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# IMPROVEMENT OF GRAUERT-RIEMENSCHNEIDER'S THEOREM FOR A NORMAL SURFACE

by Jean GIRAUD

## 1. Vanishing theorem.

**1.1.** A *surface* is a noetherian, excellent, normal scheme of dimension 2. A desingularization of X is a proper and birational map $f: \tilde{X} \to X$ such that $\tilde{X}$ is regular. The set

(1) $\quad \mathrm{Sing}\,(f) = \{x \in X,\ \dim(f^{-1}(x)) > 0\}$

is made up of finitely many closed points and $f$ is an isomorphism above

(2) $X_f = X - \mathrm{Sing}\,(f) \subset X_{\mathrm{reg}} = \{x \in X,\ 0_{X,x} \text{ is regular}\}$.

We usually denote by $E_i$ the irreducible components of

(3) $$E(f) = f^{-1}\,(\mathrm{Sing}\,(f))$$

and for $A = \mathbf{N}$, $\mathbf{Z}$ or $\mathbf{Q}$, we let

(4) $$NS(f,A) = \oplus\, A E_i .$$

We do not assume that $X_f = X_{\mathrm{reg}}$, hence X itself may be regular. For any $V = \Sigma V_i . E_i \in NS(f,\mathbf{Q})$, we write

(5) $V \geqslant 0$ when all $V_i$ are $\geqslant 0$

(6) $V \gtrsim 0$ when all $-V.E_i$ are $\geqslant 0$.

Note that the minus sign is justified by

(7) $$V \gtrsim 0 \Rightarrow V \geqslant 0 .$$

To prove (7) we let $V = V_+ - V_-$; since $V \gtrsim 0$, we have

$0 \leqslant -V.V = -V_+.V_- + V_-^2 \leqslant V_-^2$, hence $V_- = 0$, since the intersection matrix is negative definitive. We introduce the dual basis of $NS(f,\mathbf{Q})$

$$(8) \qquad E_i^* \text{ defined by } E_i^*.E_j = -\delta_{ij}$$

and we observe that

$$(9) \qquad E_i^* \geqq 0, \quad dE_i^* \in NS(f,\mathbf{N})$$

where $d$ is the absolute value of the determinant of the intersection matrix.

LEMMA 1.2. — *For any* $V \in NS(f,\mathbf{Q})$ *there exists a unique* $[V] \in NS(f,\mathbf{Z})$ *such that*

(i) $V \lesssim [V]$,

(ii) if $W \in NS(f,\mathbf{Z})$ *and if* $V \lesssim W$ *then* $[V] \leqslant W$.

We will prove that $[V]$ is the infimum for the usual order relation of $E(V) = \{W \in NS(f,\mathbf{Z}),\ V \lesssim W\}$. Let $N \in \mathbf{Z}$ be such that $dN \leqslant \inf(V.E_i)$; we have $-dN\Sigma E_i^* \in E(V)$, hence $E(V)$ is non empty. For $i = 1,2$, let $W_i = \Sigma W_{i,j}E_j \in E(V)$ and let $Z = \Sigma Z_j E_j$ with $Z_j = \inf(W_{1,j}, W_{2,j})$. By Artin's trick we prove that $Z \in E(V)$ as follows. For any $j$, we have $Z_j = W_{1,j}$ or $Z_j = W_{2,j}$. By symmetry we can assume that $Z_j = W_{1,j}$ and we get

$$Z.E_j = W_{1,j}E_j^2 + \sum_{k \neq j} Z_k E_k.E_j \leqslant W_1.E_j \leqslant V.E_j$$

hence $Z \geqq V$. To conclude, we note that the coordinates of any $W = \Sigma W_i E_i \in E(V)$ are bounded from below since $W_i = -W.E_i^* \geqslant -V.E_i^*$ since $E_i^*$ is $\geqslant 0$. Observe the obvious

$$(1) \quad [V+W] \leqslant [V] + [W];\ [V+E] = [V] + E \text{ if } E \in NS(f,\mathbf{Z}).$$

We also let

$$(2) \qquad [\underline{V}] = -[-V] \text{ in such a way that } [\underline{V}] \lesssim V \lesssim [V].$$

1.3. Let L be an invertible sheaf on $\tilde{X}$. We define

(1) $e_f(L) \in NS(f,\mathbf{Q})$ by $e_f(L).E_i = \deg(L|E_i)$ for any $i$.

We also write $L \gtrsim 0$ instead of $e_f(L) \gtrsim 0$ and this means $L.E_i \leqslant 0$ for all $i$. We will often drop the subscript $f$. Sending V to $0_{\tilde{X}}(V)$ we identify $NS(f,Z)$ to a subgroup of $Pic(\tilde{X})$ and since $V = e_f(0_{\tilde{X}}(V))$, $0_{\tilde{X}}(V) \gtrsim 0$ is equivalent to $V \gtrsim 0$. Hence, when we write $Pic(\tilde{X})$ additively, we can safely write V in place of $0_{\tilde{X}}(V)$ and $L + V$ in place of $L(V) = L \otimes 0_{\tilde{X}}(V)$. We will sometimes write [L] insteated of $[e_f(L)]$.

1.4. We can also give an algorithmic description of [V] as follows. Start with $Z \in NS(f,Z)$ such that $Z \leqslant [V]$. For instance, if $V = \Sigma V_i E_i$ let $Z = \Sigma V'_i E_i$ where $V'_i$ is the smallest integer $\geqslant V_i$. If $Z \neq [V]$ there must exist a $i$ such that $Z.E_i > V.E_i$ and we still have $Z + E_i \leqslant [V]$. In fact, since $V \lesssim [V]$, we have $([V]-Z).\ E_i \leqslant (V-Z).E_i < 0$, hence $([V]-Z) \geqslant E_i$ since $[V] - Z$ is effective with integral coefficients. We now replace Z by $Z + E_i$ and reach [V] in a finite number of steps.

VANISHING THEOREM 1.5. – *Let* $f: \tilde{X} \to X$ *be a desingularization of a normal surface* X, *let* $E = f^{-1}(\mathrm{Singl}(f))$ *and let* L *be an invertible sheaf on* $\tilde{X}$.

(i) *If* $[L] \gtrsim 0$ *then* $H^1_E(\tilde{X},L) = 0$.

(ii) *If* $[L] \geqslant 0$ *then* $f_*(L)$ *is reflexive.*

(iii) *Let* K *be the dualizing sheaf of* $\tilde{X}$. *If* $[K-L] \gtrsim 0$ *then* $\mathbf{R}^1 f_*(L) = 0$.

1.5.1. To prove (i) we let $M = [L]$ and $L' = L(-M)$ in such a way that $[L'] = 0$ and $M \gtrsim 0$, $M \in NS(f,N)$. For any $V \in NS(f,N)$, $V \neq 0$, there exists an $E_i$ such that $(L'+V).\ E_i < 0$. Otherwise we would have $L' + V \lesssim 0$ hence $L' \lesssim -V$, hence $0 = [L'] \leqslant -V < 0$ which is impossible. We observe that $E_i$ must be contained in the support of V, otherwise we would have $V.E_i \geqslant 0$, hence

$$(L'+V).E_i \geqslant ([L']+V).E_i = V.E_i \geqslant 0.$$

Furthermore, since $M \gtrsim 0$, we have

$$(L+V).E_i = (L'+M+V).E_i \leqslant (L'+V).E_i < 0.$$

As a consequence we get

$$(1) \qquad V - E_i \in NS(f,N) \quad \text{and} \quad (L+V).E_i < 0.$$

As a consequence we get $H^0(E_i, L(V)|E_i) = 0$ hence the map

$$(2) \qquad H^0(V-E_i;\ L(V-E_i)|(V-E_i)) \to H^0(V;L(V)|V)$$

is surjective. By induction on $V$, we conclude that, if $[L] \gtrsim 0$, we have

(3) $H^0(V,L(V)|V) = 0$ for any $V \in NS(f,N)$

hence $H^1_E(\tilde{X};L) = \varinjlim H^0(V;L(V)|V) = 0$. This proves (i) and we get (iii) by duality.

1.5.2. To prove (ii), we can assume that $[L] = 0$ since $f_*(L)$ reflexive implies that, for any $V \in NS(f,N)$, the map $f_*(L) \to f_*(L(V))$ is an isomorphism. Let $u : f_*(L) \to f_*(L)^{vv}$ be the map from $f_*(L)$ to it's bidual. Since $L$ is invertible, we know that $u$ is an isomorphism over the open subset $X_f$ of $X$. Since $X$ is normal, we know that coker$(u)$ is finite and since $f$ is proper, this implies the existence of some $V \in NS(f;N)$ such that $f_*(L)^{vv} = f_*(L(V))$. Since $[L] = 0$, we know that $H^0(V,L(V)|V) = 0$ hence $f_*(L) \to f_*(L(V))$ is an isomorphism and this concludes the proof.

1.5.3. We do not really need duality for surfaces to state and prove (iii). In fact, we can define

(1) $K_f \in NS(f,\mathbf{Q})$ by $(K_f+E_i).E_i = -2\chi(0_{E_i})$ for all $i$,

and write the hypothesis $[K_f-e_f(L)] \gtrsim 0$. As for the proof it runs parallel to the proof of (i) and uses the fact that $H^1(E_i,M) = 0$ if $M$ is an invertible sheaf on the reduced and irreducible Gorenstein curve $E_i$ with $\deg(M) > -2\chi(0_{E_i})$; details are left to the reader. We define $C(f)$ and $C_+$ in $NS(f,N)$ by

(2) $$[K_f] = C_+ - C(f).$$

Observe that if we denote by $K_{\tilde{X}}$ and $K_X$ the dualizing sheaves of $\tilde{X}$ and $X$ we have

(3) $$K_f = e_f(K_{\tilde{X}}) \quad \text{and} \quad K_X = f_*(K_{\tilde{X}}(C(f))).$$

The first formula comes from (1). For the second observe that $[K_{\tilde{X}}(C(f))] = K_{\tilde{X}}] + C(f) = C_+ \geqslant 0$ hence its direct image is reflexive by (1.5(ii)) and coïncide with $K_{\tilde{X}}$ over $X_f$, hence it must be $K_X$.

Corollary 1.6. — *Under the hypothesis of* (1.5), *let* $L$ *be an invertible sheaf on* $\tilde{X}$ *such that* $[L] = 0$. *Then* $f_*(L)$ *is reflexive and the map* $u : \mathbf{R}^1 f_*(L) \to H^1(C(f),L|C(f))$ *is an isomorphism.*

We know that $u$ is surjective. Let us introduce $V \in NS(f,\mathbf{Z})$ such that $[K_f + C(f) - e_f(L) - V] = 0$. We claim that $V \geqslant 0$. In fact $0 = [K_f + C(f) - e_f(L) - V] \geqq K_f + C(f) - e_f(L) - V$ hence $e_f(L) + V \geqq K_f + C(f)$ hence $V = [e_f(L) + V] \geqslant [K_f + C(f)] = C_+ \geqslant 0$. We have a diagram

$$\begin{array}{ccc} \mathbf{R}^1 f_*(L) & \xrightarrow{\ u\ } & \mathbf{R}^1 f_*(L|C(f)) \\ {\scriptstyle v}\downarrow & & \downarrow \\ \mathbf{R}^1 f_*(L(V)) & \xrightarrow{\ w\ } & \mathbf{R}^1 f_*(L(V)|C(f)). \end{array}$$

By (1.5.1(3)), the morphism $v$ is injective hence it is enough to show that $w$ is injective. This follows from $\mathbf{R}^1 f_*(L(V - C(f))) = 0$ which comes from (1.5 (iii)) since $[K_f - e_f(L) - V + C(f)] = 0$.

COROLLARY 1.7. – *We have* $\mathbf{R}^1 f_*(\mathcal{O}_{\tilde{X}}) \simeq H^1(C(f); \mathcal{O}_{C(f)})$ *and* $\mathbf{R}^1 f_*(\mathcal{O}_{\tilde{X}}) = 0$ *is equivalent to* $C(f) = 0$.

We get the isomorphism by (1.6) applied to $L = \mathcal{O}_{\tilde{X}}$. Hence $C(f) = 0$ implies $\mathbf{R}^1 f_*(\mathcal{O}_{\tilde{X}}) = 0$. Conversely, if $\mathbf{R}^1 f_*(\mathcal{O}_{\tilde{X}}) = 0$ and $C(f) \neq 0$, we have $\chi(\mathcal{O}_{C(f)}) > 0$ which means

$$\begin{aligned} 0 > (K_{\tilde{X}} + C(f)).C(f) &= (K_f + C(f)).C(f) \\ &\geqslant ([K_f] + C(f)).C(f) = C_+.C(f) \geqslant 0 \end{aligned}$$

a contradiction.

PROPOSITION 1.8. – *Let* $f : \tilde{X} \to X$ *be a desingularization of a normal surface* $X$ *and let* $M$ *be a reflexive sheaf of rank one on* $X$. *There exists a pair* $(L,u)$ *where* $L$ *is an invertible sheaf on* $\tilde{X}$ *such that* $[e_f(L)] = 0$ *and* $u : f_*(L)|X_f \simeq M|X_f$ *is an isomorphism. The pair* $(L,u)$ *is unique up to a unique isomorphism. Furthermore* $M = f_*(L)$.

1.8.1. It is clear that there exists a pair $(L',u')$, where $L'$ is invertible on $\tilde{X}$ and $u' : f_*(L')|X_f \simeq M|X_f$ is an isomorphism. If $(L'',u'')$ is another solution, we canonically have $L'' = L'(V)$, $V \in NS(f,\mathbf{Z})$, hence we get existence and uniqueness since $[e_f(L'(V))] = [e_f(L')] + V$. By (1.5(ii)), $f_*(L)$ is reflexive since $[e_f(L)] = 0$, hence $f_*(L) \simeq M$ since both are reflexive and coïncide over $X_f$.

1.8.2. We denote by $f^v(\mathrm{M})$ the invertible sheaf on $\tilde{\mathrm{X}}$ characterized by $[f^v(\mathrm{M})] = 0$ and $f_*(f^v(\mathrm{M})) = \mathrm{M}$. We observe that we have

$$(1) \qquad e_f(f^v(\mathrm{M})) \in \mathrm{NS}(f,\mathbf{Q}), \quad e_f(f^v(\mathrm{M})) \lesssim 0,$$

but this element is not necessarily zero. However, if M is *invertible*, we obviously have $f^v(\mathrm{M}) = f^*(\mathrm{M})$ since $e_f(f^*(\mathrm{M})) = 0$. More generally, it is useful to compare $f^v(\mathrm{M})$ with another lifting $\tilde{\mathrm{M}}$ defined as follows

$$(2) \qquad \mathrm{M}' = f^*(\mathrm{M})/\text{torsion} \quad \tilde{\mathrm{M}} = \mathrm{M}'^{vv} = \text{bidual of } \mathrm{M}'.$$

COROLLARY 1.8.3. — *Let* M *be a reflexive sheaf of rank one on* X. *Then* $\mathrm{M} \lesssim 0$ *and* $[\tilde{\mathrm{M}}] \leqslant 0$. *We have* $f^v(\mathrm{M}) = \tilde{\mathrm{M}}(-[\tilde{\mathrm{M}}])$.

Since M′ is torsion free of rank one it is invertible except at finitely many closed points; hence $\tilde{\mathrm{M}}$ is invertible. To prove that $\tilde{\mathrm{M}} \lesssim 0$, assume that there exists $\mathrm{E}_i$ such that $\tilde{\mathrm{M}}.\mathrm{E}_i < 0$. Then $f_*(\tilde{\mathrm{M}}(-\mathrm{E}_i)) = f_*(\tilde{\mathrm{M}}) = \mathrm{M}$. In a neighborhood U of the generic point of $\mathrm{E}_i$, we have $\mathrm{M}' = \tilde{\mathrm{M}}$, hence $\tilde{\mathrm{M}}$ is generated on a possibly smaller neighborhood U′ by sections of M, hence we cannot have $f_*(\tilde{\mathrm{M}}(-\mathrm{E}_i)) = f_*(\tilde{\mathrm{M}})$. By definition of $[\tilde{\mathrm{M}}]$, we get $[\tilde{\mathrm{M}}] \leqslant 0$ out of $\tilde{\mathrm{M}} \lesssim 0$. We deduce $f^v(\mathrm{M}) = \tilde{\mathrm{M}}(-[\tilde{\mathrm{M}}])$ from $[\tilde{\mathrm{M}}(-[\tilde{\mathrm{M}}])] = 0$.

COROLLARY 1.8.4. *Assume that* $\tilde{\mathrm{X}}$ *dominates some desingularization* X′ *of* X. *We have* $f = gh$ *with* $\tilde{\mathrm{X}} \xrightarrow{h} \mathrm{X}' \xrightarrow{g} \mathrm{X}$. *For any reflective sheaf of rank one* M *on* X *we have* $f^v(\mathrm{M}) = h^*(g^v(\mathrm{M}))$.

Since $\tilde{\mathrm{X}}$ and X′ are regular and $h$ proper and birational, we have $h_*h^*(g^v(\mathrm{M})) = g^v(\mathrm{M})$ hence $f_*h^*(g^v(\mathrm{M}))) = \mathrm{M}$, hence we only have to prove that $[e_f(h^*(g^v(\mathrm{M}))] = 0$. We use the map

$$(1) \qquad h^* : \mathrm{NS}(g,\mathbf{Q}) \to \mathrm{NS}(f,\mathrm{Q})$$

which preserves integrality, positivity and the intersection numbers. We still have to prove that we have, for any $\mathrm{V} \in \mathrm{NS}(g,\mathbf{Q})$

$$(2) \qquad h^*([\mathrm{V}]) = [h^*(\mathrm{V})].$$

For any $\mathrm{E} \in \mathrm{NS}(f,\mathrm{N})$, we have $h^*(\mathrm{V}).\mathrm{E} = \mathrm{V}.h_*(\mathrm{E}) \geqslant [\mathrm{V}].h_*(\mathrm{E}) = h^*([\mathrm{V}]).\mathrm{E}$, hence $h^*(\mathrm{V}) \lesssim h^*([\mathrm{V}])$, hence $[h^*(\mathrm{V})] \leqslant h^*([\mathrm{V}])$, in other words $h^*([\mathrm{V}]) = [h^*(\mathrm{V})] + \mathrm{A}$, $\mathrm{A} \in \mathrm{NS}(f,\mathrm{N})$.

From $h^*(V) \lesssim [h^*(V)]$, we deduce $V \lesssim h_*([h^*(V)]) = h_*h^*([V]) - h_*(A) = [V] - h_*(A)$. By definition of $[V]$, we deduce that $[V] \leqslant [V] - h_*(A)$, hence $h_*(A) = 0$, hence $A \in NS(h,N)$. We get $0 = h^*(V)$. $A \geqslant [h^*(V)]$. $A = h^*([V])$. $A - A^2 = - A^2$, hence $A = 0$.

PROPOSITION 1.9. — *Let* $f\colon \tilde{X} \to X$ *and assume that* $\mathbf{R}^1 f_*(O_{\tilde{X}}) = 0$.

(i) *Let* M *be a reflexive sheaf of rank one on* X. *We have* $f^v(M) = f^*(M)/torsion$ *and* $\mathbf{R}^1 f_*(f^v(L)) = 0$.

(ii) *Let* L *be an invertible sheaf on* $\tilde{X}$ *such that* $L \lesssim 0$. *The map* $f^* f_*(L) \to L$ *is surjective and* $\mathbf{R}^1 f_*(L) = 0$.

We first prove (ii). We let $M = f_*(L)$, $L_0 = \mathrm{Im}(f^*(M) \to L)$, $L_1 =$ bidual of $L_0$ and we get $L_0 \subset L_1 \subset L$ and $M \subset f_*(L_0) \subset f_*(L_1) \subset f_*(L) = M$. Since $\mathbf{R}^1 f_*(L_0) = 0$, we get $f_*(L_1/L_0) = 0$ and this implies $L_1/L_0 = 0$ since $L_1/L_0$ has finite support. Let us define $V \in NS(f,N)$ by $L = L_0(V)$. We have $f_*(L|V) = 0$, hence $\chi(K_V) = \chi(L|V) - L.V = - h^1(L|V) - L.V \leqslant - L.V$. Since $L \lesssim 0$, we get $- L.V \leqslant 0$ hence $\chi(O_V) \leqslant 0$, hence $V = 0$ since $h^1(O_V) = 0$. This means that $L_0 = L$, from which $\mathbf{R}^1 f_*(L) = 0$ follows.

To prove (i) we let $L = f^v(M)$ and apply (ii) to L (see (1.8.3)); recall that $M = f_* f^v(L)$ by (1.8).

As an exercise, we now deduce some well known facts about rational singularities.

PROPOSITION 1.10. *Let* $f\colon \tilde{X} \to X$ *be a desingularization and assume that* $\mathbf{R}^1 f_*(O_{\tilde{X}}) = 0$. *Let* I *be an ideal of* $O_X$. *The following conditions are equivalent*

(i) I *is integrally closed and* $IO_{\tilde{X}}$ *is invertible,*

(ii) $I = f_*((IO_{\tilde{X}})^{vv})$,

(iii) *There exists an effective divisor* D *on* $\tilde{X}$, *with* $O_{\tilde{X}}(-D) \gtrsim 0$ *such that* $I = f_*(O_{\tilde{X}}(-D))$.

*Furthermore, if we have* (iii), *we necessarily have* $IO_{\tilde{X}} = M_{\tilde{X}}(-D)$.

If $IO_{\tilde{X}}$ is invertible, then $\tilde{X}$ dominates the normalized blowing up of I, hence $f_*(IO_{\tilde{X}})$ is the integral closure of I. Hence (i) ⇒ (ii), since in that case $IO_{\tilde{X}} = IO_{\tilde{X}}^{vv}$. Since $(IO_{\tilde{X}})^{vv} \lesssim 0$, we have $IO_{\tilde{X}}^{vv} = O_{\tilde{X}}(-D)$, with D effective (not necessarily vertical) and $D \gtrsim 0$; hence (ii) ⇒ (iii). If we assume (iii), then I is integrally closed and (1.9 (ii)) implies that

$IO_{\tilde{X}} = O_{\tilde{X}}(-D)$, hence (iii) $\Rightarrow$ (i) and we have also proven the last assertion.

It follows that we have a 1-1-correspondance between ideals I of $O_X$ which satisfy the above conditions and effective divisors D on X with $D \gtrsim 0$. We have that I is primary if and only if D is vertical ($\dim f(D) = 0$) and I is reflexive (i.e. the ideal of a Weil divisor) if and only if $[D] = 0$. Observe that (1.9(i)) tells us that a reflexive I satisfy (i). Observe that if I is the maximal ideal of some closed point $x$, then we must have (ii), hence the corresponding D must be the connected component of the fundamental cycle corresponding to $x$. To complete the picture, recall Lipman's result saying that the set of ideals satisfying (i) is stable by multiplication, which means that $f_*(O_{\tilde{X}}(-D-E)) = f_*(O_{\tilde{X}}(-D))f_*(O_{\tilde{X}}(-E))$ if D and E are effective and $D \gtrsim 0$, $E \gtrsim 0$.

*Example* 1.11. — We now assume that $f: \tilde{X} \to X$ is the *minimal* desingularization and that X is the spectrum of a local ring R with algebraically closed residue field, in such a way that $K_{\tilde{X}} \lesssim 0$; this implies $[K_f] = -C(f)$. Assume that $K_X$ *is invertible* which means that R is a *Gorenstein ring*. Since $f^*(K_X) = K_{\tilde{X}}(V)$ for some vertical V and $e_f(f^*(K_X)) = 0$, we conclude that $V = K_f$, hence $K_f$ has integral coefficients, hence $K_f = -C(f)$ and $K_{\tilde{X}}(C(f)) = f^*(K_X) \approx O_{\tilde{X}}$.

If we have rational singularity, we know that $C(f) = 0$, hence $K_f = 0$, hence we get the well known result that $E_i^2 = -2$ for all $i$. If $C(f) \neq 0$, we still have that the dualizing sheaf $K_{C(f)} = K_{\tilde{X}}(C(f)) \otimes O_{C(f)}$ is isomorphic to $O_{C(f)}$. The converse is also true, see for instance [2].

## 2. Genus formula.

2.1. Let $k$ be a field and X be a proper $k$-scheme of dimension 2 which is normal. We want to study Weil divisors of X, or equivalently reflexive sheaves of rank one on X. Such a sheaf M is determined by the invertible sheaf $i^*(M)$ since $M \to i_* i^*(M)$ is an isomorphism where $i: X_{reg} \to X$ is the inclusion of the open set $X_{reg}$ made up of regular points of X. In other words, we study $\mathrm{Pic}(X_{reg})$. Let $f: \tilde{X} \to X$ be a desingularization of X, we have an exact sequence

$$(1) \qquad 0 \to NS(f,\mathbf{Z}) \xrightarrow{a} \mathrm{Pic}(\tilde{X}) \xrightarrow{b} \mathrm{Pic}(X_{reg}) \to 0$$

where $a(\mathrm{D})$ is the class of $\mathrm{O}_{\tilde{\mathrm{X}}}(\mathrm{D})$ and $b$ is induced by the inclusion $j : \mathrm{X}_{\mathrm{reg}} \to \mathrm{X}$. The canonical lifting $f^v(\mathrm{M})$ of a reflexive sheaf of rank one M on X defined in (1.8.2) gives us a *non-linear* section of $b$. By composition with the usual map

$$(2)\ e_f : \mathrm{Pic}(\tilde{\mathrm{X}}) \to \mathrm{NS}(f,\mathbf{Z})^* \subset \frac{1}{d}\mathrm{NS}(f,\mathbf{Z}) \subset \mathrm{NS}(f,\mathbf{Q}),\ (1.3)$$

we get a class

$$(3) \qquad e_f(f^v(\mathrm{M})) \in \frac{1}{d}\ \mathrm{NS}(f,\mathbf{Z})$$

which can only take *a finite number of values* since $[e_f(f^v(\mathrm{M}))] = 0$. Of course, this is still non linear. To recover the classical linear theory of [6], we recall that, for $\mathrm{A} = \mathbf{Z}$ or $\mathbf{Q}$, the quadratic module $\mathrm{NS}(f, \mathrm{A})$ lies inside the Néron-Severi group $\mathrm{NS}(\tilde{\mathrm{X}},\mathrm{A})$ and we define

$$(4) \qquad \mathrm{NS}(\mathrm{X},\mathrm{A}) = \text{orthogonal of } \mathrm{NS}(f,\mathrm{A}) \text{ inside } \mathrm{NS}(\tilde{\mathrm{X}},\mathrm{A})$$

which gives an orthogonal decomposition

$$(5) \qquad \mathrm{cl}(f^v(\mathrm{M})) = \mathrm{cl}(\mathrm{M}) + e_f(f^v(\mathrm{M}))$$

inside $\mathrm{NS}(\tilde{\mathrm{X}},\mathbf{Q}) = \mathrm{NS}(\mathrm{X},\mathbf{Q}) \oplus \mathrm{NS}(f,\mathbf{Q})$. We also have another linear invariant

$$(6) \qquad d_f(\mathrm{M}) = \text{class of } e_f(f^v(\mathrm{M})) \text{ in } \mathrm{NS}(f,\mathbf{Z})^*/\mathrm{NS}(f,\mathbf{Z}).$$

It is clear that the two linear invariants $\mathrm{cl}(\mathrm{M})$ and $d_f(\mathrm{M})$ can be computed with any lifting L of M, namely $\mathrm{cl}(\mathrm{M})$ is the orthogonal projection on $\mathrm{NS}(\mathrm{X},\mathbf{Q})$ of $\mathrm{cl}(\mathrm{L})$ and $d_f(\mathrm{M})$ is the image of $e_f(\mathrm{L})$; proof: $\mathrm{L} = f^v(\mathrm{M})(\mathrm{D})$ for some $\mathrm{D} \in \mathrm{NS}(f,\mathbf{Z})$. For instance, if $\mathrm{K}_{\mathrm{X}}$ and $\mathrm{K}_{\tilde{\mathrm{X}}}$ are the dualizing sheaves of X and $\tilde{\mathrm{X}}$ we have an orthogonal decomposition

$$(7) \qquad \mathrm{cl}(\mathrm{K}_{\tilde{\mathrm{X}}}) = \mathrm{cl}(\mathrm{K}_{\mathrm{X}}) + \mathrm{K}_f \quad (1.5.3)$$

and

$$(8) \qquad e_f(\mathrm{K}_{\mathrm{X}}) = \mathrm{K}_f - [\mathrm{K}_f]\ .$$

If we introduce the effective divisor $\mathrm{C}(f) = [\mathrm{K}_f]_-$ as in (1.5.3) we know that the multi-degree of $f^v(\mathrm{M})|\mathrm{C}(f)$ can only take a finite number of values, hence the same holds for the length of

(9) $\mathbf{R}^1 f_*(f^v(\mathrm{M})) = \mathrm{H}^1(\mathrm{C}(f);\ f^v(\mathrm{M})|\mathrm{C}(f))$, (1.6).

THEOREM 2.2. — *Let* M *be a reflexive sheaf of rank one on* X. *We have*

(1) $\chi(\mathrm{M}) = \frac{1}{2}(\mathrm{cl}(\mathrm{M}), \mathrm{cl}(\mathrm{M}) - \mathrm{cl}(\mathrm{K_X})) + \chi(\mathrm{O_X}) + \frac{1}{2}\, e(\mathrm{M})\, d(\mathrm{M})$

*where the scalar product is computed in* NS(X,**Q**) *and for any desingularization* $f : \tilde{\mathrm{X}} \to \mathrm{X}$ of X *we have*

(2) $e(\mathrm{M}) = (e_f(f^v(\mathrm{M})),\ e_f(f^v(\mathrm{M})) - \mathrm{K}_f)$

(3) $d(\mathrm{M}) = \lg \mathbf{R}^1 f_*(f^v(\mathrm{M})) - \lg \mathbf{R}^1 f_*(\mathrm{O}_{\tilde{\mathrm{X}}})$
$= h^1(\mathrm{C}(f);\ f^v(\mathrm{M})|\mathrm{C}(f)) - h^1(\mathrm{C}(f);\ \mathrm{O}_{\mathrm{C}(f)})$ (1.5.3).

*Proof.* — Apply the usual Riemann-Roch formula to $f^v(\mathrm{M}) = \mathrm{L}$. Since $\mathrm{M} = f_*(f^v(\mathrm{M}))$, we get

$$\begin{aligned}\chi(\mathrm{M}) &= \chi(\mathrm{L}) + \lg \mathbf{R}^1 f_*(\mathrm{L}) = (\mathrm{L}, \mathrm{L} - \mathrm{K}_{\tilde{\mathrm{X}}})/2 + \chi(\mathrm{O}_{\tilde{\mathrm{X}}}) + \lg \mathbf{R}^1 f_*(\mathrm{L}) \\ &= \chi(\mathrm{L}, \mathrm{L} - \mathrm{K}_{\tilde{\mathrm{X}}})/2 + \chi(\mathrm{O_X}) + \lg \mathbf{R}^1 f_*(\mathrm{L}) - \lg \mathbf{R}^1 f_*(\mathrm{O}_{\tilde{\mathrm{X}}})\end{aligned}$$

and split the scalar product $(\mathrm{L}, \mathrm{L} - \mathrm{K}_{\tilde{\mathrm{X}}})$ according to the orthogonal decomposition $\mathrm{NS}(\tilde{\mathrm{X}}, \mathbf{Q}) = \mathrm{NS}(\mathrm{X}, \mathbf{Q}) + \mathrm{NS}(f, \mathbf{Q})$.

According to (1.8.4), the terms $e(\mathrm{M})$ and $d(\mathrm{M})$ do not depend on the choice of the desingularization. Furthermore we have

(4) $e(\mathrm{M}) = \sum_{x \in \mathrm{Sing}(\mathrm{X})} e(\mathrm{M}, x), \quad d(\mathrm{M}) = \sum_{x \in \mathrm{Sing}(\mathrm{X})} d(\mathrm{M}, x)$

where $e(\mathrm{M},x)$ and $d(\mathrm{M},x)$ are defined by replacing X by $\mathrm{Spec}(\mathrm{O}_{\mathrm{X},x})$, or even by $\mathrm{Spec}(\hat{\mathrm{O}}_{\mathrm{X},x})$ as is easily seen. Furthermore $e(\mathrm{M},x) = d(\mathrm{M},x) = 0$ if M *is invertible in a neighborhood of* $x$. Furthermore $d(\mathrm{M},x) = 0$ if $\mathrm{O}_{\mathrm{X},x}$ is a *rational singularity* (1.7). We also know that $e(\mathrm{M})$ and $d(\mathrm{M})$ can only take a finite number of values.

For $n \in \mathbf{Z}$, we let $\mathrm{M}^n = i_*(i^*(\mathrm{M})^n)$ = bidual of $\mathrm{M}^{\otimes n}$ and we have

(5) $\chi(\mathrm{M}^n) = \frac{n^2}{2}(\mathrm{cl}(\mathrm{M}), \mathrm{cl}(\mathrm{M})) - \frac{n}{2}(\mathrm{cl}(\mathrm{M}), \mathrm{cl}(\mathrm{K_X}))$
$+ \chi(\mathrm{O_X}) + e(\mathrm{M}^n)/2 + d(\mathrm{M}^n).$

Observe that $e(M^n) = 0$ if the determinant of the intersection matrix divides $n$. In fact, in that case, we have $d_f(M^n) = 0$ hence $e_f(f^v(M)) = [e_f(f^v(M))] = 0$. For instance, if X is the Satake compactification of some Hilbert-Blümenthal surface and $M = K_X$, we can get an a priori proof of the formula for the rank of the vector spaces $H^0(X, K_X^n)$ of automorphic forms [3].

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