqslant n\alpha \geqslant 2$. This shows that for all compact K of $\mathbf{R}$ there exists $u \notin K$ with $\sum_{i \leqslant n} L_{t_i} f(u) \geqslant 2$. Hence there exists $1 \leqslant i_1 < i_2 \leqslant n$ such that $t_{i_1} + u \in A$, $t_{i_2} + u \in A$. The construction of A shows that there exists p with $t_{i_1} + u$, $t_{i_2} + u \in a_p + P_p + [0, \epsilon_p]$. Moreover for all p_0, one can take K large enough such that the above condition implies $p \geqslant p_0$.

Let us denote by λ the canonical measure of L. An elementary estimate using Fubini's theorem shows that

$$\lambda \otimes \lambda (\{(t_1, t_2) \in L^2 ; \exists u \in \mathbf{R}, t_1 + u, t_2 + u \in a_p + P_p + [0, \epsilon_p]\}) \leqslant 2\epsilon_p (\operatorname{card} P_p)^2$$

we have hence shown that for all p_0 :

$$\lambda_n^*\Big(\Big\{(t_1,\ldots,t_n)\in \mathrm{L}^n\,;\ \varphi\Big(\sum_{i\leqslant n} \mathrm{L}_{t_i}f\Big)\geqslant 2\Big\}\Big) \leqslant \mathrm{C}_n^2 \sum_{p\geqslant p_0} 2\epsilon_p\,(\mathrm{card}\ \mathrm{P}_p)^2$$

where λ_n^* is the outer measure associated to $\lambda^{\otimes n}$. We hence have $\beta^n = \lambda_n^*(\mathrm{B}^n) = 0$, so $\beta = 0$, which proves c).

The condition c) shows that to prove b) it is enough to show that there exists a non-localizable character θ on $\mathrm{L}^\infty(\mathbf{R})$ with $\theta(\mathrm{L}_t f) = 1$ for $t \in \mathbf{Q}$. But this follows easily from the fact that for each n, and each $t \in \mathrm{P}_n$, $\mathrm{L}_t f = 1$ on $a_n + [0, \epsilon_n]$.

Theorem 18 and proposition 19 suggest an other extension of theorem 16. This next result is proved in a similar way as theorem 18, except that we use theorem 8 instead of theorem 5.

THEOREM 20. – *Let* G *be a locally compact abelian group and* $f \in \mathrm{L}^\infty(\mathrm{G})$. *The following are equivalent:*

a) *For all character* θ *on* $\mathrm{L}^\infty(\mathrm{G})$, $t \longrightarrow \theta(\mathrm{L}_t f)$ *is Riemann-measurable.*

b) *For all* $\varphi \in \mathrm{L}^{\infty *}(\mathrm{G})$, $t \longrightarrow \varphi(\mathrm{L}_t f)$ *is Riemann-measurable.*

c) *There exist* $h' \in \mathcal{L}^\infty(\mathrm{G})$, $h' \in f$, *such that* $\{\mathrm{L}_t h'\,;\ t \in \mathrm{G}\}$ *is* τ_p*-relatively compact in the set of Riemann-measurable functions.*

d) *There exist* $h \in \mathcal{L}^\infty(\mathrm{G})$, $h \in f$, *such that* $\overline{c}\{\mathrm{L}_t h\,;\ t \in \mathrm{G}\}$ *consists of Riemann-measurable functions.*

Though the proof of the following theorem follows easily from a powerful result of D.H. Fremlin, and is not related to the methods of this paper, we include it here since it is obviously akin to the preceeding results.

THEOREM 21. – *Let* G *be a locally compact abelian group and* $f \in \mathrm{L}^\infty(\mathrm{G})$. *The following are equivalent*

a) *There is a left uniformly continuous function* h *in the class of* f.

b) *The map* $t \longrightarrow \mathrm{L}_t f$ *is measurable for the Haar measure when* $\mathrm{L}^\infty(\mathrm{G})$ *is provided with the Borel* σ*-algebra generated by the norm.*

Proof. – $a \Longrightarrow b$ is obvious, since this map is then continuous. $b \Longrightarrow a$: Let K be a compact of positive measure. Since the map $t \longrightarrow L_t f$ is measurable, the result of Fremlin [3] tells it is Lusin measurable, i.e. there is a closed $L \subset K$, with $|L| > 0$, on which this map is continuous.

So for each $\epsilon > 0$, there exists a neighborhood V of the unit of G such that for $t, u \in L$, $tu^{-1} \in V$, we have $\|L_t f - L_u f\| \leqslant \epsilon$. It is known that LL^{-1} is a neighborhood of the unit of G. (For example, if ρ is a left invariant lifting of G, $L^{-1}L$ contains the open set $\rho(L^{-1})\,\rho(L^{-1})^{-1})$. For $x, y \in G$ with $xy^{-1} \in V \cap LL^{-1}$, one can write $xy^{-1} = tu^{-1}$, for $t, u \in L$. Since $tu^{-1} \in V$, one has

$$\|L_x f - L_y f\| = \|L_{xy^{-1}} f - f\| = \|L_{tu^{-1}} f - f\| = \|L_t f - L_u f\| \leqslant \epsilon .$$

We have shown that $t \longrightarrow L_t f$ is continuous on G. The result follows easily, e.g. by taking $h = \rho(f)$ for a left invariant lifting ρ. (It is continuous since $\rho(f)(t) = \rho(L_t f)(e)$ where e denotes the unit of G.)

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