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COMPLETELY CONTINUOUS MULTIPLIERS FROM $L_1(G)$ INTO $L_\infty(G)$

by G. CROMBEZ and W. GOVAERTS (*)

1. Introduction.

When (S,Σ,μ) is a positive measure space and T a weakly compact linear map from $L_1(S,\Sigma,\mu)$ into a Banach space B, then T transforms weakly convergent sequences in $L_1(S,\Sigma,\mu)$ in norm convergent sequences in B, i.e., T is completely continuous [9, 1.6.1]; indeed, $L_1(S,\Sigma,\mu)$ has the Dunford-Pettis property [5, 9.4]. In particular, for a locally compact Hausdorff group G with left Haar measure, all weakly compact convolution operators T_g from $L_1(G)$ into $L_\infty(G)$, induced by functions g in $L_\infty(G)$, are completely continuous. However, those g in $L_\infty(G)$ which induce weakly compact T_g are precisely the weakly almost periodic functions [10], and they form a proper subspace of $L_\infty(G)$; hence the question arises whether or not all g in $L_\infty(G)$ give rise to a completely continuous T_g.

Whenever G is a discrete group, every T_g is completely continous, since then in $L_1(G)$ norm convergence and weak convergence of sequences coincide [4, IV.8.13]. For nondiscrete G this last property is not true : for instance, the sequence of functions $t \to e^{int}$ $(n \in N)$ on the circle group T is not norm convergent in $L_1(T)$, but converges weakly to zero by the Riemann-Lebesgue lemma. This suggests that complete continuity of a convolution operator T_g is not trivial if G is not discrete.

In section 2 we give an example of a (continuous) function g on the additive group $\mathscr{R}$ of real numbers such that T_g is not completely

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continuous. In section 3 we introduce the notion of a uniformly measurable function in $L_\infty(G)$; we show that such a function induces a completely continuous T_g; conversely, for metrizable nondiscrete G there are no other functions g which induce completely continuous T_g. In section 4 the set of uniformly measurable functions is investigated. All g in $L_\infty(G)$ give rise to completely continuous T_g if G is either discrete (already known) or compact. From theorem VI.8.14 in [4] it may be derived that T_g is completely continuous if the function $y \to {}_{y^{-1}}g$ from G to $(L_\infty(G), \| \ \|_\infty)$ is measurable; in section 5 we show that such g are in fact uniformly measurable, but that the converse does not hold. Finally, in section 6 we use our results on completely continuous T_g to obtain more information on convolution operators.

Definitions and notations.

We always denote by G a locally compact Hausdorff group with identity e. With respect to left Haar measure m on G, $L_1(G)$ and $L_\infty(G)$ are the usual corresponding Banach spaces. A function g in $L_\infty(G)$ gives rise to a convolution operator T_g from $L_1(G)$ into $L_\infty(G)$ by means of $T_g(f) = f * g$, where $(f * g)(x) = \int_G f(xy)g(y^{-1})\, dy$. T_g is called completely continuous if T_g maps any weakly (i.e., $\sigma(L_1(G), L_\infty(G))$) convergent sequence onto a norm convergent sequence. For a function g on G, we use $\tilde{g}$ for the function defined by $\tilde{g}(x) = g(x^{-1})$; for $a \in G$, left translation is defined by $({}_a g)(x) = g(ax)$. The characteristic function of a set A in G is denoted by χ_A. $C_c(G)$ is the set of continuous-complex-valued functions on G with compact support, and $C_c^+(G)$ denotes the non-negative functions in $C_c(G)$. The linear space of all right uniformly continuous, bounded, complex-valued functions on G is denoted by $C_{ru}(G)$. All other non-explained notation about groups is taken from [7].

2. Example of a not completely continuous convolution operator.

Take the additive group $\mathscr{R}$ of real numbers. For $n = 1, 2, \ldots,$ put

$$f_n(x) = \begin{cases} e^{inx}, & \text{for } 0 \leqslant x \leqslant 2\pi \\ 0, & \text{for } x \in \mathscr{R} \setminus [0, 2\pi]. \end{cases}$$

Each f_n belongs to $L_1(\mathscr{R})$, and $\|f_n\|_1 = 2\pi$. However, the sequence $\{f_n\}_{n=1}^{\infty}$ converges weakly to zero in $L_1(\mathscr{R})$. For, given h in $L_\infty(\mathscr{R})$, denote by k the restriction of h to the interval $[0, 2\pi]$. Then k may be considered as a bounded function on the circle group T, while the function $x \to e^{inx}$ is a character on T. Hence

$$\int_{\mathscr{R}} f_n(x)h(x)\, dx = \int_0^{2\pi} e^{inx}k(x)\, dx = \hat{k}(-n),$$

and this tends to zero as n tends to infinity. Let g be the function defined on $\mathscr{R}$ by means of $g(t + 2\pi m) = e^{imt}$ for $0 < t \leqslant 2\pi$, $m \in Z$. The function g is bounded and continuous, and so belongs to $L_\infty(\mathscr{R})$. We prove that the convolution operator T_g from $L_1(\mathscr{R})$ to $L_\infty(\mathscr{R})$ is not completely continuous by showing that, for each positive integer n, there exists x_n in $\mathscr{R}$ such that $|(f_n * g)(x_n)| = 2\pi$. Put $x_n = 2\pi(n + 1)$. For $0 \leqslant t \leqslant 2\pi$, $x_n - t = (2\pi - t) + 2\pi n$, and $0 < 2\pi - t \leqslant 2\pi$ as soon as $t \neq 2\pi$; hence $g(x_n - t) = e^{in(2\pi - t)}$ almost everywhere on the interval $[0, 2\pi]$.

So we obtain

$$(f_n * g)(x_n) = \int_0^{2\pi} e^{int} e^{in(2\pi - t)}\, dt = 2\pi e^{2\pi i n},$$

from which the result follows.

3. Uniformly measurable functions in $L_\infty(G)$ and completely continuous convolution operators.

DEFINITION. — *For a measurable set* A *in* G *and a finite measurable partition* $\mathscr{A} = \{A_i\}_{i=1}^n$ *of* A, *we call a function* h *on* A *an* $\mathscr{A}$*-step function if* h *is constant on each set* A_i. *The set of all* $\mathscr{A}$*-step functions is denoted by Step* $\mathscr{A}$.

THEOREM 3.1. — *For a function* g *in* $L_\infty(G)$, *the following are equivalent :*

(i) $\forall \varepsilon > 0$, $\forall \delta > 0$, $\forall$ *measurable* A *in* G *with* $m(A) < \infty$, *there exists a measurable partition* $\mathscr{A} = \{A_i\}_{i=1}^n$ *of* A *such that, to each* x in G *there corresponds a subset* A_x *of* A *with* $m(A_x) < \delta$ *and to each* $i \in \{1, \ldots, n\}$ *there corresponds a complex number* $c_{x,i}$ *such that* $|g(a^{-1}x) - c_{x,i}| < \varepsilon$ *for all* $a \in A_i \backslash A_x$.

(ii) *the same as* (i), *but only for those* x *in a dense subset of* G.

(iii) *the same as* (ii), *but only for all compact subsets* A *of* G.

(iv) $\forall \varepsilon > 0$, $\forall$A compact in G, *there exists a measurable partition* $\mathscr{A} = \{A_i\}_{i=1}^n$ *of* A *and a dense subset* D *of* G *such that, to each* x *in* D *there corresponds a function* h_x *in* Step $\mathscr{A}$ *such that* $\int_A |g(a^{-1}x) - h_x(a)|\,da < \varepsilon$.

(v) *the same as* (iv), *but now for all* x *in* G.

(vi) *the same as* (v), *but for all measurable* A *in* G *with* $m(A) < \infty$.

Proof. – (i) $\Leftrightarrow$ (ii) and (ii) $\Rightarrow$ (iii) are trivial.

(iii) $\Rightarrow$ (iv) : Let $\varepsilon > 0$ and A compact in G be given. Put $\delta = \dfrac{\varepsilon}{6\|g\|_\infty}$, and consider in (iii) the partition $\mathscr{A} = \{A_i\}_{i=1}^n$ of A corresponding to $\dfrac{\varepsilon}{2m(A)}$ and δ. For x in D and a in A_i, put $h_x(a) = c_{x,i}$; then h_x clearly belongs to Step $\mathscr{A}$. Since we may assume that $|c_{x,i}| \leqslant 2\|g\|_\infty$, we obtain the result by writting $A = \bigcup_{i=1}^{n} (A_i \backslash A_x) \cup A_x$.

(iv) $\Rightarrow$ (v) : Let y be an arbitrary point in G and A compact. It is sufficient to prove that there exists a neighborhood V of y such that $\int_A |g(a^{-1}z) - g(a^{-1}y)|\,da$ is arbitrarily small for all z in V; for, such neighborhood contains a point x for which the inequality to be proved is already true, and putting then $h_y(a) \equiv h_x(a)$ we obtain the result by the triangle inequality.

Choose an open neighborhood U of A with compact closure $\bar{U}$. There exists a symmetric neighborhood W_1 of e such that $AW_1 \cup W_1A \subset U$; then $y^{-1}A \subset y^{-1}U$ and $z^{-1}A \subset y^{-1}U$ for all z in W_1y. Choose a function f in $C_c(G)$ such that $0 \leqslant f \leqslant 1$, and $f = 1$ on $y^{-1}\bar{U}$; put $h = f\tilde{g}$, where $\tilde{g}(t) = g(t^{-1})$; then $h \in L_1(G)$. Given $\varepsilon > 0$, there exists a symmetric neighborhood W_2 of e such that

$\|{}_{z^{-1}}h - {}_{y^{-1}}h\|_1 < \varepsilon$ if $z \in yW_2$. If $W = W_1 \cap W_2$, and $z \in yW \cap Wy$, then we obtain

$$\int_A |g(a^{-1}z)-g(a^{-1}y)|\, da = \int_A |({}_{z^{-1}}\tilde{g})(a) - ({}_{y^{-1}}\tilde{g})(a)|\, da$$

$$\leqslant \|{}_{z^{-1}}h - {}_{y^{-1}}h\|_1 < \varepsilon .$$

(v) $\Rightarrow$ (vi) : Given a measurable A in G with $m(A) < \infty$, there exists a compact $K \subset A$ such that $m(A\backslash K)$ is arbitrarily small. For this K the condition to be proved is already true. So it suffices to join $A\backslash K$ to the partition of A, and to put $h_x(a) = 0$ for all a in $A\backslash K$.

(vi) $\Rightarrow$ (i). If (i) is not true, then there exist $\varepsilon_0 > 0$, $\delta_0 > 0$, and a measurable subset A of G with $m(A) < \infty$, such that for any measurable partition $\mathscr{A} = \{A_i\}_{i=1}^n$ of A a point x in G may be found such that for no constants $c_{x,i}$ we have that $|g(a^{-1}x) - c_{x,i}| < \varepsilon_0$ for all a in $A_i\backslash A_x$ as soon as $m(A_x) < \delta_0$. Let then $\mathscr{A} = \{A_i\}_{i=1}^n$ be an arbitrary measurable partition of A, h an arbitrary function in Step $\mathscr{A}$, and A_x a measurable subset of A with $m(A_x) < \delta_0$. For the mentioned point x we then have

$$\int_A |g(a^{-1}x) - h(a)|\, da \geqslant \sum_{i=1}^n \int_{A_i\backslash A_x} |g(a^{-1}x) - h(a)|\, da$$

$$\geqslant \varepsilon_0(m(A) - \delta_0),$$

which is clearly a contradiction with the assumption of (vi).

□

Definition. — *A function* $g \in L_\infty(G)$ *is uniformly measurable if it satisfies one of the equivalent conditions of theorem* 3.1.

Theorem 3.2. — *If* g *in* $L_\infty(G)$ *is uniformly measurable, then* T_g *is completely continuous.*

Proof. — Let $\{f_j\}_{j=1}^\infty$ be a weak zero-sequence in $L_1(G)$; then $\|f_j\|_1 \leqslant M$, $\forall j$. Given $\varepsilon > 0$ there exists a compact set A in G such that $\int_{G\backslash A} |f_j(y)|\, dy < \frac{\varepsilon}{\|g\|_\infty}$ for all j [5, 4.21.2]. With that ε there corresponds $\delta > 0$ such that $\int_T |f_j(y)|\, dy < \frac{\varepsilon}{\|g\|_\infty}$ for all j, whenever $m(T) < \delta$.

Given a point x of G, use $\frac{\varepsilon}{M}$ and δ in (iii) of theorem 3.1 to obtain a partition $\mathscr{A} = \{A_i\}_{i=1}^n$ of the compact set A; then $m(A_x) < \delta$ and $|g(a^{-1}x) - c_{x,i}| < \frac{\varepsilon}{M}$ for all $a \in A_i \backslash A_x$. Since

$$(f_j * g)(x) = \int_A f_j(y)g(y^{-1}x)\,dy + \int_{G\backslash A} f_j(y)g(y^{-1}x)\,dy,$$

and $\left|\int_{G\backslash A} f_j(y)g(y^{-1}x)\,dy\right| \leqslant \varepsilon$ for all j, we only have to investigate the integral on A. This integral may first be written as $\sum_{i=1}^n \int_{A_i\backslash A_x} + \int_{A_x}$ with $m(A_x) < \delta$, so that $\left|\int_{A_x} f_j(y)g(y^{-1}x)\,dy\right| \leqslant \varepsilon$ for all j. Finally,

$$(1)\quad \sum_{i=1}^n \int_{A_i\backslash A_x} f_j(a)g(a^{-1}x)\,da = \sum_{i=1}^n \int_{A_i\backslash A_x} f_j(a)c_{x,i}\,da + \sum_{i=1}^n \int_{A_i\backslash A_x} f_j(a)(g(a^{-1}x) - c_{x,i})\,da.$$

We may assume that $|c_{x,i}| \leqslant 2\|g\|_\infty$. Since $\{f_j\}_{j=1}^\infty$ is weakly convergent to zero in $L_1(G)$, we have

$$\left|\int_{A_i\backslash A_x} f_j(a)\,da\right| < \frac{\varepsilon}{2n\|g\|_\infty}$$

for sufficiently large values of j. This leads to

$$\left|\sum_{i=1}^n \int_{A_i\backslash A_x} f_j(a)c_{x,i}\,da\right| < \varepsilon$$

for sufficiently large values of j. The second member on the right hand side of (1) is bounded by $\frac{\varepsilon}{M}\sum_{i=1}^n \int_{A_i\backslash A_x} |f_j(a)|\,da$, and so is smaller than $\frac{\varepsilon}{M}\|f_j\|_1 \leqslant \varepsilon$ for all j. Hence $|(f_j * g)(x)| < 4\varepsilon$ for sufficiently large values of j, independant of x; so $\{f_j * g\}_{j=1}^\infty$ is norm-convergent to zero in $L_\infty(G)$.

□

Theorem 3.2 gives us a sufficient condition for a function g in $L_\infty(G)$ to induce a completely continuous T_g. To show that for metrizable groups G this condition is also necessary, we first state two lemmas.

LEMMA 3.3. — *Let* G *be nondiscrete,* A *a relatively compact subset of* G, g *a real-valued function in* $L_\infty(G)$. *Then there exist a real number* d *and a partition* $A = B \cup C$ *of* A *such that* $g \geqslant d$ *on* B, $g \leqslant d$ *on* C, *and* $m(B) = m(C) = \frac{1}{2} m(A)$.

Proof. — We may assume that $m(A) \neq 0$. Since $m(A) < \infty$, g takes almost everywhere on A values between $- \|g\|_\infty$ and $+ \|g\|_\infty$. Consider

$$U_1 = \left\{ c \in [-\|g\|_\infty - 1, \|g\|_\infty + 1] : m(\{x \in A : g(x) > c\}) \leqslant \frac{1}{2} m(A) \right\}$$

$$U_2 = \left\{ c \in [-\|g\|_\infty - 1, \|g\|_\infty + 1] : m(x \in A : g(x) < c\}) \leqslant \frac{1}{2} m(A) \right\}.$$

Neither U_1 nor U_2 is empty, each element of U_2 is not greater than each element of U_1, and each element of the interval $[-\|g\|_\infty - 1, \|g\|_\infty + 1]$ belongs either to U_1 or to U_2. So there exists a cut d in the interval such that

$$m(\{x \in A : g(x) > d\}) \leqslant \frac{1}{2} m(A), \qquad m(\{x \in A : g(x) < d\}) \leqslant \frac{1}{2} m(A).$$

Putting

$$S = \{x \in A : g(x) > d\}, \quad T = \{x \in A : g(x) < d\}, \quad D = \{x \in A : g(x) = d\},$$

we have that $A = S \cup T \cup D$, $m(S) \leqslant \frac{1}{2} m(A)$, $m(D) \leqslant \frac{1}{2} m(A)$.

So the lemma will be proved if we show that, for any real t with $0 < t < 1$, there exists a partition $D = E \cup F$ such that

$$m(E) = tm(D), \qquad m(F) = (1-t)m(D).$$

Given $\varepsilon > 0$, there exists a neighborhood V of e such that $m(V) < \varepsilon$. Since D is relatively compact, there exists a finite cover $\bigcup_{i=1}^{n} x_i V$ of D where the x_i belong to the closure $\bar{D}$ of D. From this cover we obtain a partition $\bigcup_{i=1}^{n} D_i$ of D, and $m(D_i) < \varepsilon$ for all i. Let then B_1, respectively B_2 be the union of those D_i such that

$$tm(D) - \varepsilon < m(B_1) \leqslant tm(D), \quad (1-t)m(D) - \varepsilon < m(B_2) \leqslant (1-t)m(D).$$

Then $D = B_1 \cup B_2 \cup B_3$, and $m(B_3) < 2\varepsilon$. If $m(B_3)$ is not zero, we repeat the foregoing procedure to obtain $B_3 = B'_1 \cup B'_2 \cup B'_3$, with $m(B'_3) < 2\varepsilon^2$. After a denumerable number of steps we put $B = S \cup B_1 \cup B'_1 \cup \ldots, C = A\backslash B$; this leads to the wanted partition $A = B \cup C$. □

LEMMA 3.4. — *Let* G *be non-discrete,* A *a relatively compact subset of* G. *Let* $\mathscr{A} = \{A_i\}_{i=1}^{n}$ *be a partition of* A, $\varepsilon_0 > 0$, $g \in L_\infty(G)$ *real-valued such that* $\int_A |g(a)-h(a)|\, da \geqslant \varepsilon_0$ *for each* $h \in \text{Step}\,\mathscr{A}$. *Then each* A_i *has a partition* $A_i = B_i \cup C_i$ *such that* $m(B_i) = m(C_i) = \frac{1}{2} m(A_i)$ *and such that*

$$\int_A g(a)\,(\chi_{\bigcup_i B_i}(a) - \chi_{\bigcup_i C_i}(a))\, da \geqslant \varepsilon_0 .$$

Proof. — Applying lemma 3.3, choose for each i a number c_i and a partition $A_i = B_i \cup C_i$ such that $g \geqslant c_i$ on B_i, $g \leqslant c_i$ on C_i, and $m(B_i) = m(C_i)$. Defining f on A by putting $f(y) = c_i$ for all y in A_i we obtain

$$\begin{aligned}\int_A g(a)(\chi_{\bigcup_i B_i}(a) - \chi_{\bigcup_i C_i}(a))\, da &= \sum_{i=1}^{n} \int_{A_i} g(y)(\chi_{B_i}(y) - \chi_{C_i}(y))\, dy \\ &= \sum_{i=1}^{n} \int_{A_i} (g(y)-c_i)(\chi_{B_i}(y) - \chi_{C_i}(y))\, dy \\ &= \sum_{i=1}^{n} \int_{A_i} |g(y)-c_i|\, dy \\ &= \int_A |g(a)-f(a)|\, da \geqslant \varepsilon_0 .\end{aligned}$$ □

THEOREM 3.5. — *If* G *is metrizable and nondiscrete, and* $g \in L_\infty(G)$ *is not uniformly measurable, then* T_g *is not completely continuous.*

Proof. — First, assume that g is real-valued. By assumption there is an $\varepsilon_0 > 0$ and a compact subset A of G such that corresponding to any measurable partition $\mathscr{A} = \{A_i\}_{i=1}^n$ of A a point x in G may be found such that

$$\int_A |g(a^{-1}x) - h(a)|\, da \geqslant \varepsilon_0$$

whenever $h \in \operatorname{Step} \mathscr{A}$.

We shall construct a sequence $\{\mathscr{A}_n\}_{n=1}^\infty$ of partitions of A, a sequence $\{x_n\}_{n=1}^\infty$ of points in G, and a sequence $\{\varphi_n\}_{n=1}^\infty$ of real-valued functions on G, with the following properties:

(i) $\mathscr{A}_n$ contains no sets of diameter $> 2^{-n}$.

(ii) $\mathscr{A}_{n+1}$ is a refinement of $\mathscr{A}_n$.

(iii) φ_{n+1} takes a constant value on each member of $\mathscr{A}_{n+1}$ and $\int_B \varphi_{n+1}(y)\, dy = 0$ whenever $B \in \mathscr{A}_n$.

(iv) $\{\varphi_n\}_{n=1}^\infty$ is weakly convergent in $L_1(G)$ to zero.

(v) $|(\varphi_n * g)(x_n)| > \varepsilon_0$ for all n, and so $\varphi_n * g$ is not norm convergent in $L_\infty(G)$ to zero; this will prove that T_g is not completely continuous.

Choose a partition $\mathscr{A}_1$ of A such that (i) is true, choose a real-valued function φ_1 in Step $\mathscr{A}_1$ that belongs to $L_1(G)$ and a point x_1 in G such that (v) holds. Assume that for some n we have already constructed $\{\mathscr{A}_j\}_{j=1}^n$, $\{x_j\}_{j=1}^n$, $\{\varphi_j\}_{j=1}^n$ such that the conditions (i), (ii), (iii) and (v) are true respectively for all n, until $n-1$, until $n-1$, until n. Choose a new partition $\mathscr{A}_n'$ of A that refines $\mathscr{A}_n$ and contains no sets of diameter $> 2^{-n-1}$. By assumption, corresponding to $\mathscr{A}_n'$ we may find a point x_{n+1} in G such that $\int_A |g(a^{-1}x_{n+1}) - h(a)|\, da \geqslant \varepsilon_0$ whenever h belongs to Step $\mathscr{A}_n'$. If $\mathscr{A}_n' = \{B_i\}_{i=1}^m$, apply lemma 3.4 to obtain a partition $B_i = C_i \cup D_i$ for each i such that

$$m(C_i) = m(D_i) = \frac{1}{2} m(B_i),$$

and

$$\int_A g(a^{-1}x_{n+1})(\chi_{\bigcup_i C_i}(a) - \chi_{\bigcup_i D_i}(a))\, da \geqslant \varepsilon_0 .$$

Now define the partition $\mathscr{A}_{n+1}$ wanted for the induction as

$$\mathscr{A}_{n+1} = \{C_1, \ldots, C_m, D_1, \ldots, D_m\},$$

and define φ_{n+1} on G such that $\varphi_{n+1}(y) = 1$ for $y \in \bigcup_{i=1}^{m} C_i$, $\varphi_{n+1}(z) = -1$ for $z \in \bigcup_{i=1}^{m} D_i$, and $\varphi_{n+1}(t) = 0$ for $t \in G\backslash A$. Then the conditions (i), (ii) and (iii) are clearly true for $\mathscr{A}_{n+1}$ and φ_{n+1}. Also (v) is true for $n+1$, since

$$(\varphi_{n+1} * g)(x_{n+1}) = \int_A \varphi_{n+1}(a) g(a^{-1}x_{n+1})\, da \geqslant \varepsilon_0 ,$$

by definition of $\mathscr{A}_{n+1}$ and φ_{n+1}.

Finally, to prove (iv) we have to show that for any k in $L_\infty(G)$ the sequence $\left\{\int_G \varphi_n(y)k(y)\, dy\right\}_{n=1}^{\infty}$ converges to zero.

Since any k in $L_\infty(G)$ may be approximated by a sequence of simple functions, and by the outer regularity of the Haar measure, it is sufficient to prove that the sequence $\left\{\int_A \varphi_n(y)\chi_U(y)\, dy\right\}_{n=1}^{\infty}$ converges to zero for any open set U.

Let d be the metric on G, N a positive integer, and put

$$F_N = \left\{x \in G : d(x, A\backslash U) < \frac{1}{N}\right\}.$$

Since $A\backslash U$ is compact, given $\delta > 0$ we may choose a natural number N sufficiently large such that $m(F_N\backslash(A\backslash U)) < \delta$. For $n > N$, all sets of the partition $\mathscr{A}_n$ have a diameter smaller than $\frac{1}{N}\cdot$ Denote by D the union of all sets in the partition $\mathscr{A}_n$ that contain a point of $A\backslash F_N$; we have $A\backslash F_N \subset D \subset A \cap U$, and $m((A\cap U)\backslash D) < \delta$.

Moreover, by property (iii) the integral $\int_D \varphi_p(y)\,dy$ is zero whenever $p > n$. The result then follows since $|\varphi_p(y)| \leqslant 1$ for all p.

If g is complex-valued and not uniformly measurable, then the real part or the imaginary part of g is not uniformly measurable, and by the foregoing procedure we again arrive at the result since the constructed sequence $\{\varphi_n\}_{n=1}^{\infty}$ consists of real-valued functions. □

4. Examples and properties of uniformly measurable functions.

THEOREM 4.1. – *If* $g \in C_{ru}(G)$, *then* g *is uniformly measurable.*

Proof. – Given $\varepsilon > 0$, there exists a symmetric open neighborhood $V = V^{-1}$ of e such that $|g(y) - g(sy)| < \varepsilon$, $\forall x \in G$, $\forall s \in V$. Given a compact set A in G, there exists a partition $\mathscr{A} = \{A_i\}_{i=1}^{n}$ of A such that $A_i \subset x_i V$ for suitable points $x_i\,(i=1,\ldots,n)$ in A. For x in G, choose $A_x = \varnothing$, and $c_{x,i} = g(x_i^{-1}x)$; then we clearly have that $|g(a^{-1}x) - c_{x,i}| < \varepsilon$ for all $a \in A_i$. □

From theorem 3.2 and the foregoing result we conclude that all T_g are completely continuous if G is discrete (this follows directly from [4, IV.8.13]).

THEOREM 4.2. – *If* $g \in L_{\infty}(G)$ *has compact support, then* g *is uniformly measurable.*

Proof. – Let S be the compact support of g, and A a given compact set. If $x \notin AS$ then $g(a^{-1}x) = 0$ for all a in A. Hence we only have to investigate condition (iii) of theorem 3.1 for those x in a compact subset $L = AS$ of G.

Since g may be uniformly approximated by simple functions, for given x in L and $\varepsilon > 0$ there exist numbers $c_{x,i}$ and a partition $\mathscr{A}_x = \{A_i^x\}_{i=1}^{n}$ of A such that $|g(a^{-1}x) - c_{x,i}| < \varepsilon$ for all a in A_i^x.

Given $\delta > 0$ there exists a compact subset B_i^x of A_i^x such that

$$m(A_i^x \backslash B_i^x) < \frac{\delta}{2n}.$$

For each i choose an open neighborhood U_i of B_i^x such that $m(U_i \backslash B_i^x) < \frac{\delta}{4n}$, and a neighborhood V_i of e such that $B_i^x V_i \cup V_i B_i^x \subset U_i$. For each v in V_i the set $(B_i^x \backslash v B_i^x) \cup (v B_i^x \backslash B_i^x)$ has a measure smaller than $\frac{\delta}{2n}$. Doing the same for each i, we obtain a neigborhood $V = \bigcap_{i=1}^{n} V_i$ of e such that, for all y in Vx, the same partition A_i^x and the same constants $c_{x,i}$ may be used to obtain $|g(a^{-1}y) - c_{x,i}| < \varepsilon$ for all a in $A_i \backslash A_y$, where $m(Ay) < \delta$; indeed,

$$A_y = \left(\bigcup_{i=1}^{n} (A_i^x \backslash B_i^x) \right) \cup \left(\bigcup_{i=1}^{n} ((B_i^x \backslash v B_i^x) \cup (v B_i^x \backslash B_i^x)) \right),$$

for some v in V. Repeating this procedure for each x in L, we find m points x_j of L and m neighborhoods V_j of e such that $L = \bigcup_{j=1}^{m} V_j x_j$. A common refinement of the $\mathscr{A}_{x_j}$ then gives a suitable partition of A.

□

From this theorem and theorem 3.2 we deduce that all T_g are completely continuous if G is compact.

THEOREM 4.3. — *The set of all uniformly measurable functions in* $L_\infty(G)$ *is a norm-closed linear subspace of* $L_\infty(G)$.

Proof. — It is almost trivial that the set of uniformly measurable functions is closed under addition and scalar multiplication. If $\{g_j\}_{j=1}^{\infty}$ is a sequence of uniformly measurable functions which is norm convergent in $L_\infty(G)$ to a function g, then given $\varepsilon > 0$ choose a natural number N such that $|g(y) - g_N(y)| < \frac{\varepsilon}{2}$ for all y in G except on a locally null set. Given $\varepsilon > 0$, $\delta > 0$, A compact, the triangle inequality shows that the partition of A for g_N and the numbers $c_{x,i}$ for g_N corresponding to a given point x in G may also be used for g.

□

If $g \in L_\infty(G)$ and there exist a compact set K and a uniformly measurable function h such that $g = h$ on $G \backslash K$, then g is uniformly measurable. Indeed, $g = h + (g - h)$. In the same way a function in $L_\infty(G)$ which vanishes at infinity is uniformly measurable.

If G is unimodular some additional results hold.

THEOREM 4.4. – *Let* G *be unimodular,* $g \in L_\infty(G)$. *Assume that for each* $\varepsilon' > 0$ *there exists a uniformly measurable* $g_{\varepsilon'}$ *and a measurable set* T *with* $m(T) < \varepsilon'$ *such that* $g = g_{\varepsilon'}$ *on* $G \setminus T$. *Then* g *is uniformly measurable.*

Proof. – Given $\delta > 0$, choose $g_{\varepsilon'}$ with $\varepsilon' = \frac{\delta}{2}\cdot$ Given $\varepsilon > 0$ and A compact, choose a partition $\mathscr{A} = \{A_i\}_{i=1}^n$ of A for $g_{\varepsilon'}$ such that $|g_{\varepsilon'}(a^{-1}x) - c_{x,i}| < \varepsilon \ \forall a \in A_i \setminus A_x$, where $m(A_x) < \frac{\delta}{2}\cdot$ Now $g(a^{-1}x) = g_{\varepsilon'}(a^{-1}x)$ except for $a \in xT^{-1}$. Since $m(xT^{-1}) < \frac{\delta}{2}$, it suffices to replace A_x by $B_x = A_x \cup (xT^{-1} \cap A)$.

□

COROLLARY 4.5. – *If* G *is unimodular, and* $g \in L_1(G) \cap L_\infty(G)$, *then* G *is uniformly measurable.*

Proof. – This follows from theorems 4.3, 4.4 and 4.2 if we remark that g may be uniformly approximated by simple functions, each simple function being expressed by means of characteristic functions of measurable sets with finite measure, and then use the inner regularity of the Haar measure for those measurable sets.

□

5. Connection with measurable vector-valued functions.

It is not true that for given g in $L_\infty(G)$ the function $y \to {}_{y^{-1}}g$ from G to $(L_\infty(G), \|\ \|_\infty)$ is always measurable in the sense of [5, 8.14.1]; e.g. the function g used in the example in section 2 does not have this property. We first show that, whenever $y \to {}_{y^{-1}}g$ is measurable, then g is uniformly measurable.

THEOREM 5.1. – *Let* g *be a function in* $L_\infty(G)$ *such that* $F : G \to (L_\infty(G), \|\ \|_\infty)$, $F(y) = {}_{y^{-1}}g$ *is measurable. Then* g *is uniformly measurable.*

Proof. – We show that condition (iv) of theorem 3.1 is true. Given A compact, there exists a sequence of simple vector-valued functions which is almost uniformly convergent on A to F. So, given $\varepsilon > 0$, there exists a partition $\mathscr{A} = \{A_i\}_{i=1}^n$ of A, functions $\{h_i\}_{i=1}^n$ in $L_\infty(G)$, and subsets B_i of A_i such that

$$m\left(\bigcup_{i=1}^n B_i\right) < \frac{\varepsilon}{6\|g\|_\infty}, \quad \text{and} \quad \|_{a^{-1}}g - h_i\|_\infty < \frac{\varepsilon}{2m(A)}$$

for all $a \in A_i \backslash B_i$; we may suppose that $\|h_i\|_\infty < 2\|g\|_\infty$ for each i. Given an open set U in G with $m(U) < \infty$, we define the complex-valued function h on $A \times U$ by putting $h(a,x) = h_i(x)$ if $a \in A_i$. For fixed a in $\bigcup_{i=1}^n (A_i \backslash B_i)$, there exists a null set C_a in U such that

$$|g(a^{-1}x) - h(a,x)| < \frac{\varepsilon}{2m(A)} \text{ for } x \in U \backslash C_a,$$

and so

$$\int_{A \backslash \bigcup_{i=1}^n B_i} \left(\int_U |g(a^{-1}x) - h(a,x)|\, dx \right) da < \frac{\varepsilon}{2m(A)} m(U)m(A) = \frac{\varepsilon}{2} m(U).$$

Since we can repeat the above for any measurable subset V of U instead of U itself we obtain, by an application of Fubini's theorem,

$$\int_{A \backslash \bigcup_{i=1}^n B_i} |g(a^{-1}x) - h(a,x)|\, da < \frac{\varepsilon}{2}$$

for almost all $x \in U$ (and hence on a dense subset of U). For such x we also have that

$$\int_A |g(a^{-1}x) - h(a,x)|\, da = \int_{A \backslash \bigcup_{i=1}^n B_i} + \int_{\bigcup_{i=1}^n B_i} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

Finally, putting $h_x(a) = h(a,x)$ for all such x, h_x belongs to Step $\mathscr{A}$, and condition (iv) of theorem 3.1 is fulfilled.

□

The converse of theorem 5.1 is not true. Indeed, consider the additive group $\mathscr{R}$ of real numbers and, for $x \in \mathscr{R}$, put $g(x) = 1$ if $x \geqslant 0$ and $g(x) = 0$ if $x < 0$. Since we can always change the values of g on a compact interval to make the function a uniformly continuous one (e.g., by going linearly from the value 0 to 1 on the interval $[-1, +1]$), it follows from theorem 4.1 and the remark following theorem 4.3 that g is uniformly measurable. However, the function $F : \mathscr{R} \to (L_\infty(\mathscr{R}), \| \|_\infty)$, $F(y) = {}_{y^{-1}}g$ is not measurable; indeed, if $y \neq z$ then $\|{}_{y^{-1}}g - {}_{z^{-1}}g\|_\infty = 1$, and this implies that the set $\{{}_{y^{-1}}g : y \in A\}$ is not contained in a separable linear subspace of $L_\infty(\mathscr{R})$ if A is an uncountable subset of $\mathscr{R}$ with $m(A) < \infty$; this contradicts a well-known property of measurable vector-valued functions (e.g. [5, th. 8.15.2]; [3, II th. 2]).

6. Remarks.

6.1. A bounded linear operator T from a Banach space X to a Banach space Y is called strictly singular [almost weakly compact] if, whenever T has a bounded inverse on a closed subspace M of X, then M is finite-dimensional [reflexive] (see [8], [6]). Denoting by STR and AWC the sets of strictly singular and almost weakly compact operators respectively, we have STR $\subset$ AWC. It was shown in [2] that, for the additive group Z of integer numbers with the discrete topology, the sets of multipliers from $L_1(Z)$ to $L_\infty(Z)$ belonging to STR and AWC are equal. By our previous results, this property about multipliers is also valid for any discrete and any compact group G, due to the following.

LEMMA 6.1. — *A bounded linear operator* T *from* X *to* Y *which is completely continuous and almost weakly compact, is strictly singular.*

Proof. — Let T have a bounded inverse on a closed subspace M of X. If T belongs to AWC, then M is reflexive, and so the unit ball $b(M)$ of M is weakly sequentially compact. Every sequence $\{x_n\}_{n=1}^\infty$ in $b(M)$ has a subsequence $\{x_{n_k}\}_{k=1}^\infty$ that weakly converges to a point x. If T is also completely continuous, then $\{Tx_{n_k}\}_{k=1}^\infty$ is norm convergent in Y to Tx; making use now of the existence of a bounded inverse of T, we derive that $\{x_{n_k}\}_{k=1}^\infty$ is norm convergent in X to x, which means that $b(M)$ is compact; hence M is finite-dimensional.

□

6.2. The introductory example in Section 2 does not stand alone. Indeed, for every non-compact, non-discrete metrizable group G there exist non-completely continous multipliers $T_g(g \in L_\infty(G))$. To construct such g consider any compact $A \subseteq G$ with $m(A) > 0$. As in Theorem 3.5 we define a sequence of real-valued measurable functions $\{\varphi_n\}_{n=1}^\infty$ on G with $|\varphi_n| \equiv 1$ on A, $|\varphi_n| \equiv 0$ on $G \backslash A$ such that φ_n weakly converges to zero in $L_1(G)$. Choose a sequence $\{x_n\}_{n=1}^\infty$ in G such that $A^{-1}x_n \cap A^{-1}x_m = \varnothing$ if $n \neq m$. Define $g \in L_\infty(G)$ by putting $g(y^{-1}x_n) = \varphi_n(y)$ for $y \in A$ and $g(z) = 0$ if $z \notin \bigcup_n A^{-1}x_n$. Then

$$\|\varphi_n * g\|_\infty \geqslant |(\varphi_n * g)(x_n)| = \left|\int_G \varphi_n(y) g(y^{-1}x_n)\, dy\right| = m(A)$$

for all n. It would be easy, too, to make g a continuous function.

6.3. Let k be a function in $L_1(G)$, H_k the convolution operator from $L_1(G)$ to $L_1(G)$, induced by k. For noncompact G, the only weakly compact H_k is the zero-operator [1]. However, all H_k are completely continuous, as the following theorem shows (for compact Abelian G, see [3, p. 90]).

THEOREM 6.2. – *Let $k \in L_1(G)$. If $\{f_n\}_{n=1}^\infty$ weakly converges to zero in $L_1(G)$, then $\{f_n * k\}_{n=1}^\infty$ norm converges to zero in $L_1(G)$.*

Proof. – Since $C_c(G)$ is dense in $L_1(G)$, we may suppose that $k \in C_c(G)$; let K be the support of k. Given $\varepsilon > 0$, there exists a compact set K_1 in G such that $\int_{G\backslash K_1} |f_n(x)|\, dx < \frac{\varepsilon}{2\|k\|_1}$, $\forall n \in Z^+$. Let K_2 be a compact set in G such that $K_2 \supset K_1 K$. We have

$$(1) \quad \|f_n * k\|_1 = \int_{G\backslash K_2} |(f_n * k)(x)|\, dx + \int_{K_2} |(f_n * k)(x)\, dx.$$

Since $k \in L_\infty(G)$, we know from theorems 4.2. and 3.2. that $f_n * k$ is $\| \ \|_\infty$-convergent to zero; hence

$$\int_K |(f_n * k)(x)|\, dx < \frac{\varepsilon}{2}, \quad \forall n \geqslant N_1.$$

Further,

$$\begin{aligned}\int_{G\setminus K_2} |(f_n * k)(x)|\,dx &\leqslant \int_{G\setminus K_2}\left(\int_G |f_n(y)|\,|k(y^{-1}x)|\,dy\right)dx\\ &= \int_{G\setminus K_1} |f_n(y)|\left(\int_{G\setminus K_2} |k(y^{-1}x)|\,dx\right)dy\\ &+ \int_{K_1} |f_n(y)|\left(\int_{G\setminus K_2} |k(y^{-1}x)|\,dx\right)dy.\end{aligned}$$

Clearly,

$$\int_{G\setminus K_1} |f_n(y)|\left(\int_{G\setminus K_2} |k(y^{-1}x)|\,dx\right)dy < \frac{\varepsilon}{2},$$

$\forall n \in Z^+$ by our choice of K_1, while

$$\int_{K_1} |f_n(y)|\left(\int_{G\setminus K_2} |k(y^{-1}x)|\,dx\right)dy = 0$$

since, whenever $y \in K_1$ and $x \in G\setminus K_2$, then $y^{-1}x \notin K$ by the choice of K_2. Hence $\|f_n * k\|_1 < \varepsilon$, $\forall n \geqslant N_1$, from which the result follows.

□

Since any g in $C_{ru}(G)$ may be written as $g = k * h$ with $k \in L_1(G)$ and $h \in L_\infty(G)$, we again deduce that any g in $C_{ru}(G)$ induces a completely continuous T_g from $L_1(G)$ to $L_\infty(G)$.

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