ANNALES DE L'INSTITUT FOURIER

MANUEL VALDIVIA

A class of locally convex spaces without $\mathcal{C}$-webs

Annales de l'institut Fourier, tome 32, nº 2 (1982), p. 261-269
<http://www.numdam.org/item?id=AIF_1982__32_2_261_0>

Ann. Inst. Fourier, Grenoble
32, 2 (1982), 261-269.

A CLASS OF LOCALLY CONVEX SPACES WITHOUT $\mathscr{C}$-WEBS

by Manuel VALDIVIA

The linear spaces we use are defined over the field K of the real or complex numbers. Space means non-zero separated locally convex topological linear space. Given a space E we denote by E′ its topological dual. The dimension of E is denoted by dim E. $\hat{E}$ is the completion of E. If E_n is a one-dimensional space, $n = 1, 2, \ldots$, we put $\varphi = \bigoplus_{n=1}^{\infty} E_n$.

A space E is unordered Baire-like if given a sequence (A_n) of closed absolutely convex sets of E, with $E = \bigcup_{n=1}^{\infty} A_n$, there is a positive integer n_0 such that A_{n_0} is a neighbourhood of the origin in E.

A space E is suprabarrelled if given an increasing sequence (E_n) of subspaces of E covering E there is a positive integer n_0 such that E_{n_0} is barrelled and dense in E, [7].

A space E is Baire-like if given an increasing sequence (A_n) of closed absolutely convex sets of E covering E there is a positive integer n_0 such that A_{n_0} is a neighbourhood of the origin in E.

If A is a bounded absolutely convex subsets of a space E we denote by E_A the linear hull of A endowed with the topology derived from the gauge of A. A is completing if E_A is a Banach space.

In what follows E and F are spaces and n is any non negative integer. We denote by $E \otimes_\pi F$ and $E \otimes_\varepsilon F$ the tensor product $E \otimes F$ endowed with the projective topology and the topology of the by-equicontinuous convergence respectively. $E \hat{\otimes}_\varepsilon F$ is the completion of $E \otimes_\varepsilon F$. We set

$$R_n(E, F) = \{z \in E \otimes F : \text{rank } z \leqslant n\}.$$

We write $\hat{R}_n(E, F)$ to denote the closure of $R_n(E, F)$ in $E \hat{\otimes}_\varepsilon F$.

MAIN LEMMA. – $\hat{R}_n(E, F) = R_n(\hat{E}, \hat{F})$.

Proof. – We proceed by complete induction. The result is obvious for $n = 0$. We suppose that the property is true for a positive integer p. Let

$$\left\{ \sum_{j=1}^{p+1} x_j^i \otimes y_j^i : i \in I, \geqslant \right\} \tag{1}$$

be a net in $R_{p+1}(E,F)$ converging in $E \hat{\otimes}_\varepsilon F$ to an element $z \neq 0$. Since $E' \otimes F'$ separate points in $E \hat{\otimes}_\varepsilon F$, we can select $u \in E'$, $v \in F'$ such that $u \otimes v(z) \neq 0$. Obviously

$$\lim \left\{ \sum_{j=1}^{p+1} u(x_j^i)v(y_j^i) : i \in I, \geqslant \right\} = u \otimes v(z) \neq 0. \tag{2}$$

Since (1) is a Cauchy net in $E \otimes_\varepsilon F$ given $\delta > 0$ and an equicontinuous subset V of F' there is an $i_0 \in I$ such that

$$\sup \left\{ \left| \sum_{j=1}^{p+1} (u(x_j^h)w(y_j^h) - u(x_j^k)w(y_j^k)) \right| : w \in V \right\} < \delta, \quad \text{for} \quad h, k \geqslant i_0$$

and therefore, according to (2),

$$\left\{ \sum_{j=1}^{p+1} u(x_j^i)y_j^i : i \in I, \geqslant \right\} \tag{3}$$

converges in $\hat{F}$ to $y \neq 0$.

From (2) we deduce the existence of a subnet of (1), for which we use the same notation, such that at least one of the nets

$$\{u(x_j^i) : i \in I, \geqslant\}, \qquad j = 1, 2, \ldots, p+1,$$

has all its elements different from zero. Without loss of generality we suppose that $u(x_1^i) \neq 0, \forall i \in I$.

Setting $\sum_{j=1}^{p+1} u(x_j^i)y_j = v_1^i, i \in I$, we have that

$$\sum_{j=1}^{p+1} x_j^i \otimes y_j^i = \frac{1}{u(x_1^i)} x_1^i \otimes \left(v_1^i - \sum_{j=2}^{p+1} u(x_j^i)y_j^i \right) + \sum_{j=2}^{p+1} x_j^i \otimes y_j^i,$$

and setting

$$z_1^i = \frac{1}{u(x_1^i)} x_1^i, \quad z_j^i = x_j^i - \frac{u(x_j^i)}{u(x_1^i)} x_1^i, \quad v_j^i = y_1^i, \quad j = 2, 3, \ldots, p+1,$$

we have that the net (1) can be expressed as

$$\left\{\sum_{j=1}^{p+1} z_j^i \otimes v_j^i \; : \; i \in \mathrm{I}, \geqslant\right\}.$$

Reasoning in the same way as we have done for the net (3) it follows that $\left\{\sum_{j=1}^{p+1} v(v_j^i)z_j^i : i \in \mathrm{I}, \geqslant\right\}$ converges in $\hat{\mathrm{E}}$ to an element $x \neq 0$. According to (2) again we suppose that $v(v_1^i) \neq 0$, $\forall i \in \mathrm{I}$, and setting $\sum_{j=1}^{p+1} v(v_j^i)z_j^i = u_1^i$ it follows that

$$\begin{aligned}\sum_{j=1}^{p+1} x_j^i \otimes y_j^i &= \sum_{j=1}^{p+1} z_j^i \otimes v_j^i \\ &= \frac{1}{v(v_1^i)}\left(u_1^i - \sum_{j=2}^{p+1} v(v_j^i)z_j^i\right) \otimes v_1^i + \sum_{j=2}^{p+1} z_j^i \otimes v_j^i \\ &= \frac{1}{v(v_1^i)} u_1^i \otimes v_1^i + \sum_{j=2}^{p+1} z_j^i \otimes \left(v_j^i - \frac{v(v_j^i)}{v(v_1^i)} v_1^i\right).\end{aligned}$$

The net $\left\{\frac{1}{v(v_1^i)} u_1^i \otimes v_1^i : i \in \mathrm{I}, \geqslant\right\}$ converges to $\frac{1}{u \otimes v(z)} x \otimes y$ in $\mathrm{E} \hat{\otimes}_\varepsilon \mathrm{F}$ and according to the induction hypothesis

$$\left\{\sum_{j=2}^{p+1} z_j^i \otimes \left(v_j^i - \frac{v(v_j^i)}{v(v_1^i)} v_1^i\right) : i \in \mathrm{I}, \geqslant\right\}$$

converges to an element of $\mathrm{R}_p(\hat{\mathrm{E}},\hat{\mathrm{F}})$ and therefore

$$\hat{\mathrm{R}}_{p+1}(\mathrm{E},\mathrm{F}) \subset \mathrm{R}_{p+1}(\hat{\mathrm{E}},\hat{\mathrm{F}}).$$

Finally, since $\mathrm{R}_{p+1}(\mathrm{E},\mathrm{F})$ is obviously dense in $\mathrm{R}_{p+1}(\hat{\mathrm{E}},\hat{\mathrm{F}})$ it follows that $\hat{\mathrm{R}}_{p+1}(\mathrm{E},\mathrm{F}) = \mathrm{R}_{p+1}(\hat{\mathrm{E}},\hat{\mathrm{F}})$. q.e.d.

COROLLARY 1. – *If* E *and* F *are complete then* $\mathrm{R}_n(\mathrm{E},\mathrm{F})$ *is complete in* $\mathrm{E} \otimes_\varepsilon \mathrm{F}$.

COROLLARY 2. – $\mathrm{R}_n(\mathrm{E},\mathrm{F})$ *is closed in* $\mathrm{E} \otimes_\varepsilon \mathrm{F}$.

Proof. – $\mathrm{R}_n(\mathrm{E},\mathrm{F})$ coincides with $\mathrm{R}_n(\hat{\mathrm{E}},\hat{\mathrm{F}}) \cap (\mathrm{E} \otimes \mathrm{F})$ and according to Main Lemma $\mathrm{R}_n(\hat{\mathrm{E}},\hat{\mathrm{F}})$ is complete in $\mathrm{E} \hat{\otimes}_\varepsilon \mathrm{F}$, hence $\mathrm{R}_n(\mathrm{E},\mathrm{F})$ is closed in $\mathrm{E} \otimes_\varepsilon \mathrm{F}$. q.e.d.

THEOREM 1. — $E \otimes_\varepsilon F$ *has a $\mathscr{C}$-web if and only if one of the two following conditions is satisfied :*

1. E *has a $\mathscr{C}$-web and* $\dim F \leqslant \chi_0$;
2. F *has a $\mathscr{C}$-web and* $\dim E \leqslant \chi_0$.

Proof. — If F is finite dimensional and if E has a $\mathscr{C}$-web then $E \otimes_\varepsilon F$ is isomorphic to $E^{\dim F}$ which has a $\mathscr{C}$-web. If F has countable infinite dimension and if E has a $\mathscr{C}$-web then $E \otimes_\varepsilon F$ has a $\mathscr{C}$-web since the topology of $E \otimes_\varepsilon F$ is coarser than the topology ot $E \otimes_\varepsilon \varphi$ and this space has a topology coarser than the topology of $E \oplus E \oplus E \oplus \dots$. We reach the same conclusion changing the roles of E and F.

We suppose now that $\dim E > \chi_0$, $\dim F > \chi_0$ and that $E \otimes_\varepsilon F$ has a $\mathscr{C}$-web $(C_{n_1, n_2, \dots, n_p})$. Let $\{x_i : i \in I\}$ and $\{y_i : i \in I\}$ be families of linearly independent vectors in E and F respectively such that $\operatorname{card} I > \chi_0$. Since $\bigcup_{n=1}^{\infty} C_n = E \otimes F$ and $\bigcup_{n_{p+1}=1}^{\infty} C_{n_1, n_2, \dots, n_{p+1}} = C_{n_1, n_2, \dots, n_p}$ there is a sequence (n_p) of positive integers and there is a decreasing sequence (J_p) of subsets of I such that $x_i \otimes y_i \in C_{n_1, n_2, \dots, n_p}$, $i \in J_p$, $\operatorname{card} J_p > \chi_0$, $p = 1, 2, \dots$. It follows that we can take linearly independent elements $u_p \in E$, $v_p \in F$, $p = 1, 2, \dots$, such that

$$u_p \otimes v_p \in C_{n_1, n_2, \dots, n_p}.$$

We select a sequence (λ_p) of strictly positive real numbers such that $\{\lambda_p u_p \otimes v_p : p = 1, 2, \dots\}$ is contained in a bounded completing absolutely convex subset A of $E \otimes_\varepsilon F$ [3, p. 75]. Since $(E \otimes F)_A$ is a Banach space and since $R_n(E,F)$ is closed in $E \otimes_\varepsilon F$ there is a positive integer n_0 such that $R_{n_0}(E,F) \cap (E \otimes F)_A$ has interior point in $(E \otimes F)_A$ and therefore

$$R_{2n_0}(E,F) \cap (E \otimes F)_A = (R_{n_0}(E,F) + R_{n_0}(E,F)) \cap (E \otimes F)_A$$

is a neighbourhood of the origin in $(E \otimes F)_A$ and since $R_{2n_0}(E,F)$ contains any scalar multiple of every vector $z \in R_{2n_0}(E,F)$ it follows that

$$A \subset R_{2n_0}(E,F).$$

But $\sum_{p=1}^{2n_0+1} \frac{1}{2^p} \lambda_p u_p \otimes v_p$ belongs to A and its rank is $2n_0 + 1$. This is a contradiction.

We suppose now that $E \otimes_\varepsilon F$ has a $\mathscr{C}$-web and $\dim F \leqslant \chi_0$. We take $y_0 \in \mathrm{F}$, $y_0 \neq 0$. Let T be the mapping from E into $\mathrm{E} \otimes_\varepsilon \mathrm{F}$ such that $\mathrm{T}(x) = x \otimes y_0$, $x \in \mathrm{E}$. Then T(E) is isomorphic to E (cf. [3], § 44, 1.(4)). Since T(E) is contained in $\mathrm{R}_1(\mathrm{E,F})$ it is easy to prove that T(E) is closed in $\mathrm{E} \otimes_\varepsilon \mathrm{F}$ and thus has a $\mathscr{C}$-web. Analogously, if $\mathrm{E} \otimes_\varepsilon \mathrm{F}$ has a $\mathscr{C}$-web and $\dim \mathrm{E} \leqslant \chi_0$, F has a $\mathscr{C}$-web. q.e.d.

THEOREM 2. — $\mathrm{E} \otimes_\pi \mathrm{F}$ *has a $\mathscr{C}$-web if and only if one of the two following conditions is satisfied :*

1. E *has a $\mathscr{C}$-web and* $\dim \mathrm{F} \leqslant \chi_0$;
2. F *has a $\mathscr{C}$-web and* $\dim \mathrm{E} \leqslant \chi_0$.

Proof. — On $\mathrm{E} \otimes \mathrm{F}$ the π-topology is finer than the ε-topology. Hence, if $\mathrm{E} \otimes_\pi \mathrm{F}$ has a $\mathscr{C}$-web then $\mathrm{E} \otimes_\varepsilon \mathrm{F}$ has a $\mathscr{C}$-web. Thus we apply the former result to obtain that 1 or 2 is satisfied.

If F is finite dimensional and if E has a $\mathscr{C}$-web then $\mathrm{E} \otimes_\pi \mathrm{F} = \mathrm{E} \otimes_\varepsilon \mathrm{F}$ has a $\mathscr{C}$-web. If F has countable infinite dimension and if E has a $\mathscr{C}$-web then $\mathrm{E} \otimes_\pi \mathrm{F}$ has a $\mathscr{C}$-web since the topology of $\mathrm{E} \otimes_\pi \mathrm{F}$ is coarser than the topology of $\mathrm{E} \otimes_\pi \varphi = \mathrm{E} \otimes_\varepsilon \varphi$. We reach the same conclusion changing the roles of E and F. q.e.d.

THEOREM 3. — *If* E *and* F *are Suslin spaces then* $\mathrm{E} \otimes_\pi \mathrm{F}$ *is a Suslin space.*

Proof. — Let $\psi : \mathrm{E} \times \mathrm{F} \to \mathrm{E} \otimes_\pi \mathrm{F}$ be the canonical bilinear mapping which is continuous and thus $\psi(\mathrm{E,F}) = \mathrm{R}_1(\mathrm{E,F})$ is a Suslin topological subspace of $\mathrm{E} \otimes_\pi \mathrm{F}$. On the other hand the mapping

$$\theta : \mathrm{R}_1(\mathrm{E,F}) \times \mathrm{R}_1(\mathrm{E,F}) \times \ldots \times \mathrm{R}_1(\mathrm{E,F}) \to \mathrm{E} \otimes_\pi \mathrm{F}$$

so that

$$\theta(x_1 \otimes y_1, x_2 \otimes y_2, \ldots, x_n \otimes y_n) = \sum_{j=1}^{n} x_j \otimes y_j.$$

is π-continuous and therefore

$$\theta(\mathrm{R}_1(\mathrm{E,F}) \times \mathrm{R}_1(\mathrm{E,F}) \times \ldots \times \mathrm{R}_1(\mathrm{E,F})) = \mathrm{R}_n(\mathrm{E,F})$$

is a Suslin topological subspace of $\mathrm{E} \otimes_\pi \mathrm{F}$. Finally, $\mathrm{E} \otimes_\pi \mathrm{F}$ is a Suslin space since coincides with $\bigcup_{n=1}^{\infty} \mathrm{R}_n(\mathrm{E,F})$, (cf. [2], § 6, N° 2, Prop. 8).

q.e.d.

COROLLARY 1.3. — *If* E *and* F *are Suslin spaces then* $E \otimes_\varepsilon F$ *is a Suslin space.*

Note 1. — If we consider two Suslin spaces E and F of dimension larger than χ_0 we can apply the former results to obtain that $E \otimes_\pi F$ is a Suslin space without a $\mathscr{C}$-web. Thus the closed graph theory of De Wilde [4] does not contain the closed graph theorem of L. Schwartz, [7] and [8].

Bourbaki [1, p. 43] proves that if E and F are metrizable and F is barrelled then every separately equicontinuous set of bilinear forms on $E \times F$ is equicontinuous. Since every barrelled metrizable space is Baire-like [1], Theorem 4 generalizes this result.

THEOREM 4. — *If* E *is metrizable and* F *is a Baire-like space then every separately equicontinuous set* $\mathscr{B}$ *of bilinear forms on* $E \times F$ *is equicontinuous.*

Proof. — Let (U_n) be a decreasing basis of closed absolutely convex neighbourhoods of the origin in E. We set

$$V_n = \{y \in F : |B(x,y)| \leqslant 1, x \in U_n, B \in \mathscr{B}\}, \quad n = 1, 2, \ldots,$$

which is a closed absolutely convex subset of F and the sequence (V_n) is increasing. On the other hand if $z \in F$ there is a positive integer p such that $|B(x,z)| \leqslant 1$, $x \in U_p$, $B \in \mathscr{B}$, according to the separate equicontinuity of $\mathscr{B}$. Thus $z \in V_p$ and since F is Baire-like there is a positive integer q such that V_q is a neighbourhood of the origin in F. Therefore $|B(x,y)| \leqslant 1$, $(x,y) \in U_q \times V_q$, $B \in \mathscr{B}$. q.e.d.

COROLLARY 1.4. — *If* E *is metrizable and if* F *is a barrelled space whose completion is Baire, then every set separately equicontinuous of bilinear forms on* $E \times F$ *is equicontinuous.*

Proof. — It is an immediate consequence of the former theorem and from the fact that every barrelled space whose completion is Baire is a Baire-like, [6]. q.e.d.

Note 2. — It is immediate that our Theorem 4 can be easily generalized to bilinear mapping with range in a third locally convex space. Then this theorem contains another result due to Bourbaki [3, Ex. 1, Chap. III, § 4, p. 44] for locally convex spaces.

PROPOSITION 1. — *If* E *is a barrelled metrizable space and if* F *is a Baire-like space then* $E \otimes_\pi F$ *is Baire-like.*

Proof. — Let (W_n) be an increasing sequence of closed absolutely convex subsets of $E \otimes_\pi F$ covering $E \otimes F$ and let (U_n) be a decreasing basis of closed absolutely convex neighbourhoods of the origin in E. We set

$$V_n = \{y \in F : x \otimes y \in W_n, x \in U_n\}, \qquad n = 1, 2, \ldots,$$

which is a closed absolutely convex subsets of F and (V_n) is an increasing sequence.

$$Z_n = \{x \in E : x \otimes z \in W_n\}, \qquad n = 1, 2, \ldots,$$

which is a closed absolutely convex subset of E and (Z_n) is an increasing sequence covering E. Then there is a positive integer n_0 such that Z_{n_0} is a neighbourhood of the origin in E. We can find a positive integer $p \geqslant n_0$ such that $U_p \subset Z_{n_0}$ and then $z \in V_p$ and so (V_n) covers F. Since F is a Baire-like there is a positive integer q such that V_q is a neighbourhood of the origin in E and thus W_q is a neighbourhood of the origin in $E \otimes_\pi F$.

q.e.d.

PROPOSITION 2. — *If* E *and* F *are unordered Baire-like spaces and if* E *is metrizable then* $E \otimes_\pi F$ *is unordered Baire-like.*

Proof. — Let (W_n) be a sequence of closed absolutely convex subsets of $E \otimes_\pi F$ covering $E \otimes F$ and let (U_n) be a decreasing basis of closed absolutely convex neighbourhood of the origin in E. We set

$$V_{n,m} = \{y \in F : x \otimes y \in W_n, \forall x \in U_m\}, \qquad , m = 1, 2, \ldots .$$

Obviously $V_{n,m}$ is a closed absolutely convex subset of F. We shall see that $\bigcup_{n,m=1}^{\infty} V_{n,m} = F$.

Indeed, let $z \in F$ and set

$$Z_n = \{x \in E : x \otimes z \in W_n\}, \quad n = 1, 2, \ldots,$$

which is a closed absolutely convex subset of E such that $\bigcup_{n=1}^{\infty} Z_n = E$. There is a positive integer p such that Z_p is a neighbourhood of the origin in E. We find a positive integer q such that $U_q \subset Z_p$. Then

$z \in V_{p,q}$. We can find positive integers r, s such that $V_{r,s}$ is a neighbourhood of the origin in F. If $\psi : E \times F \to E \otimes F$ is the canonical bilinear mapping then $\psi(U_s \times V_{r,s}) \subset W_r$ and thus W_r is a neighbourhood of the origin in $E \otimes_\pi F$. q.e.d.

PROPOSITION 3. — *If* E *is a metrizable suprabarrelled space and if* F *is an unordered Baire-like space, then* $E \otimes_\pi F$ *is suprabarrelled.*

Proof. — Let (G_n) be an increasing sequence of subspaces of $E \otimes_\pi F$ covering $E \otimes F$ and let us suppose G_n is not barrelled, $n = 1, 2, \ldots$. We can find in each G_n a barrel T_n which is not a neighbourhood of the origin in G_n. Let W_n be the closure of T_n in $E \otimes_\pi F$, $n = 1, 2, \ldots$. We use $V_{n,m}$ and Z_n with the same meaning as before. We set

$$S_n = \{x \in E : x \otimes z \in G_n\}, \quad n = 1, 2, \ldots.$$

(S_n) is an increasing sequence of subspace of E covering E and therefore there is a positive integer p such that S_p is barrelled and dense in E. If $x \in S_p$ there is a real number $h > 0$ such that $h(x \otimes z) \in T_p \subset W_p$ and thus $hz \in Z_p$, i.e., Z_p is absorbing and thus a barrel in E. It follows that $Z_p \cap S_p$ is a barrel in S_p and therefore a neighbourhood of 0. But S_p is dense in E, so $Z_p = \overline{Z_p \cap S_p}$ is a neighbourhood of the origin in E. Reasoning as we did in the proposition above, we obtain a positive number r such that W_r is a neighbourhood of the origin in $E \otimes_\pi F$, which is a contradiction. According to Proposition 1 there is a positive integer n_0 such that G_n is dense in $E \otimes_\pi F$, $n \geqslant n_0$. q.e.d.

PROPOSITION 4. — $E \otimes_\pi F$ *is a Baire space if and only if one of the following two conditions is satisfied :*

1. $\dim E < \infty$ *and* $F^{\dim E}$ *is a Baire Space*;
2. $\dim F < \infty$ *and* $E^{\dim F}$ *is a Baire space.*

Proof. — If the dimension of E is finite then $E \otimes_\pi F$ is isomorphic to $F^{\dim E}$. Analogously if F is finite dimensional then $E \otimes_\pi F$ is isomorphic to $E^{\dim F}$. Suppose that E and F are infinite dimensional and $E \otimes_\pi F$ is a Baire space. We can find a positive integer n_0 such that $R_{n_0}(E,F)$ is of second category in $E \otimes_\pi F$. According to Corollary 2.1, $R_{n_0}(E,F)$ is closed in $E \otimes_\varepsilon F$ and thus it is closed in $E \otimes_\pi F$ hence $R_{2n_0}(E,F)$ is a neighbourhood of the origin in $E \otimes_\pi F$ which is a contradiction since $R_{2n_0}(E,F)$ is not absorbing in $E \otimes F$.

The following Theorem is an immediate consequence of the preceding results, having in mind that every infinite dimensional Fréchet spaces has dimension larger or equal than $2^{\aleph_0}$, [6].

THEOREM 5. — *If* E *and* F *are infinite dimensional Fréchet spaces the following is true for* $E \otimes_\pi F$:

1. *is unordered Baire-like;*
2. *is not a Baire space;*
3. *has not a* $\mathscr{C}$*-web.*

If E *and* F *are separable then*

4. $E \otimes_\pi F$ *is a Suslin space.*

We gratefully thank Prof. H. Horváth for his valuable suggestions.

BIBLIOGRAPHY

[1] I. AMEMIYA and Y. KOMURA, Über nich-vollständinge Montelräume, *Math. Ann.*, 177 (1968), 273-277.
[2] N. BOURBAKI, Éléments de Mathématique, Topologie Générale Chap. IX, Hermann, Paris, 1974.
[3] N. BOURBAKI, Éléments de Mathématique, Espaces vectoriels topologiques, *Act. Sci. et Ind.*, Paris, vol. 1929 (1955).
[4] M. DE WILDE, Réseaux dans les espaces linéaires à semi-normes, *Mém. Soc. Royale Sci. Liège*, 18 (5), 2 (1969), 1-144.
[5] G. KÖTHE, Topological Vector Spaces II, Springer-Verlag, New York-Heidelberg-Berlin, 1979.
[6] H. LÖWIG, Über Die dimension linearer Räume, *Studia Math.*, (1934), 18-23.
[7] A. MARTINEAU, Sur le théorème du graphe fermé, *C. R. Acad. Sci.*, Paris, 263 (1966), 870-871.
[8] L. SCHWARTZ, Sur le théorème de graphe fermé. *C.R. Acad. Sci.*, Paris 263 (1966), 602-605.
[9] M. VALDIVIA, Absolutely convex sets in barrelled spaces, *Ann. Inst. Fourier*, Grenoble, 21, 2 (1971), 3-13.
[10] M. VALDIVIA, On suprabarrelled spaces. Functional Analysis, Holomorphy and Approximation Theory. Proocedings, Rio de Janeiro 1978, *Lecture Notes in Math.*, 843 (1981), 572-580, Springer-Verlag, New York-Heidelberg-Berlin.

Manuscrit reçu le 20 août 1981.

Manuel VALDIVIA,
Facultad de Matemáticas
Dr. Moliner s.n.
Burjasot (Valencia) Espagne.